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Logarithmic Solutions of the Fifth Painlevé Equation near the Origin

Dedicated to Professor Ken-ichi SHINODA

Shun SHIMOMURA

Keio University

Abstract. For the fifth Painlevé equation near the origin we present two kinds of logarithmic solutions, which are represented, respectively, by convergent series with multipliers admitting asymptotic expansions in descending logarithmic powers and by those with multipliers polynomial in logarithmic powers. It is conjectured that the asymptotic multipliers are also polynomials in logarithmic powers. These solutions are constructed by iteration on certain rings of exponential type series.

1. Introduction

The Painlevé transcendents as nonlinear special functions are expected to be of great use in a variety of problems in pure mathematics and mathematical physics. For the sixth Painlevé equation

$$\text{(VI)}\qquad \frac{d^2y}{dx^2}=\frac{1}{2}\Big(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\Big)\Big(\frac{dy}{dx}\Big)^2-\Big(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\Big)\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{2x^2(x-1)^2}\Big((\theta_\infty-1)^2-\frac{\theta_0^2x}{y^2}+\frac{\theta_1^2(x-1)}{(y-1)^2}+\frac{(1-\theta_x^2)x(x-1)}{(y-x)^2}\Big)$$

with $\theta_\infty,\theta_0,\theta_1,\theta_x\in\mathbb{C}$, Guzzetti [8] provided the tables of the critical behaviours of solutions near $x=0,\ 1,\ \infty$ as well as the parametric connection formulas, in which the solutions are classified into five groups: complex power types, inverse oscillatory types, Taylor series types, logarithmic types, and inverse logarithmic types. The logarithmic and the inverse logarithmic solutions near $x=0$ are derived from two basic logarithmic ones represented by series expansions, whose leading terms

$$y(x)\sim x((\theta_x^2-\theta_0^2)(a+\log x)^2/4+\theta_0^2/(\theta_0^2-\theta_x^2))\qquad \text{if } \theta_x^2\neq\theta_0^2\,,$$
$$y(x)\sim x(a\pm\theta_0\log x)\qquad \text{if } \theta_x^2=\theta_0^2$$

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were found by Guzzetti [5], [6] by applying a matching procedure to a Fuchsian system related to (VI) through isomonodromy deformation [11]; and the series expansions are convergent in a sector [17]. Here a is an integration constant. The logarithmic leading terms above as well as complex power ones also follow from asymptotic results on coefficient matrices of the Fuchsian system by Jimbo [10], who obtained connection formulas of the τ-functions for some Painlevé equations including (VI).

The fifth Painlevé equation normalized in the form

$$\text{(V)}\qquad \frac{d^2y}{dx^2} = \Big(\frac{1}{2y} + \frac{1}{y-1}\Big)\Big(\frac{dy}{dx}\Big)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{(y-1)^2}{x^2}\Big(\alpha y + \frac{\beta}{y}\Big) + \frac{\gamma y}{x} - \frac{y(y+1)}{2(y-1)}$$

with $\alpha, \beta, \gamma \in \mathbb{C}$ has the critical points at $x = 0$ and $x = \infty$. Near $x = 0$ equation (V) admits a two-parameter family of complex power solutions [16]. Applying WKB analysis to a linear system associated with (V) through isomonodromy deformation, Andreev and Kitaev [1] derived asymptotic solutions near $x = 0$ and $x = \infty$ with connection formulas. Kaneko and Ohyama [12] gave Taylor series solutions, and showed that the corresponding linear system at $x = 0$ is solvable in terms of the hypergeometric function and that the monodromy may be explicitly calculated. The solutions having logarithmic terms that follow from the results for (V) in [10] are of inverse logarithmic types, which are asymptotic to $c \log^{-2} x$ or $c \log^{-1} x$ and contain summands of negative logarithmic powers. By the method of power geometry, Bruno and Parusnikova [2] obtained formal series solutions of logarithmic types that have leading terms, respectively, quadratic and linear in $\log x$ like those of the basic logarithmic solutions of (VI); in the quadratic case the series apparently contains negative logarithmic powers.

In this paper we present convergent series representations for the logarithmic-type solutions of (V) mentioned above, clarifying the structure of higher order terms. The quadratic-logarithmic (respectively, linear-logarithmic) solutions are represented by convergent series with multipliers admitting asymptotic expansions in descending powers of $\log x$ (respectively, multipliers polynomial in $\log x$), which are given in Theorem 2.1 (respectively, Theorems 2.2 and 2.3). It is conjectured that the asymptotic multipliers for the quadratic-logarithmic solutions are also polynomials in $\log x$ (cf. Remark 2.1). This conjectured quadratic-logarithmic solution is regarded as one of the counterparts of the basic logarithmic solutions of (VI). These solutions of (VI), which are free of negative logarithmic powers, are obtained from matrix solutions of the associated Schlesinger equation [17], while a parallel argument for (V) yields solutions of inverse logarithmic types. For the possible basic nature as a logarithmic solution of (V), in Theorem 2.1 we describe a representation of the form derived rigorously from (V) itself; and it is also a convincing clue to the conjecture above. As degenerate cases of the linear-logarithmic solutions, we obtain families of Taylor series solutions in Remark 2.2 under a condition different from that in [12]. The linear-logarithmic solutions in Theorems 2.2 and 2.3 are constructed by iteration on a ring of exponential type series with polynomial multipliers in Sections 3 and 4. In the final section, considering a ring of exponential type series with asymptotic multipliers, we prove Theorem 2.1.

2. Main results

Let $D_0 \subset \mathbb{C} \setminus \{0\}$ be a given simply connected bounded domain such that distance$(D_0, \{0\}) > 0$, and let Θ_0 and B_0 be given positive numbers. Denote by $\Sigma(\varepsilon, \Theta_0)$ the sector $0 < |x| < \varepsilon$, $|\arg x| < \Theta_0$ on the universal covering of $\mathbb{C} \setminus \{0\}$.

THEOREM 2.1. *Suppose that* $|\alpha| + |\beta| + |\gamma| + |\gamma^{-1}| < B_0$. *Then* (V) *has a one-parameter family of solutions* $\{y_{\mathrm{quad}}(\rho, x) \mid \rho \in D_0\}$ *with the properties*:

(i) $y_{\mathrm{quad}}(\rho, x)$ *is holomorphic in* $(\rho, x) \in D_0 \times \Sigma(\varepsilon_0, \Theta_0)$, $\varepsilon_0 = \varepsilon_0(D_0, \Theta_0, B_0)$ *being a sufficiently small positive number depending only on* D_0, Θ_0 *and* B_0;

(ii) $y_{\mathrm{quad}}(\rho, x)$ *is expanded into a convergent series written in the form*

$$y_{\mathrm{quad}}(\rho, x) = 1 + \sum_{n=1}^{\infty} x^n \Big(p_n(-\log(\rho x)) + p_n^-(-\log(\rho x)) \Big)$$

with $(p_n(s), p_n^-(s))$ *such that, for each* n,

(ii.a) $p_n(s)$ *is given by*

$$p_1(s) = -\frac{\gamma}{2}s^2 + \frac{1}{2\gamma}, \quad p_n(s) = \sum_{j=0}^{2n} p_j^n s^j, \quad p_j^n \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}],$$

and that

(ii.b) $p_n^-(s)$ *is holomorphic in* $S(R_0, \widehat{\Theta}_0)$ *and admits an asymptotic representation of the form*

$$p_n^-(s) \sim \sum_{j=1}^{\infty} p_{-j}^n s^{-j}, \quad p_{-j}^n \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$$

as $s \to \infty$ *through* $S(R_0, \widehat{\Theta}_0)$, *in particular*, $p_1^-(s) \equiv 0$, *where* $S(R_0, \widehat{\Theta}_0)$ *is a strip* $\mathrm{Re}\, s > R_0$, $|\mathrm{Im}\, s| < \widehat{\Theta}_0$ *with the property that* $-\log(\rho x) \in S(R_0, \widehat{\Theta}_0)$ *holds for every* $(\rho, x) \in D_0 \times \Sigma(\varepsilon_0, \Theta_0)$.

REMARK 2.1. As a matter of fact $p_2^-(s) = p_3^-(s) \equiv 0$, and $p_2(s)$ and $p_3(s)$ are written in terms of $\phi(s) := p_1(s)$:

$$p_2(s) = \Big(\alpha + \beta + \frac{1}{2}\Big)\phi(s)^2 + 2(\alpha + \beta)(\phi(s)\phi'(s) - 2\gamma\phi(s) - \gamma\phi'(s) + 1);$$

$$\begin{aligned} p_3(s) =& \Big(\frac{3}{4}(\alpha + \beta)^2 + \frac{5}{4}\alpha + \frac{3}{4}\beta + \frac{3}{16}\Big)\phi(s)^3 + \Big(3(\alpha + \beta)^2 + \frac{5}{2}\alpha + \frac{3}{2}\beta\Big)\phi(s)^2\phi'(s) \\ & - \Big(\frac{185}{16}(\alpha + \beta)^2 + \frac{83}{16}\alpha + \frac{45}{16}\beta + \frac{1}{64}\Big)\gamma\phi(s)^2 \end{aligned}$$

$$
\begin{aligned}
&- \Big(\frac{209}{16}(\alpha+\beta)^2 + \frac{47}{16}\alpha + \frac{17}{16}\beta + \frac{1}{64}\Big)\gamma\phi(s)\phi'(s) \\
&+ \Big(5(\alpha+\beta)^2 + \frac{5}{2}\alpha + \frac{3}{2}\beta + \Big(\frac{273}{16}(\alpha+\beta)^2 + \frac{15}{16}\alpha - \frac{15}{16}\beta + \frac{1}{64}\Big)\gamma^2\Big)\phi(s) \\
&+ \Big(\frac{5}{2}(\alpha+\beta)^2 + \frac{\alpha}{4} - \frac{\beta}{4} + \Big(\frac{401}{64}(\alpha+\beta)^2 + \frac{15}{64}\alpha - \frac{15}{64}\beta + \frac{1}{256}\Big)\gamma^2\Big)\phi'(s) \\
&- \Big(\frac{209}{32}(\alpha+\beta)^2 + \frac{15}{32}\alpha - \frac{15}{32}\beta + \frac{1}{128}\Big)\gamma - 2(\alpha+\beta)^2\gamma^3 .
\end{aligned}
$$

As will be mentioned in Section 5.4, $p_n(s) + p_n^-(s)$ $(n \geq 2)$ are computed by solving linear differential equations recursively. At least for $n = 2, 3$, the inhomogeneous part of each equation has the form such that $p_n^-(s) \equiv 0$ may be derived. Even in finding $p_3(s)$ above and in showing $p_3^-(s) \equiv 0$, Maple is used, and it seems difficult to check whether $p_n^-(s) \equiv 0$ or not for all n by such a method. In view of the logarithmic solutions of (VI) (cf. [8], [17]) together with the fact $p_n^-(s) \equiv 0$ for $n = 1, 2, 3$, we may conjecture that $p_n^-(s) \equiv 0$ for every $n \geq 1$, that is, $y_{\rm quad}(\rho, x) = 1 + \sum_{n=1}^{\infty} x^n p_n(-\log(\rho x))$.

If $\gamma \in \mathbb{Z}$, then there exists another class of logarithmic solutions. Let κ_0, κ_∞, θ and κ be parameters such that

$$
\alpha = \kappa_\infty^2/2, \quad \beta = -\kappa_0^2/2, \quad \gamma = -\theta - 1, \quad \kappa = ((\kappa_0 + \theta)^2 - \kappa_\infty^2)/4 , \tag{2.1}
$$

which are coefficients of the Hamiltonian function associated with (V) given by Okamoto [15].

THEOREM 2.2. *Suppose that* $|\kappa_0| + |\kappa| < B_0$. *If* $\theta = N \in \mathbb{Z}$, *then* (V) *has a one-parameter family of solutions* $\{y_{\rm lin}^{0(N)}(\rho, x) \mid \rho \in D_0\}$, *where* $y_{\rm lin}^{0(N)}(\rho, x)$ *is holomorphic in* $(\rho, x) \in D_0 \times \Sigma(\varepsilon_0', \Theta_0)$ *for sufficiently small* $\varepsilon_0' = \varepsilon_0'(D_0, \Theta_0, B_0, N)$ *and is expanded into a convergent series described as follows*:

(i) *if* $N = 0$, *then*

$$
y_{\rm lin}^{0(0)}(\rho, x) = 1 - x - x^2\Big(\kappa \log(\rho x) - \frac{1}{2}(\kappa - \kappa_0 + 1)\Big) + \sum_{n=3}^{\infty} x^n P_n^0(\log(\rho x))
$$

with $P_n^0(s) \in \mathbb{Q}[\kappa_0, \kappa][s]$ *such that* $\deg_s P_n^0(s) \leq n - 1$;

(ii) *if* $N = -1$, *then*

$$
y_{\rm lin}^{0(-1)}(\rho, x) = 1 - x\log(\rho x) + \sum_{n=2}^{\infty} x^n P_n^{-1}(\log(\rho x))
$$

with $P_n^{-1}(s) \in \mathbb{Q}[\kappa_0, \kappa][s]$ *such that* $\deg_s P_n^{-1}(s) \leq n$;

(iii) *if* $N \le -2$, *then*

$$y_{\rm lin}^{0(N)}(\rho, x) = 1 + \sum_{n=1}^{|N|-1} c_n x^n + c_{|N|} x^{|N|} \log(\rho x) + \sum_{n=|N|+1}^{\infty} x^n P_n^N(\log(\rho x))$$

with c_n, $c_{|N|} \in \mathbb{Q}[\kappa_0, \kappa]$ *and* $P_n^N(s) \in \mathbb{Q}[\kappa_0, \kappa][s]$ *such that* $\deg_s P_n^N(s) \le n - |N|$;

(iv) *if* $N \ge 1$, *then*

$$y_{\rm lin}^{0(N)}(\rho, x) = 1 + \sum_{n=1}^{N} c_n x^n + \sum_{n=N+1}^{\infty} x^n P_n^N(\log(\rho x))$$

with $c_n \in \mathbb{Q}[\kappa_0, \kappa]$ *and* $P_n^N(s) \in \mathbb{Q}[\kappa_0, \kappa][s]$ *such that* $\deg_s P_n^N(s) \le n - N$.

Furthermore we have

THEOREM 2.3. *Suppose that* $|\kappa_0| + |\kappa_\infty| + |\theta| + |\kappa_\infty^{-1}| < B_0$.

(1) *If* $\kappa_0 - \kappa_\infty = N \in \mathbb{Z} \setminus \{0\}$, *then* (V) *has a one-parameter family of solutions* $\{y_{\rm lin}^{-(N)}(\rho, x) \mid \rho \in D_0\}$, *where* $y_{\rm lin}^{-(N)}(\rho, x)$ *is holomorphic in* $(\rho, x) \in D_0 \times \Sigma(\varepsilon_0'', \Theta_0)$ *for sufficiently small* $\varepsilon_0'' = \varepsilon_0''(D_0, \Theta_0, B_0, N)$ *and is expanded into a convergent series of the form*

$$y_{\rm lin}^{-(N)}(\rho, x) = \frac{\kappa_0}{\kappa_\infty} + \sum_{n=1}^{|N|-1} c_n x^n + x^{|N|}(c_{|N|} \log(\rho x) + c'_{|N|}) + \sum_{n=|N|+1}^{\infty} x^n P_n^N(\log(\rho x))$$

with c_n, $c_{|N|}$, $c'_{|N|} \in \mathbb{Q}[\kappa_0, \kappa_\infty, \theta, \kappa_\infty^{-1}]$ *and* $P_n^N(s) \in \mathbb{Q}[\kappa_0, \kappa_\infty, \theta, \kappa_\infty^{-1}][s]$ *such that* $\deg_s P_n^N(s) \le n - |N|$ *for* $|N| \ge 2$ (*respectively*, $\le n$ *for* $|N| = 1$).

(2) *If* $\kappa_0 + \kappa_\infty = N \in \mathbb{Z} \setminus \{0\}$, *then* (V) *has a one-parameter family of solutions* $\{y_{\rm lin}^{+(N)}(\rho, x) \mid \rho \in D_0\}$, *where the series expansion for* $y_{\rm lin}^{+(N)}(\rho, x)$ *is given by replacing* κ_∞ *with* $-\kappa_\infty$ *in the representation for* $y_{\rm lin}^{-(N)}(\rho, x)$.

REMARK 2.2. In each series expansion of Theorem 2.2 or 2.3, if the first logarithmic term vanishes, then the solution belongs to a one-parameter family of Taylor series solutions (cf. Remark 3.3). For example, if $\theta = N = 0$ and if $\kappa = 0$, then there exists a one-parameter family of solutions $\{y_*(a, x) \mid a \in \mathbb{C}\}$ with

$$y_*(a, x) = 1 - x + \Big(a + \frac{1}{2}(1 - \kappa_0)\Big)x^2 + \sum_{n=3}^{\infty} P_n^*(a) x^n ,$$

where $P_n^*(s) \in \mathbb{Q}[\kappa_0][s]$ and $\deg_s P_n^*(s) \le n - 1$, and $y_{\rm lin}^{0(0)}(\rho, x)$ satisfies

$$y_{\rm lin}^{0(0)}(\rho, x) \equiv y_*(0, x) = 1 - x + \frac{1}{2}(1 - \kappa_0)x^2 + \sum_{n=3}^{\infty} P_n^*(0) x^n .$$

REMARK 2.3. Gromak [3] found Bäcklund transformations of the form

$$\hat{y} = 1 - \frac{2\nu_1 xy}{xy' - \nu_\infty\kappa_\infty y^2 + (\nu_\infty\kappa_\infty - \nu_0\kappa_0 + \nu_1 x)y + \nu_0\kappa_0},$$

where each ν_* ($* = 0, 1, \infty$) takes 1 or -1. Suitable composites of them make substitutions of parameters including

$$(\kappa_\infty, \kappa_0, \theta) \mapsto (\kappa_\infty, \kappa_0 - m, \theta + m) \quad \text{for any } m \in \mathbb{Z},$$

$$(\kappa_\infty, \kappa_0, \theta) \mapsto ((\kappa_\infty + \kappa_0 - \theta)/2, (\kappa_\infty + \kappa_0 + \theta)/2, \kappa_0 - \kappa_\infty),$$

$$(\kappa_\infty, \kappa_0, \theta) \mapsto ((\kappa_0 - \kappa_\infty - \theta)/2, (\kappa_0 - \kappa_\infty + \theta)/2, \kappa_0 + \kappa_\infty)$$

[4, Theorem 39.9] (see also [15], [12, §3.1]). This fact together with [12, Theorem 3] suggests that, any two solutions in $\{y_{\text{lin}}^{0(N)}(\rho, x) \mid N \in \mathbb{Z}\} \cup \{y_{\text{lin}}^{\pm(N)}(\rho, x) \mid N \in \mathbb{Z} \setminus \{0\}\}$ are related by some successive application of such Bäcklund transformations, and the author believes that every linear-logarithmic solution is derived from a seed solution, say $y_{\text{lin}}^{0(-1)}(\rho, x)$. Since there exist quadratic-logarithmic and inverse logarithmic types as well simultaneously with a linear-logarithmic solution, to verify this correspondence it seems necessary to know the resultant of the transformation to some extent. However it is not easy to check its form as in Theorem 2.2 or 2.3 by a practical computation. Indeed even the substitution $(\kappa_\infty, \kappa_0, \theta) \mapsto (\kappa_\infty, \kappa_0 + 1, \theta - 1)$ needs a three-step application of Bäcklund transformations (cf. [4, p. 200]).

REMARK 2.4. For (VI) with $\theta_0 = \theta_1 = \theta_x = 0$ near $x = 0$ Guzzetti [7] derived an inverse logarithmic solution $y_{\text{VI}}^{\text{inverse}}(x) \sim -4(\theta_\infty - 1)^{-2}(\log x + a)^{-2}$ from

$$v_{\text{III}}(t) = -t(\log(t/4) + c_{\text{E}}) + O(t^5 \log^3 t)$$

appearing in [13], which satisfies the third Painlevé equation

$$v_{tt} = (v_t)^2/v - v_t/t + v^3 - 1/v \qquad (v_t = dv/dt).$$

This is equivalent to (V) with $\alpha = \beta = \gamma = 0$ through the change of variables $x = -4t$, $y = (1 - v)^2/(1 + v)^2$, so that $v_{\text{III}}(t)$ corresponds to $y_{\text{lin}}^{0(-1)}(-e^{c_{\text{E}}}/16, x)$. On the other hand $y_{\text{VI}}^{\text{inverse}}(x)$ is related to the basic quadratic-logarithmic solution of (VI) by a symmetric transformation [8].

3. Proof of Theorem 2.2

3.1. Ring of exponential type series with polynomial multipliers. Suppose that $|\kappa_0| + |\kappa| < B_0$, where B_0 is the constant in Section 2. Let $\widehat{\mathfrak{P}}$ be the ring of formal series of the form

$$\varphi = \varphi(a, t) = \sum_{n=1}^{\infty} e^{-nt} \sum_{j=0}^{2n} p_j^n t_a^j, \quad t_a := t - a, \quad p_j^n \in \mathbb{Q}[\kappa_0, \kappa],$$

where a is a complex parameter. Let $\varDelta_0 \subset \mathbb{C}$ be a given bounded domain, and let $\varTheta$ be a given positive number. In what follows we suppose that $R = R(\varDelta_0, \varTheta) > 8$ is so large that, for every $a \in \varDelta_0$,

$$|t_a| \le |e^{t/4}|, \quad |e^{-t/4}| \le 1/4 \quad \text{in} \quad S(R, \varTheta) : \operatorname{Re} t > R,\ |\operatorname{Im} t| < \varTheta. \tag{3.1}$$

For $\varphi \in \widehat{\mathfrak{P}}$ with $a \in \varDelta_0$ written as above, define the norm of φ by

$$\|\varphi\| = \|\varphi(a,t)\| = \|\varphi\|(t) := \sum_{n=1}^{\infty} |e^{-t}|^n \sum_{j=0}^{2n} |p_j^n||e^{t/4}|^j = \sum_{n=1}^{\infty}\sum_{j=0}^{2n} |p_j^n||e^{-t}|^{n-j/4},$$

and set, for each (κ_0, κ),

$$\mathfrak{P}(R,\varTheta) = \mathfrak{P}(\varDelta_0, R, \varTheta) := \left\{\varphi \in \widehat{\mathfrak{P}} \;\middle|\; \|\varphi\|(t) < \infty \text{ for every } t \in S(R,\varTheta)\right\}.$$

Then we have

PROPOSITION 3.1. (1) *If $\varphi \in \mathfrak{P}(R,\varTheta)$, then $\varphi = \varphi(a,t)$ is holomorphic in $(a,t) \in \varDelta_0 \times S(R,\varTheta)$ and satisfies $|\varphi(a,t)| \le \|\varphi\|(t)$.*

(2) $\|\varphi\| \equiv 0$ *if and only if* $\varphi \equiv 0$.

(3) *Let φ, $\psi \in \mathfrak{P}(R,\varTheta)$ and let $c \in \mathbb{Q}[\kappa_0,\kappa]$. Then $\varphi+\psi$, $\varphi\psi$, $c\varphi \in \mathfrak{P}(R,\varTheta)$, and*

$$\|\varphi+\psi\| \le \|\varphi\| + \|\psi\|, \quad \|\varphi\psi\| \le \|\varphi\|\,\|\psi\|, \quad \|c\varphi\| \le |c|\,\|\varphi\|.$$

PROOF. Let $K_0 \subset \varDelta_0 \times S(R,\varTheta)$ be a given compact set. Then we may choose $t_* \in S(R,\varTheta)$ and $\delta_0 > 0$ in such a way that $|e^{-t}| \le (1-\delta_0)|e^{-t_*}|$ for every $(a,t) \in K_0$. Suppose that $\varphi = \sum_{n=1}^{\infty} e^{-nt} \sum_{j=0}^{2n} p_j^n t_a^j \in \mathfrak{P}(R,\varTheta)$. Since $\|\varphi\|(t_*) < \infty$, we have $|p_j^n| \le |e^{t_*}|^{n-j/4}$ if n is sufficiently large, and then, by (3.1),

$$|e^{-nt} p_j^n t_a^j| \le |p_j^n||e^{-t}|^{n-j/4} \le |e^{-t}/e^{-t_*}|^{n-j/4} \le (1-\delta_0)^{n/2}$$

for $(a,t) \in K_0$, which implies the uniform convergence of φ on K_0. Thus the first assertion is verified. The remaining ones are checked easily. □

PROPOSITION 3.2. *Suppose that each $\varphi_\nu \in \mathfrak{P}(R,\varTheta)$ consists of summands for $n \ge \nu$, and that $\sum_{\nu=1}^{\infty} \|\varphi_\nu\| < \infty$ in $S(R,\varTheta)$. Then $\varphi^\infty = \sum_{\nu=1}^{\infty} \varphi_\nu \in \mathfrak{P}(R,\varTheta)$, and $\|\varphi^\infty\| \le \sum_{\nu=1}^{\infty} \|\varphi_\nu\|$.*

PROOF. Write $\varphi_\nu = \sum_{n=\nu}^{\infty}\sum_{j=0}^{2n} p_j^{\nu n} e^{-nt} t_a^j$. Then $\sum_{\nu=1}^{\infty}\sum_{n=\nu}^{\infty}\sum_{j=0}^{2n} p_j^{\nu n} e^{-nt} t_a^j$ converges absolutely in $\varDelta_0 \times S(R,\varTheta)$, and hence it is possible to rearrange the summands. Thus we have

$$\varphi^\infty = \sum_{n=1}^{\infty}\sum_{j=0}^{2n} p_j^{\infty n} e^{-nt} t_a^j, \qquad p_j^{\infty n} := \sum_{\nu=1}^{n} p_j^{\nu n} \in \mathbb{Q}[\kappa_0,\kappa],$$

which satisfies $\|\varphi^\infty\| \le \sum_{\nu=1}^{\infty} \|\varphi_\nu\| < \infty$ in $S(R,\varTheta)$ and $\varphi^\infty \in \mathfrak{P}(R,\varTheta)$. □

For $(n, j) \in \mathbb{N} \times (\mathbb{N} \cup \{0\})$ set

$$\mathcal{I}[e^{-nt}t_a^j] := -\frac{e^{-nt}}{n}\Big(t_a^j + \frac{j}{n}t_a^{j-1} + \cdots + \frac{j(j-1)\cdots(j-k+1)}{n^k}t_a^{j-k} + \cdots + \frac{j!}{n^j}\Big),$$

which induces the linear operator $\mathcal{I} : \widehat{\mathfrak{P}} \to \widehat{\mathfrak{P}}$. Note that the right-hand member is a primitive function of $e^{-nt}t_a^j$. If $j \le 2n$, then, by (3.1),

$$\begin{aligned}\left\|\mathcal{I}[e^{-nt}t_a^j]\right\| &= \frac{|e^{-t}|^n}{n}\sum_{k=0}^{j}\frac{j(j-1)\cdots(j-k+1)}{n^k}|e^{t/4}|^{j-k} \\ &\le |e^{-t}|^n|e^{t/4}|^j\sum_{k=0}^{j}2^k|e^{-t/4}|^k \le 2|e^{-t}|^{n-j/4}\end{aligned}$$

in $\varDelta_0 \times S(R, \Theta)$. Hence we have

LEMMA 3.3. *Suppose that* $\varphi \in \mathfrak{P}(R, \Theta)$. *Then*
(i) $\mathcal{I}[\varphi] \in \mathfrak{P}(R, \Theta)$, (ii) $\|\mathcal{I}[\varphi]\| \le 2\|\varphi\|$, (iii) $(d/dt)\mathcal{I}[\varphi] = \varphi$.

LEMMA 3.4. *If* $\varphi \in \mathfrak{P}(R, \Theta)$, *then, for each* $k \in \mathbb{N}$

$$e^{kt}\mathcal{I}[e^{-kt}\varphi] \in \mathfrak{P}(R, \Theta), \qquad \left\|e^{kt}\mathcal{I}[e^{-kt}\varphi]\right\| \le 2\|\varphi\|.$$

Let $\widehat{\mathfrak{P}}_0$ be the subring of $\widehat{\mathfrak{P}}$ consisting of formal series of the form

$$\varphi = \sum_{n=1}^{\infty} e^{-nt}\sum_{j=0}^{n} p_j^n t_a^j, \qquad p_j^n \in \mathbb{Q}[\kappa_0, \kappa].$$

LEMMA 3.5. *If* $\varphi \in \widehat{\mathfrak{P}}_0$, *then, for each* $k \in \mathbb{N} \cup \{0\}$, $e^{kt}\mathcal{I}[e^{-kt}\varphi] \in \widehat{\mathfrak{P}}_0$. *If* $\varphi \in \widehat{\mathfrak{P}}_0$ *has the form* $\varphi = \sum_{n=2}^{\infty} e^{-nt}\sum_{j=0}^{n} p_j^n t_a^j$, *then* $e^{-t}\mathcal{I}[e^t\varphi] \in \widehat{\mathfrak{P}}_0$.

We write $\mathfrak{P}_0(R, \Theta) := \mathfrak{P}(R, \Theta) \cap \widehat{\mathfrak{P}}_0$, which is a subring of $\mathfrak{P}(R, \Theta)$.

3.2. System of equations equivalent to (V). Equation (V) is equivalent to the Hamiltonian system

(3.2) $$dy/dx = \partial H_{\mathrm{V}}/\partial z, \quad dz/dx = -\partial H_{\mathrm{V}}/\partial y$$

with $H_{\mathrm{V}} = H_{\mathrm{V}}(x, y, z)$ such that

$$xH_{\mathrm{V}}(x, y, z) = y(y-1)^2z^2 - (\kappa_0(y-1)^2 + \theta y(y-1) + xy)z + \kappa(y-1),$$

which was obtained by Okamoto [15], [14] from isomonodromy deformation of a second-order linear differential equation, where κ_0, κ, θ are parameters satisfying (2.1).

REMARK 3.1. Kaneko and Ohyama [12] pointed out that another polynomial Hamiltonian system

$$dy/dx = \partial H_*/\partial w, \quad dw/dx = -\partial H_*/\partial y \tag{3.3}$$

with

$$\begin{aligned} xH_*(x, y, w) = &- y(y-1)^2w^2 + \Big(\frac{1}{2}(-\theta_0+\theta_1+\theta_\infty)(y-1)^2 \\ &+ (\theta_0+\theta_1)y(y-1) + xy + (\theta_0+\theta_1)\Big)w - \frac{\theta_1}{2}(\theta_0+\theta_1+\theta_\infty)y \end{aligned}$$

follows from isomonodromy deformation of a Schlesinger system by Jimbo and Miwa [11], and that (3.2) is related to (3.3) by

$$\kappa_0 = \frac{1}{2}(\theta_0-\theta_1-\theta_\infty), \quad \kappa_\infty = \frac{1}{2}(\theta_0-\theta_1+\theta_\infty), \quad \theta = \theta_0+\theta_1-2$$

and

$$w = -z + \frac{\theta+1}{y-1} + \frac{x}{(y-1)^2}.$$

Note that $\partial(xH_{\rm V})/\partial z\,|_{x=0}$ and $\partial(xH_{\rm V})/\partial y\,|_{x=0}$ simultaneously vanish at

$$(y, z) = (1, \kappa/\theta), \quad \Big(\pm\frac{\kappa_0}{\kappa_\infty}, -\frac{\kappa_\infty(\kappa_0+\theta\mp\kappa_\infty)}{2(\kappa_\infty\mp\kappa_0)}\Big). \tag{3.4}$$

System (3.2) with $x = e^{-t}$ is written in the form

$$y' = \partial H/\partial z, \quad z' = -\partial H/\partial y \tag{3.5}$$

($y' = dy/dt$) with

$$H := -e^{-t}H_{\rm V} = -y(y-1)^2z^2 + (\kappa_0(y-1)^2 + \theta y(y-1) + e^{-t}y)z - \kappa(y-1).$$

We would like to construct a family of solutions of this system (for a general Briot-Bouquet system see [9]).

Suppose that $\theta = N \in \mathbb{Z}\setminus\{0\}$. To write (3.5) around $(y, z) = (1, \kappa/\theta) = (1, \kappa/N)$ put

$$y = 1 + \tilde{y}, \quad z = \kappa/N + \tilde{z}.$$

Then

$$\begin{aligned} H = &- (1+\tilde{y})\tilde{y}^2\tilde{z}^2 + \Big(-\frac{2\kappa}{N}\tilde{y}^3 + \Big(\kappa_0 + N - \frac{2\kappa}{N}\Big)\tilde{y}^2 + (N+e^{-t})\tilde{y} + e^{-t}\Big)\tilde{z} \\ &- \frac{\kappa^2}{N^2}\tilde{y}^3 - \frac{\kappa}{N}\Big(\frac{\kappa}{N} - \kappa_0 - N\Big)\tilde{y}^2 + \frac{\kappa}{N}e^{-t}\tilde{y} + \frac{\kappa}{N}e^{-t}. \end{aligned}$$

The canonical transformation

$$\tilde{y} = \eta, \quad \tilde{z} = \zeta + \frac{\kappa}{N^2}\Big(\frac{\kappa}{N} - \kappa_0 - N\Big)\eta$$

leads us to the Hamiltonian system with

(3.6) $$H = h_{10}e^{-t}\eta + e^{-t}\zeta + h_{20}e^{-t}\eta^2 + (N + e^{-t})\eta\zeta + \sum_{3\le k+l\le 5} h_{kl}\eta^k\zeta^l,$$

where $h_{kl} \in \mathbb{Q}[\kappa_0, \kappa]$.

Suppose that $|N| \ge 2$. The canonical transformation $\eta = \partial W/\partial\zeta$, $Z = \partial W/\partial Y$ with $W = W(t, Y, \zeta)$ changes H to $H - \partial W/\partial t$. Using this fact, we may find

$$\eta = \sum_{n=1}^{|N|-1} c_n e^{-nt} + Y, \quad \zeta = \sum_{n=1}^{|N|-1} c'_n e^{-nt} + Z$$

with $c_n, c'_n \in \mathbb{Q}[\kappa_0, \kappa]$ such that H becomes

$$\begin{aligned} &e^{-|N|t}\tilde{h}_{10}(e^{-t})Y + e^{-|N|t}\tilde{h}_{01}(e^{-t})Z + e^{-t}\tilde{h}_{20}(e^{-t})Y^2 \\ &+ (N + e^{-t}\tilde{h}_{11}(e^{-t}))YZ + e^{-t}\tilde{h}_{02}(e^{-t})Z^2 + \sum_{3\le k+l\le 5} \tilde{h}_{kl}(e^{-t})Y^k Z^l, \end{aligned}$$

where $\tilde{h}_{kl}(\tau) \in \mathbb{Q}[\kappa_0, \kappa][\tau]$. Such a transformation is the composite of $Y_{n-1} = c_n e^{-nt} + Y_n$, $Z_{n-1} = c'_n e^{-nt} + Z_n$, $Y_0 = \eta$, $Z_0 = \zeta$, $Y_{|N|-1} = Y$, $Z_{|N|-1} = Z$ for $1 \le n \le |N| - 1$, each of which reduces the coefficients of linear terms to $O(e^{-(n+1)t})$. Consequently we obtain

$$\begin{aligned} Y' =& e^{-|N|t}\tilde{h}_{01}(e^{-t}) + (N + e^{-t}\tilde{h}_{11}(e^{-t}))Y \\ &\qquad + e^{-t}h^*_{01}(e^{-t})Z + \sum_{2\le k+l\le 4} h^*_{kl}(e^{-t})Y^k Z^l, \\ Z' =& - e^{-|N|t}\tilde{h}_{10}(e^{-t}) - (N + e^{-t}\tilde{h}_{11}(e^{-t}))Z \\ &\qquad + e^{-t}h^*_{10}(e^{-t})Y + \sum_{2\le k+l\le 4} \tilde{h}^*_{kl}(e^{-t})Y^k Z^l, \end{aligned}$$

where $h^*_{kl}(\tau)$, $\tilde{h}^*_{kl}(\tau) \in \mathbb{Q}[\kappa_0, \kappa][\tau]$. By $Y = e^{-|N|t}\tilde{Y}$, $Z = e^{-|N|t}\tilde{Z}$ this is changed into

$$\begin{aligned} \tilde{Y}' =& \tilde{h}_{01}(e^{-t}) + (N + |N| + e^{-t}\tilde{h}_{11}(e^{-t}))\tilde{Y} \\ &\qquad + e^{-t}h^*_{01}(e^{-t})\tilde{Z} + \sum_{2\le k+l\le 4} e^{-(k+l-1)|N|t}h^*_{kl}(e^{-t})\tilde{Y}^k\tilde{Z}^l, \\ \tilde{Z}' =& - \tilde{h}_{10}(e^{-t}) - (N - |N| + e^{-t}\tilde{h}_{11}(e^{-t}))\tilde{Z} \\ &\qquad + e^{-t}h^*_{10}(e^{-t})\tilde{Y} + \sum_{2\le k+l\le 4} e^{-(k+l-1)|N|t}\tilde{h}^*_{kl}(e^{-t})\tilde{Y}^k\tilde{Z}^l. \end{aligned}$$

Put

$$\begin{aligned} &\tilde{Y} = \tilde{h}_{01}(0)t_a + u, \quad \tilde{Z} = \tilde{h}_{10}(0)/(2|N|) + v \qquad \text{if } N \le -2, \\ &\tilde{Y} = -\tilde{h}_{01}(0)/(2N) + u, \quad \tilde{Z} = -\tilde{h}_{10}(0)t_a + v \qquad \text{if } N \ge 2. \end{aligned}$$

Observing that $(k+l-1)|N|-(k+l) \geq 2(k+l-1)-(k+l) \geq k+l-2 \geq 0$ if $k+l \geq 2$, we have

PROPOSITION 3.6. (1) *If* $N \leq -2$, *then*, *by a transformation* $(y, z) \mapsto (u, v)$ *such that*

$$y = 1 + \sum_{n=1}^{|N|-1} c_n e^{-nt} - c_{|N|} e^{-|N|t} t_a + e^{-|N|t} u$$

with $c_n \in \mathbb{Q}[\kappa_0, \kappa]$, *system* (3.5) *is taken into*

$$\begin{aligned} u' &= q_{00}(t) + \sum_{1 \leq k+l \leq 4} q_{kl}(t) u^k v^l\,, \\ v' &= q^*_{00}(t) + 2|N|v + \sum_{1 \leq k+l \leq 4} q^*_{kl}(t) u^k v^l \end{aligned} \tag{3.7}$$

with $q_{kl}(t)$, $q^*_{kl}(t) \in \mathfrak{P}_0(R, \Theta)$ *for any positive numbers* R, Θ.

(2) *If* $N \geq 2$, *then*, *by a transformation* $(y, z) \mapsto (u, v)$ *such that*

$$y = 1 + \sum_{n=1}^{N} c_n e^{-nt} + e^{-Nt} u$$

with $c_n \in \mathbb{Q}[\kappa_0, \kappa]$, *system* (3.5) *is taken into*

$$\begin{aligned} u' &= q_{00}(t) + 2Nu + \sum_{1 \leq k+l \leq 4} q_{kl}(t) u^k v^l\,, \\ v' &= q^*_{00}(t) + \sum_{1 \leq k+l \leq 4} q^*_{kl}(t) u^k v^l \end{aligned} \tag{3.8}$$

with $q_{kl}(t)$, $q^*_{kl}(t) \in \mathfrak{P}_0(R, \Theta)$ *for any positive numbers* R, Θ.

REMARK 3.2. Obviously $q_{kl}(t)$ and $q^*_{kl}(t)$ are polynomials in e^{-t} and t_a.

In case $|N| = 1$ we have

PROPOSITION 3.7. (1) *If* $N = -1$, *then*, *by a transformation* $(y, z) \mapsto (u, v)$ *such that*

$$y = 1 + e^{-t} t_a + e^{-t} u\,,$$

system (3.5) *is taken into*

$$\begin{aligned} u' &= q_{00}(t) + \sum_{1 \leq k+l \leq 4} q_{kl}(t) u^k v^l\,, \\ v' &= q^*_{00}(t) + 2v + \sum_{1 \leq k+l \leq 4} q^*_{kl}(t) u^k v^l \end{aligned} \tag{3.9}$$

with $q_{kl}(t)$, $q^*_{kl}(t) \in \mathfrak{P}(R, \Theta)$ *such that* $e^{(k+l-1)t}q_{kl}(t)$, $e^{(k+l-1)t}q^*_{kl}(t) \in \mathfrak{P}_0(R, \Theta)$ *for any positive numbers* R, Θ.

(2) *If* $N = 1$, *then, by a transformation* $(y, z) \mapsto (u, v)$ *such that*

$$y = 1 - e^{-t}/2 + e^{-t}u\,,$$

system (3.5) *is taken into*

$$(3.10)\qquad \begin{aligned} u' &= q_{00}(t) + 2u + \sum_{1\le k+l\le 4} q_{kl}(t)u^k v^l\,,\\ v' &= q^*_{00}(t) + \sum_{1\le k+l\le 4} q^*_{kl}(t)u^k v^l \end{aligned}$$

with $q_{kl}(t)$, $q^*_{kl}(t) \in \mathfrak{P}_0(R, \Theta)$ *for any positive numbers* R, Θ.

PROOF. In case $|N| = 1$, the system corresponding to (3.6) is

$$\eta' = e^{-t} + (N + e^{-t})\eta + \sum\nolimits_{(2)}^{(4)+} h^*_{kl}\eta^k\zeta^l\,,$$

$$\zeta' = -h_{10}e^{-t} - (N + e^{-t})\zeta - 2h_{20}e^{-t}\eta + \sum\nolimits_{(2)}^{(4)-} \tilde{h}^*_{kl}\eta^k\zeta^l\,,$$

where h^*_{kl}, $\tilde{h}^*_{kl} \in \mathbb{Q}[\kappa_0, \kappa]$ and the summation $\sum_{(2)}^{(4)+}$ (respectively, $\sum_{(2)}^{(4)-}$) is over (k, l) satisfying $2 \le k + l \le 4$, $l \le 1$, $k \ge l + 1$ (respectively, $2 \le k + l \le 4$, $l \le 2$, $k \ge l - 1$). We put $\eta = e^{-t}Y$, $\zeta = e^{-t}Z$ to obtain

$$Y' = 1 + (N + 1 + e^{-t})Y + \sum\nolimits_{(2)}^{(4)+} h^*_{kl}e^{-(k+l-1)t}Y^kZ^l\,,$$

$$Z' = -h_{10} - (N - 1 + e^{-t})Z - 2h_{20}e^{-t}Y + \sum\nolimits_{(2)}^{(4)-} \tilde{h}^*_{kl}e^{-(k+l-1)t}Y^kZ^l\,.$$

If $N = 1$, we put $Y = -1/2 + u$, $Z = -h_{10}t_a + v$. Observing that, in $\sum_{(2)}^{(4)-}$, $(k + l - 1) - l = k - 1 \ge l - 2 = 0$ if $l = 2$ and that $(k + l - 1) - l \ge 1 - l = 0$ if $l = 1$, we have the assertion (2).

If $N = -1$, we put $Y = t_a + u$, $Z = h_{10}/2 + v$. It is easy to see that $q_{kl}(t)$, $q^*_{kl}(t) \in \mathfrak{P}(R, \Theta)$. Note that

$$e^{-(k+l-1)t}(t_a + u)^k(h_{10}/2 + v)^l = \sum_{k',\,l'} c_{k'l'}e^{-(k+l-1)t}t_a^{k-k'}u^{k'}v^{l'}$$

for $0 \le k' \le k$, $l' \le l$. Since $k + l - 1 - (k - k') = k' + l - 1 \ge k' + l' - 1$, we have $e^{(k'+l'-1)t}q_{k'l'}(t)$, $e^{(k'+l'-1)t}q^*_{k'l'}(t) \in \widehat{\mathfrak{P}}_0$, which implies the assertion (1). □

Suppose that $\theta = 0$. Then the Hamiltonian of (3.5) is

$$H = -y(y - 1)^2z^2 + (\kappa_0(y - 1)^2 + e^{-t}y)z - \kappa(y - 1)\,,$$

which is written in the form

$$H = -(\tilde{y}+1)\tilde{y}^2 z^2 + (\kappa_0 \tilde{y}^2 + e^{-t}(\tilde{y}+1))z - \kappa \tilde{y}$$

with $\tilde{y} = y - 1$. The corresponding system is

$$\tilde{y}' = e^{-t} + e^{-t}\tilde{y} + \kappa_0 \tilde{y}^2 - 2(1+\tilde{y})\tilde{y}^2 z\,,$$
$$z' = \kappa - e^{-t}z - 2\kappa_0 \tilde{y} z + (2+3\tilde{y})\tilde{y} z^2\,.$$

Substitution of $\tilde{y} = e^{-t}Y$ yields

$$Y' = 1 + (1+e^{-t})Y + \kappa_0 e^{-t}Y^2 - 2e^{-t}Y^2 z - 2e^{-2t}Y^3 z\,,$$
$$z' = \kappa - e^{-t}z - 2\kappa_0 e^{-t}Yz + 2e^{-t}Yz^2 + 3e^{-2t}Y^2 z^2\,.$$

We put $Y = -1 + u$, $z = \kappa t_a + v$ to obtain

PROPOSITION 3.8. *If $N = 0$, then, by a transformation $(y, z) \mapsto (u, v)$ such that*

$$y = 1 - e^{-t} + e^{-t}u\,,$$

system (3.5) *is taken into*

$$\begin{aligned} u' &= q_{00}(t) + u + Q_0(t,u) + Q_1(t,u)v\,, \\ v' &= q^*_{00}(t) + q^*_{10}(t)u + q^*_{20}(t)u^2 + Q^*_1(t,u)v + Q^*_2(t,u)v^2 \end{aligned} \tag{3.11}$$

with

$$Q_0(t,u) = \sum_{k=1}^{3} q_{k0}(t)u^k, \quad Q_1(t,u) = -2e^{-t}(u-1)^2(1+e^{-t}(u-1))\,,$$

$$Q^*_1(t,u) = q^*_{01}(t) + q^*_{11}(t)u + q^*_{21}(t)u^2, \quad Q^*_2(t,u) = e^{-t}(u-1)(2+3e^{-t}(u-1))\,,$$

where, for any $R, \Theta > 0$, the coefficients $q_{kl}(t)$, $q^*_{kl}(t)$ have the properties:

(i) $q^*_{00}(t)$, $q^*_{10}(t) \in \mathfrak{P}(R,\Theta)$ *and* $e^{-t}q^*_{00}(t)$, $e^{-t}q^*_{10}(t) \in \mathfrak{P}_0(R,\Theta)$;

(ii) $q_{kl}(t) \in \mathfrak{P}_0(R,\Theta)$, *in particular*, $q_{00}(t) = e^{-t}(\kappa_0 - 1 - 2\kappa t_a) + 2\kappa e^{-2t}t_a$;

(iii) $q^*_{kl}(t) \in \mathfrak{P}_0(R,\Theta)$ *for every* $(k,l) \ne (0,0), (1,0)$.

3.3. Proof of Theorem 2.2 in the case where $N \ge 1$ or $N \le -2$. For a given positive number B_0 suppose that $|\kappa_0| + |\kappa| < B_0$. For a given bounded domain $\Delta_0 \subset \mathbb{C}$ and for a given positive number Θ_0, let $R = R(\Delta_0, \Theta_0)$ be a positive number as in Section 3.2. Suppose that $N \ge 2$, and write (3.8) in the form

$$u' = 2Nu + \Psi(t,u,v), \qquad v' = \Psi_*(t,u,v)$$

with

$$\Psi(t,u,v) := q_{00}(t) + \Pi(t,u,v), \quad \Pi(t,u,v) := \sum_{1\le k+l\le 4} q_{kl}(t)u^k v^l\,,$$

$$\Psi_*(t,u,v) := q^*_{00}(t) + \Pi_*(t,u,v), \quad \Pi_*(t,u,v) := \sum_{1\le k+l\le 4} q^*_{kl}(t)u^k v^l \,.$$

If $(u,v) = (u(a,t), v(a,t)) \in \mathfrak{P}_0(R,\Theta_0)^2$ satisfies the system of formal integral equations

$$u = e^{2Nt}\mathcal{I}[e^{-2Nt}\Psi(t,u,v)], \quad v = \mathcal{I}[\Psi_*(t,u,v)]\,, \tag{3.12}$$

then, by Lemmas 3.3 and 3.4, it solves (3.8). In what follows we construct such a solution of (3.12). We may define the sequence $\{(u_\nu(a,t), v_\nu(a,t)) \mid \nu \ge 0\} \subset \mathfrak{P}_0(R,\Theta_0)^2$ by

$$\begin{aligned} &u_0(a,t) \equiv 0, \quad v_0(a,t) \equiv 0\,, \\ &u_{\nu+1}(a,t) = e^{2Nt}\mathcal{I}\left[e^{-2Nt}\Psi(t,u_\nu(a,t),v_\nu(a,t))\right]\,, \\ &v_{\nu+1}(a,t) = \mathcal{I}\left[\Psi_*(t,u_\nu(a,t),v_\nu(a,t))\right]\,. \end{aligned}$$

By Remark 3.2, $\|q_{kl}\|$, $\|q^*_{kl}\| = O(|e^{-t/2}|)$ uniformly in (a,κ_0,κ) such that $a \in \Delta_0$, $|\kappa_0| + |\kappa| < B_0$. Noting this fact we have

LEMMA 3.9. *There exist a positive number L_0 and a sufficiently large positive number $R_0 = R_0(\Delta_0,\Theta_0,B_0)$ such that*

(i) $\|q_{00}\|$, $\|q^*_{00}\| \le L_0|e^{-t/2}|$ *in $S(R_0,\Theta_0)$, and that*

(ii) *for any $(u,\tilde{u},v,\tilde{v}) \in \mathfrak{P}_0(R_0,\Theta_0)^4$*

$$\begin{aligned} &\|\Pi(t,u,v) - \Pi(t,\tilde{u},\tilde{v})\| \le L_0|e^{-t/2}|(\|u-\tilde{u}\| + \|v-\tilde{v}\|)\,, \\ &\|\Pi_*(t,u,v) - \Pi_*(t,\tilde{u},\tilde{v})\| \le L_0|e^{-t/2}|(\|u-\tilde{u}\| + \|v-\tilde{v}\|) \end{aligned}$$

in $S(R_0,\Theta_0)$ as long as $\|u\|$, $\|\tilde{u}\|$, $\|v\|$, $\|\tilde{v}\| < 1$.

Then, by Lemmas 3.3, 3.4 and 3.9, in $S(R_0,\Theta_0)$

$$\begin{aligned} &\|u_1\| = \left\|e^{2Nt}\mathcal{I}\left[e^{-2Nt}q_{00}(t)\right]\right\| \le 2\|q_{00}\| \le 2L_0|e^{-t/2}|\,, \\ &\|v_1\| = \left\|\mathcal{I}\left[q^*_{00}(t)\right]\right\| \le 2\|q^*_{00}\| \le 2L_0|e^{-t/2}|\,, \end{aligned}$$

and, for $\nu \ge 1$,

$$\begin{aligned} \|u_{\nu+1} - u_\nu\| &= \left\|e^{2Nt}\mathcal{I}\left[e^{-2Nt}(\Pi(t,u_\nu,v_\nu) - \Pi(t,u_{\nu-1},v_{\nu-1}))\right]\right\| \\ &\le 2\,\|\Pi(t,u_\nu,v_\nu) - \Pi(t,u_{\nu-1},v_{\nu-1})\| \\ &\le 2L_0|e^{-t/2}|\,(\|u_\nu - u_{\nu-1}\| + \|v_\nu - v_{\nu-1}\|)\,, \\ \|v_{\nu+1} - v_\nu\| &\le 2L_0|e^{-t/2}|\,(\|u_\nu - u_{\nu-1}\| + \|v_\nu - v_{\nu-1}\|) \end{aligned}$$

as long as $\|u_j\|$, $\|v_j\| < 1$ for every $j \le \nu$. These inequalities yield

$$\begin{aligned} \|u_{\nu+1} - u_\nu\| + \|v_{\nu+1} - v_\nu\| &\le 4L_0|e^{-t/2}|\,(\|u_\nu - u_{\nu-1}\| + \|v_\nu - v_{\nu-1}\|) \\ &\le \dots \le (4L_0|e^{-t/2}|)^\nu(\|u_1\| + \|v_1\|) \le (4L_0|e^{-t/2}|)^{\nu+1}\,, \end{aligned}$$

which implies

$$\|u_{\nu+1}\| + \|v_{\nu+1}\| \le \sum_{j=0}^{\nu}(\|u_{j+1} - u_j\| + \|v_{j+1} - v_j\|) \le \frac{4L_0|e^{-t/2}|}{1 - 4L_0|e^{-t/2}|} .$$

Hence, choosing R_0 again, if necessary, in such a way that $4L_0|e^{-t/2}| < 1/3$ in $S(R_0, \Theta_0)$, we have, for every $\nu \ge 1$,

$$\|u_{\nu+1} - u_\nu\| + \|v_{\nu+1} - v_\nu\| \le (4L_0|e^{-t/2}|)^{\nu+1} \le 3^{-\nu-1} ,$$
$$\|u_\nu\| + \|v_\nu\| \le 6L_0|e^{-t/2}| < 1/2$$

in $S(R_0, \Theta_0)$. By Proposition 3.2, $(u_\infty, v_\infty) := \lim_{\nu\to\infty}(u_\nu, v_\nu) \in \mathfrak{P}_0(R_0, \Theta_0)^2$, which satisfies (3.12).

For a given bounded domain D_0 as in Section 2, we may choose Δ_0 in such a way that $a = \log\rho \in \Delta_0$ for every $\rho \in D_0$. Let ε_0' be such that $t \in S(R_0, \Theta_0)$ holds for every $x = e^{-t} \in \Sigma(\varepsilon_0', \Theta_0)$. By Proposition 3.6, (2),

$$y = 1 + \sum_{n=1}^{N} c_n x^n + x^N u_\infty(\log\rho, -\log x)$$

is the desired solution of (V) when $N \ge 2$.

In the case where $N = 1$ or $N \le -2$, we may construct the solution as in Theorem 2.2 by using Proposition 3.7 or 3.6.

3.4. Proof of Theorem 2.2 in the case where $N = -1$ or 0. Suppose that $N = -1$. Write (3.9) in the form

$$u' = \Psi(t, u, v), \quad v' = 2v + \Psi_*(t, u, v)$$

with

$$\Psi(t, u, v) = \sum_{0\le k+l\le 4} q_{kl}(t)u^k v^l, \quad \Psi_*(t, u, v) = \sum_{0\le k+l\le 4} q_{kl}^*(t)u^k v^l .$$

Defining the sequence $\{(u_\nu(a, t), v_\nu(a, t)) \mid \nu \ge 0\} \subset \mathfrak{P}(R, \Theta_0)^2$ by

$$u_0(a, t) \equiv 0, \quad v_0(a, t) \equiv 0 ,$$
$$u_{\nu+1}(a, t) = \mathcal{I}[\Psi(t, u_\nu(a, t), v_\nu(a, t))] ,$$
$$v_{\nu+1}(a, t) = e^{2t}\mathcal{I}\left[e^{-2t}\Psi_*(t, u_\nu(a, t), v_\nu(a, t))\right] ,$$

we may construct, for some $R_0 > 0$, the solution $(u_\infty, v_\infty) := \lim_{\nu\to\infty}(u_\nu, v_\nu) \in \mathfrak{P}(R_0, \Theta_0)^2$ of (3.9) by the same argument as in Section 3.3. Write the system of recursive relations above in the form

$$e^{-t}u_{\nu+1}(a, t) = e^{-t}\mathcal{I}\left[e^t\widehat{\Psi}(t, e^{-t}u_\nu(a, t), e^{-t}v_\nu(a, t))\right] ,$$

$$e^{-t}v_{\nu+1}(a,t) = e^{t}\mathcal{I}\left[e^{-t}\widehat{\Psi}_*(t, e^{-t}u_\nu(a,t), e^{-t}v_\nu(a,t))\right]$$

with

$$\widehat{\Psi}(t,\hat{u},\hat{v}) = \sum_{0\le k+l\le 4} e^{(k+l-1)t}q_{kl}(t)\hat{u}^k\hat{v}^l, \quad \widehat{\Psi}_*(t,\hat{u},\hat{v}) = \sum_{0\le k+l\le 4} e^{(k+l-1)t}q^*_{kl}(t)\hat{u}^k\hat{v}^l .$$

Observing $e^{(k+l-1)t}q_{kl}(t),\ e^{(k+l-1)t}q^*_{kl}(t) \in \mathfrak{P}_0(R_0,\Theta_0)$, and using Lemma 3.5, we may inductively show that $e^{-t}u_\nu,\ e^{-t}v_\nu \in \widehat{\mathfrak{P}}_0$ for every ν. This implies $e^{-t}u_\infty \in \mathfrak{P}_0(R_0,\Theta_0)$, which yields the solution of (V) as in Theorem 2.2, (ii).

If $N=0$, define the sequence $\{(u_\nu(a,t), v_\nu(a,t)) \mid \nu \ge 0\} \subset \mathfrak{P}(R,\Theta_0)^2$ by

$$u_0(a,t) \equiv 0, \quad v_0(a,t) \equiv 0,$$
$$u_{\nu+1}(a,t) = e^{t}\mathcal{I}\left[e^{-t}\Omega(t,u_\nu(a,t),v_\nu(a,t))\right],$$
$$v_{\nu+1}(a,t) = \mathcal{I}\left[\Omega_*(t,u_\nu(a,t),v_\nu(a,t))\right],$$

where $\Omega(t,u,v)+u$ and $\Omega_*(t,u,v)$ are the right-hand members of (3.11) in Proposition 3.8. Then $(u_\infty, v_\infty) := \lim_{\nu\to\infty}(u_\nu, v_\nu) \in \mathfrak{P}(R_0,\Theta_0)^2$ solves (3.11). Writing

$$u_{\nu+1}(a,t) = e^{t}\mathcal{I}\left[e^{-t}\widehat{\Omega}(t,u_\nu(a,t),e^{-t}v_\nu(a,t))\right],$$
$$e^{-t}v_{\nu+1}(a,t) = e^{-t}\mathcal{I}\left[e^{t}\widehat{\Omega}_*(t,u_\nu(a,t),e^{-t}v_\nu(a,t))\right]$$

with

$$\widehat{\Omega}(t,u,\hat{v}) = q_{00}(t) + Q_0(t,u) - 2(u-1)^2(1+e^{-t}(u-1))\hat{v},$$
$$\widehat{\Omega}_*(t,u,\hat{v}) = e^{-t}q^*_{00}(t) + e^{-t}q^*_{10}(t)u + e^{-t}q^*_{20}(t)u^2 + Q^*_1(t,u)\hat{v} + (u-1)(2+3e^{-t}(u-1))\hat{v}^2,$$

and using Lemma 3.5, we have $u_\nu,\ e^{-t}v_\nu \in \widehat{\mathfrak{P}}_0$ for every ν, and hence $u_\infty \in \mathfrak{P}_0(R_0,\Theta_0)$. Since

$$u_1(a,t) = e^{t}\mathcal{I}[e^{-t}q_{00}(t)] = e^{-t}\Big(\kappa t_a + \frac{1}{2}(\kappa-\kappa_0+1) - \frac{2}{9}\kappa e^{-t}(3t_a+1)\Big),$$

we obtain the solution of (V) as in Theorem 2.2, (i).

REMARK 3.3. In the proof of Theorem 2.2 above, if the coefficients of t_a vanish, then logarithmic terms do not appear. For example, in the process of deriving Proposition 3.6, if $N \le -2$ and if $\tilde{h}_{01}(0) = 0$, we put $\tilde{Y} = a + u$ instead of $\tilde{h}_{01}(0)t_a + u$, and the corresponding transformation is

$$y = 1 + \sum_{n=1}^{|N|-1} c_n e^{-nt} + ae^{-|N|t} + e^{-|N|t}u .$$

In deriving Proposition 3.8, if $N = 0$ and if $\kappa = 0$, we put $z = a + v$. Then we have $q_{00}(t) = (\kappa_0 - 1 - 2a)e^{-t} + 2ae^{-2t}$ and $u_1(a,t) = e^t \mathcal{I}[e^{-t} q_{00}(t)] = (a + (1-\kappa_0)/2)e^{-t} - (2a/3)e^{-2t}$. In these degenerate cases we obtain families of Taylor series solutions as in Remark 2.2.

4. Proof of Theorem 2.3

Around the point $(y,z) = (y_0, z_0) := (\kappa_0/\kappa_\infty, -\kappa_\infty(\kappa_0 - \kappa_\infty + \theta)/(2(\kappa_\infty - \kappa_0)))$ given by (3.4), let us consider (3.2). Note that

$$\begin{pmatrix} \partial^2(xH_{\rm V})/\partial y \partial z & \partial^2(xH_{\rm V})/\partial z^2 \\ -\partial^2(xH_{\rm V})/\partial y^2 & -\partial^2(xH_{\rm V})/\partial y\partial z \end{pmatrix} (y_0, z_0) \Big|_{x=0} = \begin{pmatrix} h_{11} & h_{02} \\ -h_{20} & -h_{11} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\kappa_0}{\kappa_\infty}(\kappa_0 + \theta) - 2\kappa_0 + \kappa_\infty & \frac{2\kappa_0}{\kappa_\infty^3}(\kappa_0 - \kappa_\infty)^2 \\ -\frac{1}{2}\left(\kappa_\infty + \frac{\theta\kappa_\infty}{\kappa_0 - \kappa_\infty}\right)\left(\kappa_0 - 2\kappa_\infty + \frac{\theta\kappa_0}{\kappa_0-\kappa_\infty}\right) & -\frac{\kappa_0}{\kappa_\infty}(\kappa_0+\theta) + 2\kappa_0 - \kappa_\infty \end{pmatrix},$$

whose eigenvalues are $\pm(\kappa_0 - \kappa_\infty)$. Then putting $x = e^{-t}$ and

$$\begin{pmatrix} y - y_0 \\ z - z_0 \end{pmatrix} = \begin{pmatrix} h_{02} & h_{02} \\ \kappa_\infty - \kappa_0 - h_{11} & \kappa_0 - \kappa_\infty - h_{11} \end{pmatrix} \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix},$$

we obtain the Hamiltonian

$$H = \tilde{h}_{10} e^{-t}\tilde{y} + \tilde{h}_{01}e^{-t}\tilde{z} + \tilde{h}_{20}e^{-t}\tilde{y}^2 + (\kappa_0 - \kappa_\infty + \tilde{h}_{11}e^{-t})\tilde{y}\tilde{z} + \tilde{h}_{02}e^{-t}\tilde{z}^2 + \sum_{3 \le k+l \le 5} \tilde{h}_{kl}\tilde{y}^k \tilde{z}^l$$

with $\tilde{h}_{kl} \in \mathbb{Q}[\kappa_0, \kappa_\infty, \theta, \kappa_\infty^{-1}, (\kappa_0 - \kappa_\infty)^{-1}]$ instead of (3.6). If $\kappa_0 - \kappa_\infty = N = \pm 1$, then, this Hamiltonian system is reduced to a system whose coefficients have the same properties as of (3.9). Observing the proofs of Propositions 3.6 and 3.7, by the same arguments as in the proofs of Theorem 2.2 for $|N| \ge 2$ and for $N = -1$, we can verify the assertion (1), from which the second half of the theorem immediately follows.

5. Proof of Theorem 2.1

5.1. Ring of exponential type series with asymptotic multipliers. Let B_0, Θ and $\widehat{R}$ be given positive numbers, and let $\Delta_0 \subset \mathbb{C}$ be a given bounded domain. In what follows, let $R = R(\Delta_0, \widehat{R}, \Theta) > 8$ and $\widehat{\Theta} = \widehat{\Theta}(\Delta_0, \Theta) > \Theta$ denote sufficiently large positive numbers such that $t_a := t - a \in S(\widehat{R}, \widehat{\Theta})$ and $|t_a| < |e^{t/4}|$ hold for every $(a,t) \in \Delta_0 \times S(R, \Theta)$, where $S(R, \Theta)$ is the strip $\mathrm{Re}\, t > R$, $|\mathrm{Im}\, t| < \Theta$, and suppose that $|\alpha| + |\beta| + |\gamma| + |\gamma^{-1}| < B_0$. Let $\widehat{\mathfrak{A}}(\widehat{R}, \widehat{\Theta})$ be the ring of formal series of the form

$$\varphi = \varphi(a,t) = \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} \varphi_n(t_a)$$

with a complex parameter a, where, for each n, $\varphi_n(s)$ is holomorphic in $s \in S(\widehat{R}, \widehat{\Theta})$ and admits the asymptotic expansion

$$\varphi_n(s) \sim \sum_{j=0}^{\infty} \varphi_j^n s^{-j}, \quad \varphi_j^n \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$$

as $s \to \infty$ through $S(\widehat{R}, \widehat{\Theta})$. For $\varphi \in \widehat{\mathfrak{A}}(\widehat{R}, \widehat{\Theta})$ as above, define

$$\|\varphi\| = \|\varphi(a, t)\| = \|\varphi\|(t) := \sum_{n=1}^{\infty} M(\varphi_n)|e^{-t}|^{n/2}$$

with

$$M(\varphi_n) := \sup_{s \in S(\widehat{R}, \widehat{\Theta})} |\varphi_n(s)|$$

and set, for each (α, β, γ),

$$\mathfrak{A}(R, \Theta) := \left\{\varphi(a, t) \in \widehat{\mathfrak{A}}(\widehat{R}, \widehat{\Theta}) \;\middle|\; \|\varphi\|(t) < \infty \text{ for every } t \in S(R, \Theta)\right\}.$$

EXAMPLE 5.1. For a given bounded domain Δ_0 and for given positive numbers Θ and $\widehat{R}$, choose $R > 1$ such that $\widehat{R} < |t_a| < |e^{t/4}|$ for every $(a, t) \in \Delta_0 \times S(R, \Theta)$. If $|\varphi_j^n| \le M_0(\widehat{R}/2)^j$ for every (n, j) satisfying $n \ge 1$, $j \ge 0$, where M_0 is a positive number, then, for each n, $\varphi_n(s) = \sum_{j=0}^{\infty} \varphi_j^n s^{-j}$ converges for $|s| > \widehat{R}$, and $|\varphi_n(s)| \le \sum_{j=0}^{\infty} M_0(\widehat{R}/2)^j |s|^{-j} \le 2M_0$. Then $\varphi(a, t) = \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} \varphi_n(t_a) \in \mathfrak{A}(R, \Theta)$, since $\|\varphi\|(t) \le 2M_0 \sum_{n=1}^{\infty} |e^{-t}|^{n/2} < \infty$ in $S(R, \Theta)$.

PROPOSITION 5.1. (1) *If $\varphi \in \mathfrak{A}(R, \Theta)$, then $\varphi = \varphi(a, t)$ is holomorphic in $(a, t) \in \Delta_0 \times S(R, \Theta)$ and satisfies $|\varphi(a, t)| \le \|\varphi\|(t)$.*

(2) *$\|\varphi\| \equiv 0$ if and only if $\varphi \equiv 0$.*

(3) *Let φ, $\psi \in \mathfrak{A}(R, \Theta)$ and let $c \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$. Then $\varphi + \psi$, $\varphi\psi$, $c\varphi \in \mathfrak{A}(R, \Theta)$, and*

$$\|\varphi + \psi\| \le \|\varphi\| + \|\psi\|, \quad \|\varphi\psi\| \le \|\varphi\| \, \|\psi\|, \quad \|c\varphi\| \le |c| \, \|\varphi\|.$$

Furthermore, for each (α, β, γ), if $f(s) = \sum_{j=0}^{\infty} f_j s^{-j}$ with $f_j \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$ converges for $|s| \ge \widehat{R}$, then $f(t_a)\varphi \in \mathfrak{A}(R, \Theta)$ and $\|f(t_a)\varphi\| \le M(f)\|\varphi\|$.

PROOF. Let $K_0 \subset \Delta_0 \times S(R, \Theta)$ be a given compact set. There exist $t_* \in S(R, \Theta)$ and $\delta_0 > 0$ such that $|e^{-t}| \le (1 - \delta_0)|e^{-t_*}|$ in K_0. Since $\|\varphi\|(t_*) < \infty$, we have $M(\varphi_n) \le |e^{-t_*}|^{-n/2}$ if n is sufficiently large, and hence

$$|e^{-nt} t_a^{2n} \varphi_n(t_a)| \le M(\varphi_n)|e^{-t}|^{n/2} \le |e^{-t}/e^{-t_*}|^{n/2} \le (1 - \delta_0)^{n/2}$$

in K_0. This implies that $\varphi(a, t)$ converges uniformly on every compact set contained in $\Delta_0 \times S(R, \Theta)$ and satisfies $|\varphi(a, t)| \le \|\varphi\|(t)$. Thus the first assertion is verified. □

PROPOSITION 5.2. *Suppose that each* $\varphi_\nu \in \mathfrak{A}(R, \Theta)$ *consists of summands for* $n \geq \nu$, *and that* $\sum_{\nu=1}^{\infty} \|\varphi_\nu\| < \infty$ *in* $S(R, \Theta)$. *Then* $\varphi^\infty = \sum_{\nu=1}^{\infty} \varphi_\nu \in \mathfrak{A}(R, \Theta)$, *and* $\|\varphi^\infty\| \leq \sum_{\nu=1}^{\infty} \|\varphi_\nu\|$.

Furthermore we have

LEMMA 5.3. *Let*

$$f(a, t, u_1, \dots, u_p) = \sum_{\mathbf{k}} f_{\mathbf{k}}(a, t) u_1^{k_1} \cdots u_p^{k_p}, \quad \mathbf{k} = (k_1, \dots, k_p) \in (\mathbb{N} \cup \{0\})^p \setminus \{0\}^p$$

be such that, for each (α, β, γ),

(i) $f_{\mathbf{k}}(a, t) \in \mathfrak{A}(R, \Theta)$ (*respectively*, $f_{\mathbf{k}}(a, t) = \tilde{f}_{\mathbf{k}}(t_a)$ *with* $\tilde{f}_{\mathbf{k}}(s) = \sum_{j=0}^{\infty} \tilde{f}_j^{\mathbf{k}} s^{-j}$, $\tilde{f}_j^{\mathbf{k}} \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$ *convergent for* $|s| \geq \widehat{R}$), *and that*

(ii) $\|f_{\mathbf{k}}\|(t) \leq M_0$ *in* $S(R, \Theta)$ (*respectively*, $M(f_{\mathbf{k}}) \leq M_0$), *where* M_0 *is some positive number independent of* $\mathbf{k}$.

If $\varphi_1, \dots, \varphi_p \in \mathfrak{A}(R, \Theta)$ *satisfy* $\|\varphi_1\| < 1, \dots, \|\varphi_p\| < 1$ *in* $S(R, \Theta)$, *then* $f(a, t, \varphi_1, \dots, \varphi_p) \in \mathfrak{A}(R, \Theta)$, *and*

$$\|f(a, t, \varphi_1, \dots, \varphi_p)\| \leq \sum_{\mathbf{k}} \|f_{\mathbf{k}}\| \, \|\varphi_1\|^{k_1} \cdots \|\varphi_p\|^{k_p}$$

$$\Big(\textit{respectively}, \quad \leq \sum_{\mathbf{k}} M(f_{\mathbf{k}}) \|\varphi_1\|^{k_1} \cdots \|\varphi_p\|^{k_p}\Big).$$

LEMMA 5.4. *Let* m *be a given integer, and suppose that* $\widehat{R} > 2(|m| + 2)$. *For each* $n \geq 1$,

$$|e^{ns} s^{-2n-m}| \int_{\Gamma(s)} |e^{-n\sigma} \sigma^{2n+m}| \, |d\sigma| \leq 2\sqrt{1 + \widehat{\Theta}^2}$$

for every $s \in S(\widehat{R}, \widehat{\Theta})$, *where* $\Gamma(s)$ *is a horizontal line starting from* s *and tending to* ∞ *on which* $\operatorname{Re}\sigma \geq \operatorname{Re} s$, $\operatorname{Im}\sigma = \operatorname{Im} s$ *for every* $\sigma \in \Gamma(s)$.

PROOF. Set $e^{\tilde{\sigma}} = e^{n\sigma} \sigma^{-2n-m}$, namely $\tilde{\sigma} = n\sigma - (2n + m)\log\sigma$. For $\sigma \in \Gamma(s)$ write $\tilde{\sigma} = r + i\chi$, $\sigma = \xi + i\mu$, where $\xi \geq \operatorname{Re} s \geq \widehat{R} > 2(|m| + 2)$, $|\mu| \leq \widehat{\Theta}$. From

$$r + i\chi = n(\xi + i\mu) - (2n + m)(\log\sqrt{\xi^2 + \mu^2} + i \arctan(\mu/\xi))$$

we derive

$$r = r(\xi) = n\xi - (n + m/2)\log(\xi^2 + \mu^2),$$

$$\chi = \chi(\xi) = n\mu - (2n + m)\arctan(\mu/\xi)$$

and

$$\Big|\frac{d\chi}{dr}\Big| = \Big|\frac{d\chi/d\xi}{dr/d\xi}\Big| = \Big|\frac{(2n + m)\mu}{n(\xi^2 + \mu^2) - (2n + m)\xi}\Big|$$

$$\leq \left|\frac{(2+|m|)\mu}{\xi^2-(2+|m|)\xi}\right| \leq \frac{|\mu|}{2(2+|m|)} \leq |\mu| \leq \widehat{\Theta}\,,$$

since $\xi \geq \widehat{R} > 2(|m|+2)$. This implies $|d\tilde{\sigma}| = \sqrt{dr^2+d\chi^2} = \sqrt{1+(d\chi/dr)^2}|dr| \leq \sqrt{1+\widehat{\Theta}^2}|dr|$, which yields

$$|d\sigma| = \frac{|d\tilde{\sigma}|}{|n-(2n+m)/\sigma|} \leq \frac{|d\tilde{\sigma}|}{n|1-(2+|m|)/\widehat{R}|} \leq 2\sqrt{1+\widehat{\Theta}^2}|dr|\,.$$

Since

$$\frac{dr}{d\xi} = n - \frac{(2n+m)\xi}{\xi^2+\eta^2} \geq n - \frac{2n+|m|}{\xi} \geq n - \frac{2n+|m|}{2|m|+4} > 0\,,$$

$\Gamma(s)$ is mapped to a path joining $r(\operatorname{Re} s)$ to ∞ along which $r = \operatorname{Re}\tilde{\sigma}$ is monotone. Hence

$$\int_{\Gamma(s)}\left|\frac{e^{ns}s^{-2n-m}}{e^{n\sigma}\sigma^{-2n-m}}\right||d\sigma| \leq \int_{r(\operatorname{Re} s)}^{+\infty} 2e^{r(\operatorname{Re} s)-r}\sqrt{1+\widehat{\Theta}^2}\,dr \leq 2\sqrt{1+\widehat{\Theta}^2}\,,$$

which implies the lemma. □

Using this lemma, we have

LEMMA 5.5. *Let m, $\widehat{R}$ and $\Gamma(\cdot)$ be as in Lemma 5.4. If $\varphi(a,t) \in \mathfrak{A}(R,\Theta)$, then*

$$t_a^{-m}\int_{\Gamma(t)} \tau_a^m \varphi(a,\tau)\,d\tau \in \mathfrak{A}(R,\Theta)\,,$$

$$\left\|t_a^{-m}\int_{\Gamma(t)} \tau_a^m \varphi(a,\tau)\,d\tau\right\| \leq 2\sqrt{1+\widehat{\Theta}^2}\;\|\varphi\|(t)\,,$$

where $\tau_a := \tau - a$.

PROOF. Set

$$I_{n,m,k}(s) := e^{ns}s^{-2n-m}\int_{\Gamma(s)} e^{-n\sigma}\sigma^{2n+m-k}\,d\sigma\,.$$

For any positive integer ν, we have

$$\begin{aligned}
I_{n,m,0}(s) &= e^{ns}s^{-2n-m}\left(\frac{e^{-ns}}{n}s^{2n+m} + \frac{2n+m}{n}\int_{\Gamma(s)} e^{-n\sigma}\sigma^{2n+m-1}d\sigma\right) \\
&= \frac{1}{n} + \frac{2n+m}{n}I_{n,m,1}(s) = \frac{1}{n} + \frac{2n+m}{n}s^{-1}I_{n,m-1,0}(s) \\
&= \frac{1}{n} + \frac{2n+m}{n^2}s^{-1} + \cdots \\
&\quad + \frac{(2n+m)_{-\nu}}{n^{\nu+1}}s^{-\nu} + \frac{(2n+m)_{-(\nu+1)}}{n^{\nu+1}}s^{-\nu-1}I_{n,m-\nu-1,0}(s)
\end{aligned} \tag{5.1}$$

with $(c)_{-\nu} := c(c-1)\cdots(c-\nu+1)$. Here, by Lemma 5.4, $I_{n,m-\nu-1,0}(s) = O(1)$ in $S(\widehat{R}, \widehat{\Theta})$. Write $\varphi(a,t) \in \mathfrak{A}(R,\Theta)$ in the form $\varphi(a,t) = \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} \varphi_n(t_a)$ with $\varphi_n(s) \sim \sum_{j=0}^{\infty} \varphi_j^n s^{-j}$ as $s \to \infty$ through $S(\widehat{R}, \widehat{\Theta})$. Then, observing $I_{n,m,k}(s) = s^{-k} I_{n,m-k,0}(s)$, and using (5.1) and Lemma 5.4, we have

$$\Phi_n(s) = e^{ns} s^{-2n-m} \int_{\Gamma(s)} e^{-n\sigma} \sigma^{2n+m} \varphi_n(\sigma)\, d\sigma \sim \sum_{j=0}^{\infty} \Phi_j^n s^{-j}, \quad \Phi_j^n \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$$

as $s \to \infty$ through $S(\widehat{R}, \widehat{\Theta})$. We substitute $(s, \sigma) = (t_a, \tau_a)$ with $(a,t) \in \Delta_0 \times S(R,\Theta)$ and $\tau_a \in \Gamma(t_a)$ to obtain the formal series

$$\begin{aligned}\Phi(a,t) &:= t_a^{-m} \int_{\Gamma(t)} \tau_a^m \varphi(a,\tau)\, d\tau = \sum_{n=1}^{\infty} t_a^{-m} \int_{\Gamma(t)} e^{-n\tau} \tau_a^{2n+m} \varphi_n(\tau_a)\, d\tau \\ &= \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} \Phi_n(t_a) \in \widehat{\mathfrak{A}}(\widehat{R}, \widehat{\Theta})\,.\end{aligned}$$

By Lemma 5.4,

$$\begin{aligned}M(\Phi_n) &= \sup_{s \in S(\widehat{R},\widehat{\Theta})} |\Phi_n(s)| = \sup_{s \in S(\widehat{R},\widehat{\Theta})} \left| e^{ns} s^{-2n-m} \int_{\Gamma(s)} e^{-n\sigma} \sigma^{2n+m} \varphi_n(\sigma)\, d\sigma \right| \\ &\le \sup_{s \in S(\widehat{R},\widehat{\Theta})} |e^{ns} s^{-2n-m}| \int_{\Gamma(s)} |e^{-n\sigma} \sigma^{2n+m}|\, M(\varphi_n)\, |d\sigma| \le 2\sqrt{1+\widehat{\Theta}^2}\, M(\varphi_n)\,,\end{aligned}$$

which implies

$$\|\Phi\|(t) = \sum_{n=1}^{\infty} M(\Phi_n) |e^{-t}|^{n/2} \le 2\sqrt{1+\widehat{\Theta}^2} \sum_{n=1}^{\infty} M(\varphi_n) |e^{-t}|^{n/2} = 2\sqrt{1+\widehat{\Theta}^2}\, \|\varphi\|(t)$$

and $\Phi(a,t) \in \mathfrak{A}(R,\Theta)$. This completes the proof. □

5.2. System of integral equations. By the change of variables $x = e^{-t}$, $y = 1 + e^{-t} z$, equation (V) is taken into

$$\begin{aligned}z'' - &\frac{(z')^2}{z} - \gamma + \frac{1}{z} \\ &= e^{-t} \left(\frac{(z'-z)^2}{2(1+e^{-t}z)} + \Big(\alpha(1+e^{-t}z) + \frac{\beta}{1+e^{-t}z} \Big) z^2 + \gamma z - \frac{3}{2} - \frac{e^{-t}z}{2} \right).\end{aligned}$$

Suppose that $\gamma \neq 0$. The associated truncated equation

$$z'' - \frac{(z')^2}{z} - \gamma + \frac{1}{z} = 0$$

admits a solution of the form

$$z = \phi = \phi(t_a) = -\frac{\gamma}{2}t_a^2 + \frac{1}{2\gamma}, \qquad t_a = t - a$$

with $a \in \Delta_0$, which satisfies

$$(\phi')^2 = 1 - 2\gamma\phi, \qquad \phi'' = -\gamma\,. \tag{5.2}$$

Putting $z = \phi + w$, we have

$$\begin{aligned} w'' - \frac{2\phi'}{\phi}w' - \frac{2\gamma}{\phi}w = {} & \frac{-2\gamma w^2 - 2\phi' w w' + \phi(w')^2}{\phi(\phi + w)} \\ & + e^{-t}\biggl(\frac{(\phi' - \phi + w' - w)^2}{2(1 + e^{-t}(\phi + w))} + \Bigl(\alpha(1 + e^{-t}(\phi + w)) + \frac{\beta}{1 + e^{-t}(\phi + w)}\Bigr)(\phi + w)^2 \\ & \quad + \gamma(\phi + w) - \frac{3}{2} - \frac{e^{-t}}{2}(\phi + w)\biggr), \end{aligned}$$

which is written in the form

$$w'' - \frac{2\phi'}{\phi}w' - \frac{2\gamma}{\phi}w = G(a, t, w, w')\,, \tag{5.3}$$

where

$$\begin{aligned} & G(a, t, w, w') := g_0(a, t) + g(a, t, w, w') + g^*(a, t, w, w')\,, \\ & g_0(a, t) = e^{-t}\left(\frac{(\phi' - \phi)^2}{2(1 + e^{-t}\phi)} + \Bigl(\alpha(1 + e^{-t}\phi) + \frac{\beta}{1 + e^{-t}\phi}\Bigr)\phi^2 + \gamma\phi - \frac{3}{2} - \frac{e^{-t}\phi}{2}\right)\,, \\ & g(a, t, w, w') = \frac{\phi(w')^2 - 2\phi' w w' - 2\gamma w^2}{\phi(\phi + w)}\,, \\ & g^*(a, t, w, w') = e^{-t}\biggl(\frac{(\phi' - \phi + w' - w)^2}{2(1 + e^{-t}(\phi + w))} - \frac{(\phi' - \phi)^2}{2(1 + e^{-t}\phi)} \\ & \qquad + \alpha(1 + e^{-t}(\phi + w))(\phi + w)^2 - \alpha(1 + e^{-t}\phi)\phi^2 \\ & \qquad + \frac{\beta(\phi + w)^2}{1 + e^{-t}(\phi + w)} - \frac{\beta\phi^2}{1 + e^{-t}\phi} + \gamma w - \frac{e^{-t}w}{2}\biggr)\,. \end{aligned}$$

If $|e^{-t}\phi(t_a)|$, $|w/\phi(t_a)| < 1$ and if $|t_a|$ is sufficiently large, then $g_0(a, t)$ and $g(a, t, w, w')$ are expanded into the convergent series

$$g_0(a, t) = t_a^2 \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} g_n^0(t_a), \quad g_n^0(t_a) = \sum_{j=0}^{\infty} g_j^{0n} t_a^{-j}\,, \tag{5.4}$$

$$g(a, t, w, w') = t_a^2 \sum_{k+l \ge 2} g_{kl}(t_a)(t_a^{-2}w)^k (t_a^{-2}w')^l, \quad g_{kl}(t_a) = \sum_{j=0}^{\infty} g_j^{kl} t_a^{-j} \tag{5.5}$$

with $g_j^{0n} \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$, $g_j^{kl} \in \mathbb{Q}[\gamma, \gamma^{-1}]$. In particular,

$$t_a^4 g_1^0(t_a) = (\phi' - \phi)^2/2 + (\alpha + \beta)\phi^2 + \gamma\phi - 3/2\,,$$

$$t_a^6 g_2^0(t_a) = -\Big((\phi' - \phi)^2/2 - (\alpha - \beta)\phi^2 + 1/2\Big)\,\phi\,,$$

$$t_a^{2n+2} g_n^0(t_a) = (-1)^{n+1}\Big((\phi' - \phi)^2/2 + \beta\phi^2\Big)\,\phi^{n-1} \quad \text{for } n \geq 3.$$

Note that

$$g^*(a, t, w, w') = \frac{\phi U(\phi^{-1}, \phi'\phi^{-1}; e^{-t}\phi, \phi^{-1}w, \phi^{-1}w')}{(1 + e^{-t}\phi + e^{-t}\phi \cdot \phi^{-1}w)(1 + e^{-t}\phi)}\,,$$

where $U(X_1, X_2; Y, W_1, W_2)$ is a polynomial such that

$$U(X_1, X_2; 0, W_1, W_2) \equiv 0, \quad U(X_1, X_2; Y, 0, 0) \equiv 0\,.$$

If $|e^{-t}\phi(t_a)| + |e^{-t}w| < 1$, and if $|t_a|$ is sufficiently large, then

(5.6) $$g^*(a, t, w, w') = t_a^2 \sum_{k+l \geq 1} g_{kl}^*(a, t)(t_a^{-2}w)^k (t_a^{-2}w')^l$$

with

$$g_{kl}^*(a, t) = \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} g_n^{*kl}(t_a), \quad g_n^{*kl}(t_a) = \sum_{j=0}^{\infty} g_j^{*kln} t_a^{-j}$$

such that $g_j^{*kln} \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$. Let B_0 and Δ_0 be as in Section 5.1, and let $\widehat{R} \geq 10$. In what follows we may suppose the following, in which $R' = R'(\Delta_0, B_0) > \widehat{R}$ is a sufficiently large positive number:

(i) $|t_a| > \widehat{R}$ and $|e^{-t}t_a^2| < 1/2$ for every (a, t) satisfying $|t| > R'$, $a \in \Delta_0$;

(ii) as power series in $(e^{-t}t_a^2, s^{-1}, u, v)$

$$\sum_{n=1}^{\infty}(e^{-t}t_a^2)^n \sum_{j=0}^{\infty} g_j^{0n} s^{-j}, \quad \sum_{k+l \geq 2}\sum_{j=0}^{\infty} g_j^{kl} s^{-j} u^k v^l, \quad \sum_{k+l \geq 1}\sum_{n=1}^{\infty}(e^{-t}t_a^2)^n \sum_{j=0}^{\infty} g_j^{*kln} s^{-j} u^k v^l$$

converge for $|e^{-t}t_a^2| < 1/2$, $|s| > \widehat{R}$, $|u| < 1$, $|v| < 1$ absolutely and uniformly in (α, β, γ) satisfying $|\alpha| + |\beta| + |\gamma| + |\gamma^{-1}| < B_0$.

For the linear equation

(5.7) $$w'' - \frac{2\phi'}{\phi}w' - \frac{2\gamma}{\phi}w = 0\,,$$

using (5.2), we have

LEMMA 5.6. *Equation* (5.7) *has linearly independent solutions given by*

$$\psi_1(t_a) := 4\gamma^{-2}\phi^2 + 8\gamma^{-3}\phi - 8\gamma^{-4}, \quad \psi_2(t_a) := -\gamma^{-1}\phi'\,,$$

and the corresponding Wronskian is $\psi_{\rm w}(t_a) := -12\gamma^{-2}\phi^2$.

A solution of the system of integral equations

$$(5.8)\quad \begin{aligned} w(a,t) &= \int_{\Gamma(t)} \frac{\psi_1(t_a)\psi_2(\tau_a) - \psi_1(\tau_a)\psi_2(t_a)}{\psi_{\rm w}(\tau_a)} G(a,\tau,w(a,\tau),w'(a,\tau))\,d\tau\,, \\ w'(a,t) &= \int_{\Gamma(t)} \frac{\psi_1'(t_a)\psi_2(\tau_a) - \psi_1(\tau_a)\psi_2'(t_a)}{\psi_{\rm w}(\tau_a)} G(a,\tau,w(a,\tau),w'(a,\tau))\,d\tau \end{aligned}$$

solves (5.3), where $\Gamma(t)$ is a horizontal line starting from t and tending to ∞ on which $\mathrm{Re}\,\tau \geq \mathrm{Re}\,t$, $\mathrm{Im}\,\tau = \mathrm{Im}\,t$. We may suppose that $\widehat{R} = \widehat{R}(B_0) \geq 10$ has been given in advance in such a way that the series

$$\frac{\psi_1(s)}{\psi_{\rm w}(s)} = -\frac{1}{3} + \sum_{j=1}^{\infty} \psi_j^1 s^{-j}, \quad \frac{\psi_2(s)}{\psi_{\rm w}(s)} = s^{-3}\Big(-\frac{1}{3} + \sum_{j=1}^{\infty} \psi_j^2 s^{-j}\Big),$$

$$\psi_1(s) = s^4\Big(1 + \sum_{j=1}^{\infty} \tilde{\psi}_j^1 s^{-j}\Big), \quad \psi_2(s) = s$$

with $\psi_j^1, \psi_j^2, \tilde{\psi}_j^1 \in \mathbb{Q}[\gamma,\gamma^{-1}]$ converge for $|s| > \widehat{R}$ uniformly in γ satisfying $|\gamma| + |\gamma^{-1}| < B_0$, and that $|\psi_1(s)/\psi_{\rm w}(s)|$, $|s^3\psi_2(s)/\psi_{\rm w}(s)|$, $|s^{-4}\psi_1(s)|$, $|s^{-3}\psi_1'(s)|$ are bounded. Write (5.8) in the form

$$\begin{aligned} w(a,t) &= \psi_0(t_a)t_a^4\,\mathcal{I}\left[\frac{\psi_2(t_a)}{\psi_{\rm w}(t_a)} G(a,t,w(a,t),w'(a,t))\right] \\ &\qquad - t_a\mathcal{I}\left[\frac{\psi_1(t_a)}{\psi_{\rm w}(t_a)} G(a,t,w(a,t),w'(a,t))\right], \\ w'(a,t) &= \tilde{\psi}_0(t_a)t_a^3\,\mathcal{I}\left[\frac{\psi_2(t_a)}{\psi_{\rm w}(t_a)} G(a,t,w(a,t),w'(a,t))\right] \\ &\qquad - \mathcal{I}\left[\frac{\psi_1(t_a)}{\psi_{\rm w}(t_a)} G(a,t,w(a,t),w'(a,t))\right] \end{aligned}$$

with

$$\psi_0(s) := s^{-4}\psi_1(s), \quad \tilde{\psi}_0(s) := s^{-3}\psi_1'(s), \quad \mathcal{I}[f(t)] := \int_{\Gamma(t)} f(\tau)\,d\tau\,.$$

Then $u(a,t) = t_a^{-2}w(a,t)$ and $v(a,t) = t_a^{-2}w'(a,t)$ satisfy

$$\begin{aligned} u(a,t) &= \psi_0(t_a)t_a^2\,\mathcal{I}\Big[t_a^{-1}F_*(a,t,u(a,t),v(a,t))\Big] \\ &\qquad - t_a^{-1}\mathcal{I}\Big[t_a^2 F(a,t,u(a,t),v(a,t))\Big]\,, \\ v(a,t) &= \tilde{\psi}_0(t_a)t_a\,\mathcal{I}\Big[t_a^{-1}F_*(a,t,u(a,t),v(a,t))\Big] \end{aligned}$$

$$- t_a^{-2}\mathcal{I}\left[t_a^2 F(a,t,u(a,t),v(a,t))\right]$$

with

$$F(a,t,u,v) := \frac{\psi_1(t_a)}{\psi_{\mathrm{w}}(t_a)} t_a^{-2} G(a,t,t_a^2 u, t_a^2 v)\,,$$

$$F_*(a,t,u,v) := \frac{\psi_2(t_a)}{\psi_{\mathrm{w}}(t_a)} t_a G(a,t,t_a^2 u, t_a^2 v) = (1 + t_a^{-1}\psi_*(t_a))F(a,t,u,v)\,.$$

Here the series

$$\psi_0(s) = 1 + \sum_{j=1}^{\infty} \psi_j^0 s^{-j}, \quad \tilde{\psi}_0(s) = 4 + \sum_{j=1}^{\infty} \tilde{\psi}_j^0 s^{-j}, \quad \psi_*(s) = \sum_{j=0}^{\infty} \psi_j^* s^{-j} \tag{5.9}$$

with ψ_j^0, $\tilde{\psi}_j^0$, $\psi_j^* \in \mathbb{Q}[\gamma, \gamma^{-1}]$ converge for $|s| > \widehat{R}$ uniformly in γ satisfying $|\gamma| + |\gamma^{-1}| < B_0$. Thus the system above is written in the form

$$\begin{aligned} u(a,t) &= \psi_0(t_a) t_a^2\, \mathcal{I}\left[(t_a^{-1} + t_a^{-2}\psi_*(t_a))F(a,t,u(a,t),v(a,t))\right] \\ &\qquad - t_a^{-1}\mathcal{I}\left[t_a^2 F(a,t,u(a,t),v(a,t))\right]\,, \\ v(a,t) &= \tilde{\psi}_0(t_a) t_a\, \mathcal{I}\left[(t_a^{-1} + t_a^{-2}\psi_*(t_a))F(a,t,u(a,t),v(a,t))\right] \\ &\qquad - t_a^{-2}\mathcal{I}\left[t_a^2 F(a,t,u(a,t),v(a,t))\right]\,. \end{aligned} \tag{5.10}$$

Conversely, for the solution $(u(a,t), v(a,t))$ of this system, $w(a,t) = t_a^2 u(a,t)$ solves (5.8). Furthermore, by (5.4), (5.5) and (5.6), we have

LEMMA 5.7. *The function $F(a,t,u,v)$ is written in the form*

$$F(a,t,u,v) = f_0(a,t) + f(a,t,u,v) + f^*(a,t,u,v)$$

with $f_0(a,t)$, $f(a,t,u,v)$ and $f^*(a,t,u,v)$ given by

$$f_0(a,t) = \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} f_n^0(t_a), \quad f_n^0(s) = \sum_{j=0}^{\infty} f_j^{0n} s^{-j}\,,$$

$$f(a,t,u,v) = \sum_{k+l\ge 2} f_{kl}(t_a) u^k v^l, \quad f_{kl}(s) = \sum_{j=0}^{\infty} f_j^{kl} s^{-j}\,,$$

$$f^*(a,t,u,v) = \sum_{k+l\ge 1} f_{kl}^*(a,t) u^k v^l, \quad f_{kl}^*(a,t) = \sum_{n=1}^{\infty} e^{-nt} t_a^{2n} f_n^{*kl}(t_a)\,,$$

$$f_n^{*kl}(s) = \sum_{j=0}^{\infty} f_j^{*kln} s^{-j}\,.$$

Here the coefficients $f_j^{0n},\ f_j^{kl},\ f_j^{*kln} \in \mathbb{Q}[\alpha, \beta, \gamma, \gamma^{-1}]$ *are such that*

$$\sum_{n=1}^{\infty}(e^{-t}t_a^2)^n \sum_{j=0}^{\infty} f_j^{0n} s^{-j}, \quad \sum_{k+l\geq 2}\sum_{j=0}^{\infty} f_j^{kl} s^{-j} u^k v^l, \quad \sum_{k+l\geq 1}\sum_{n=1}^{\infty}(e^{-t}t_a^2)^n \sum_{j=0}^{\infty} f_j^{*kln} s^{-j} u^k v^l$$

as power series in $(e^{-t}t_a^2, s^{-1}, u, v)$ *converge for* $|e^{-t}t_a^2| < 1/2,\ |s| > \widehat{R},\ |u| < 1,\ |v| < 1$ *absolutely and uniformly in* (α, β, γ) *satisfying* $|\alpha| + |\beta| + |\gamma| + |\gamma^{-1}| < B_0$, *and that the sums of the absolute values of summands are uniformly bounded.*

This lemma implies $f_0(a, t),\ f_{kl}^*(a, t) \in \mathfrak{A}(R', \Theta')$ for any $\Theta' > 0$.

5.3. Construction of a solution of (5.10). In addition to B_0, Δ_0 and $\widehat{R}$, let Θ be a given positive number. Then choose $R = R(\Delta_0, \widehat{R}, \Theta) \geq R' > \widehat{R} \geq 10$ and $\widehat{\Theta} = \widehat{\Theta}(\Delta_0, \Theta) > \Theta$ as in Section 5.1. Let us define $\{(u_\nu(a, t), v_\nu(a, t)) \mid \nu \geq 0\} \subset \mathfrak{A}(R, \Theta)^2$ by

$$\begin{aligned} &u_0(a, t) \equiv 0, \quad v_0(a, t) \equiv 0\,, \\ &u_{\nu+1}(a, t) = \psi_0(t_a)t_a^2\mathcal{I}\Big[(t_a^{-1} + t_a^{-2}\psi_*(t_a))F(a, t, u_\nu(a, t), v_\nu(a, t))\Big] \\ &\qquad\qquad - t_a^{-1}\mathcal{I}\Big[t_a^2 F(a, t, u_\nu(a, t), v_\nu(a, t))\Big]\,, \\ &v_{\nu+1}(a, t) = \tilde{\psi}_0(t_a)t_a\mathcal{I}\Big[(t_a^{-1} + t_a^{-2}\psi_*(t_a))F(a, t, u_\nu(a, t), v_\nu(a, t))\Big] \\ &\qquad\qquad - t_a^{-2}\mathcal{I}\Big[t_a^2 F(a, t, u_\nu(a, t), v_\nu(a, t))\Big]\,. \end{aligned} \tag{5.11}$$

By Lemmas 5.3 and 5.7 this procedure is possible as long as $\|u_\nu(a, t)\|,\ \|v_\nu(a, t)\| < 1$. Indeed, under the supposition $u_\nu(a, t),\ v_\nu(a, t) \in \mathfrak{A}(R, \Theta)$, integrating by parts and using $\|u_\nu(a, t)\|,\ \|v_\nu(a, t)\| = O(|e^{-t}|^{1/2})$ as $t \to \infty$ along $\Gamma(t)$, we have

$$\begin{aligned} &t_a^2\mathcal{I}\Big[t_a^{-1}F(a, t, u_\nu(a, t), v_\nu(a, t))\Big] - t_a^{-1}\mathcal{I}\Big[t_a^2 F(a, t, u_\nu(a, t), v_\nu(a, t))\Big] \\ =&t_a^2\mathcal{I}\Big[t_a^{-1}F(a, t, u_\nu(a, t), v_\nu(a, t))\Big] \\ &\quad - t_a^{-1}\Big(t_a^3\mathcal{I}\Big[t_a^{-1}F(a, t, u_\nu(a, t), v_\nu(a, t))\Big] \\ &\qquad - \mathcal{I}\Big[3t_a^2\mathcal{I}\Big[t_a^{-1}F(a, t, u_\nu(a, t), v_\nu(a, t))\Big]\Big]\Big) \\ =&3t_a^{-1}\mathcal{I}\Big[t_a \cdot t_a\mathcal{I}\Big[t_a^{-1}F(a, t, u_\nu(a, t), v_\nu(a, t))\Big]\Big]\,, \end{aligned}$$

and hence the first relation in (5.11) is

$$\begin{aligned} u_{\nu+1}(a, t) =& 3t_a^{-1}\mathcal{I}\Big[t_a \cdot t_a\mathcal{I}\Big[t_a^{-1}F(a, t, u_\nu(a, t), v_\nu(a, t))\Big]\Big] \\ &+ (\psi_0(t_a) - 1)t_a^2\mathcal{I}\Big[t_a^{-1}F(a, t, u_\nu(a, t), v_\nu(a, t))\Big] \\ &+ \psi_0(t_a)t_a^2\mathcal{I}\Big[t_a^{-2}\psi_*(t_a)F(a, t, u_\nu(a, t), v_\nu(a, t))\Big]\,. \end{aligned} \tag{5.12}$$

Then we derive $(u_{\nu+1}(a,t), v_{\nu+1}(a,t)) \in \mathfrak{A}(R,\Theta)^2$ by using Lemma 5.5 with $|m| = 2$, which is applicable since $\widehat{R} > 10$.

By Lemma 5.7, if $|u|$, $|\tilde{u}|$, $|v|$, $|\tilde{v}| < 1$, then

$$f(a,t,u,v) - f(a,t,\tilde{u},\tilde{v}) = h_1(a,t,u,\tilde{u},v,\tilde{v})(u-\tilde{u}) + h_2(a,t,u,\tilde{u},v,\tilde{v})(v-\tilde{v}),$$

$$f^*(a,t,u,v) - f^*(a,t,\tilde{u},\tilde{v}) = h_1^*(a,t,u,\tilde{u},v,\tilde{v})(u-\tilde{u}) + h_2^*(a,t,u,\tilde{u},v,\tilde{v})(v-\tilde{v})$$

with

$$h_\iota(a,t,u,\tilde{u},v,\tilde{v}) = \sum_{k+k'+l+l'\geq 1} h^\iota_{kk'll'}(a,t) u^k \tilde{u}^{k'} v^l \tilde{v}^{l'},$$

$$h^*_\iota(a,t,u,\tilde{u},v,\tilde{v}) = \sum_{k+k'+l+l'\geq 0} h^{*\iota}_{kk'll'}(a,t) u^k \tilde{u}^{k'} v^l \tilde{v}^{l'}$$

($\iota = 1, 2$), where $h^\iota_{kk'll'}(a,t)$ (respectively, $h^{*\iota}_{kk'll'}(a,t)$) are polynomials in $f_{k''l''}(t_a)$ (respectively, $f^*_{k''l''}(a,t)$), in particular $h^{*1}_{0000}(a,t) = f^*_{10}(a,t)$, $h^{*2}_{0000}(a,t) = f^*_{01}(a,t) \in \mathfrak{A}(R,\Theta)$.

Hence, by Lemma 5.3, for u, $\tilde{u}$, v, $\tilde{v} \in \mathfrak{A}(R,\Theta)$ such that $\|u\|$, $\|\tilde{u}\|$, $\|v\|$, $\|\tilde{v}\| < 1/2$,

$$\begin{aligned} &\|F(a,t,u,v) - F(a,t,\tilde{u},\tilde{v})\| \\ &\leq L_0(|e^{-t}|^{1/2} + \|u\| + \|\tilde{u}\| + \|v\| + \|\tilde{v}\|)(\|u-\tilde{u}\| + \|v-\tilde{v}\|), \end{aligned} \tag{5.13}$$

where L_0 is some positive number. Since $F(a,t,0,0) \in \mathfrak{A}(R,\Theta)$, by Lemma 5.5

$$\|u_1(a,t)\| + \|v_1(a,t)\| \leq L_1|e^{-t}|^{1/2}$$

for some $L_1 > 0$. By Lemma 5.5, (5.11), (5.12) and (5.13) combined with (5.9), we have

$$\begin{aligned} &\|u_{j+1}(a,t) - u_j(a,t)\| + \|v_{j+1}(a,t) - v_j(a,t)\| \\ &\leq L_2|e^{-t}|^{1/2}(\|u_j(a,t) - u_{j-1}(a,t)\| + \|v_j(a,t) - v_{j-1}(a,t)\|) \end{aligned} \tag{5.14}$$

as long as

$$\|u_k(a,t)\| + \|v_k(a,t)\| \leq 2L_1|e^{-t}|^{1/2} \quad \text{for } k \leq j, \tag{5.15}$$

which is valid for $j = 1$, where L_2 is a positive constant independent of j. Suppose that (5.15) is valid for $j \leq \nu$. Then, for $j \leq \nu$

$$\begin{aligned} &\|u_{j+1}(a,t) - u_j(a,t)\| + \|v_{j+1}(a,t) - v_j(a,t)\| \\ &\leq (L_2|e^{-t}|^{1/2})^j(\|u_1(a,t)\| + \|v_1(a,t)\|) \leq L_1 L_2^j |e^{-t}|^{(j+1)/2}, \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} \|u_{\nu+1}(a,t)\| + \|v_{\nu+1}(a,t)\| &\leq \sum_{j=0}^{\nu}(\|u_{j+1}(a,t) - u_j(a,t)\| + \|v_{j+1}(a,t) - v_j(a,t)\|) \\ &\leq L_1|e^{-t}|^{1/2}\sum_{j=0}^{\nu}(L_2|e^{-t}|^{1/2})^j \leq \frac{L_1|e^{-t}|^{1/2}}{1 - L_2|e^{-t}|^{1/2}}. \end{aligned}$$

This implies that (5.15) and (5.16) are valid for all integers if $1 - L_2 e^{-R/2} > 1/2$. For such R we conclude that $(u_\infty(a,t), v_\infty(a,t)) = \lim_{\nu\to\infty}(u_\nu(a,t), v_\nu(a,t)) \in \mathfrak{A}(R,\Theta)$ satisfies (5.10), so that $w(a,t) = t_a^2 u_\infty(a,t)$ solves (5.3).

5.4. Completion of the proof of Theorem 2.1. For a given bounded domain D_0 as in Section 2, choose Δ_0 in such a way that $a = \log\rho \in \Delta_0$ for every $\rho \in D_0$. For given positive numbers B_0 and $\Theta = \Theta_0$, let $\widehat{R}$, R and $\widehat{\Theta}$ be as in Sections 5.2 and 5.3. Take ε_0 so small that $t \in S(R,\Theta_0)$ holds for every $x = e^{-t} \in \Sigma(\varepsilon_0,\Theta_0)$. Then

$$y = 1 + x\Big(\phi(-\log(\rho x)) + \log^2(\rho x) u_\infty(\log\rho, -\log x)\Big)$$

is the desired solution in Theorem 2.1 with $R_0 = \widehat{R}$, $\widehat{\Theta}_0 = \widehat{\Theta}$.

Substituting $w = e^{-t}w_2(t_a) + e^{-2t}w_3(t_a) + \cdots$ into (5.3), we may recursively obtain each series $w_n(s) = p_n(s) + p_n^-(s)$ in descending powers of s. The first two relations are

$$\phi w_2'' - 2(\phi+\phi')w_2' + (\phi + 2\phi' - 2\gamma)w_2 = \frac{\phi}{2}(\phi'-\phi)^2 + (\alpha+\beta)\phi^3 + \gamma\phi^2 - \frac{3}{2}\phi,$$

$$\begin{aligned}\phi w_3'' - 2(2\phi+\phi')w_3' + (4\phi + 4\phi' - 2\gamma)w_3 = {} & -\frac{\phi^2}{2}(\phi'-\phi)^2 + (\alpha-\beta)\phi^4 - \frac{\phi^2}{2} \\ & - \frac{2\gamma}{\phi}w_2^2 - \frac{2\phi'}{\phi}w_2(w_2'-w_2) + (w_2'-w_2)^2 \\ & + \phi(\phi'-\phi)(w_2'-2w_2) + 2(\alpha+\beta)\phi^2 w_2 + \gamma\phi w_2\,,\end{aligned}$$

which yield $w_2(s) = p_2(s)$ and $w_3(s) = p_3(s)$ as in Remark 2.1.

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Present Address:
DEPARTMENT OF MATHEMATICS,
KEIO UNIVERSITY,
3–14–1 HIYOSHI, KOHOKU-KU, YOKOHAMA 223–8522, JAPAN.
e-mail: shimomur@math.keio.ac.jp