A Note on the Galois Brumer-Stark Conjecture

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Abstract. In this paper, we prove, for primes l satisfying some conditions, the l-parts of the Galois Brumer-Stark conjecture, which is formulated by Dejou and Roblot for Galois CM-extensions with dihedral or generalized quaternion Galois group of specified degrees.

1. Introduction

In number theory, the study of the mysterious relation between the special values of L-functions and arithmetic objects has been a main subject. Concerning this subject, the Brumer-Stark conjecture has been studied for many years. This conjecture is formulated for abelian extensions of number fields and concerned with the Galois module structure of ideal class groups. There exists a large body of evidence in support of this conjecture, see, for example, [6], [9] and [5].

In the last several years, non-abelian generalizations of the Brumer-Stark conjecture have been formulated independently by D. Burns [1], [2] and A. Nickel [10]. On the other hand, in [4], G. Dejou and X.-F. Roblot formulated another non-abelian generalization, the "Galois Brumer-Stark conjecture", which does not seem to be equivalent, in general, to the conjectures by Burns and Nickel. An important difference between the Galois Brumer-Stark conjecture and conjectures by Burns and Nickel is that the former has a claim concerning a field extension but the latter do not have such a claim. The main result of this paper is a proof of the Galois Brumer-Stark conjecture in a special case.

We briefly review the formulation here. Let K/k be a finite Galois extension of number fields with Galois group G and S a finite set of places of k which contains all infinite places and all finite places which ramify in K. Then the "Stickelberger element" $\theta_{K/k,S}$ is defined by using the special values of L-functions attached to K/k, for the precise definition, see §3.

We set

$$K^\circ := \{\alpha \in K^* \mid |\alpha^\sigma| = 1, \text{ for all embeddings } \sigma : K \hookrightarrow \mathbf{C}\}\,.$$

Moreover, we write $\mu(K)$, w_K and $\mathrm{Ann}_{\mathbf{Z}[G]}(\mu(K))$ for the group of roots of unity in K, the order of $\mu(K)$ and $\mathbf{Z}[G]$-left annihilator of $\mu(K)$, respectively. Finally, we define the natural number d_G to be the least common multiple among the order of the commutator subgroup of G and the cardinalities of the conjugacy classes of G. Then the following is the statement of the Galois Brumer-Stark conjecture:

CONJECTURE 1.1.   *Assume* $|S| \geq 2$. *Then*

- $d_G\,\mathrm{Ann}_{\mathbf{Z}[G]}(\mu(K))\theta_{K/k,S} \subset \mathbf{Z}[G]$.
- *For each fractional ideal* $\mathfrak{A}$ *of* K, *there exists* $\alpha \in K^\circ$ *such that*

$$\mathfrak{A}^{d_G w_K \theta_{K/k,S}} = (\alpha)\,,$$

  and $K(\alpha^{1/w_K})/k$ *is a "strong central extension" of* K/k.

For the definition of the strong central extension, see §4. For a prime number p, we can formulate the following "p-part" of the above conjecture:

CONJECTURE 1.2.   *Assume* $|S| \geq 2$. *Then*

- $d_G\,\mathrm{Ann}_{\mathbf{Z}[G]}(\mu(K))\theta_{K/k,S} \subset \mathbf{Z}[G]$.
- *For each fractional ideal* $\mathfrak{A}$ *of* K *whose class in the ideal class group of* K *has* p-power *order, there exists* $\alpha \in K^\circ$ *such that*

$$\mathfrak{A}^{d_G w_{K,p} \theta_{K/k,S}} = (\alpha)\,,$$

  and $K(\alpha^{1/w_{K,p}})/k$ *is a strong central extension of* K/k, *where* $w_{K,p}$ *is the order of* $\mu(K) \otimes \mathbf{Z}_p$.

We easily see that Conjecture 1.1 is true if and only if Conjecture 1.2 is true for all primes p.

In what follows we assume K/k is a finite Galois CM-extension, that is, k is totally real and K is an imaginary quadratic extension of some totally real field. For an odd prime p and a nonzero natural number n, we write D_{4p} and $Q_{2^{n+2}}$ for the dihedral group of order $4p$ and the generalized quaternion group of order 2^{n+2}, respectively. Then the main results of this paper are the followings, which are Theorems 5.4 and 5.8 in §5.

THEOREM 1.3.   *Assume* G *is isomorphic to* D_{4p}. *Then the* l-part *of the Galois Brumer-Stark conjecture is true for* K/k, S, *where* l *is an odd prime whose mod* p *class generates* $(\mathbf{Z}/p\mathbf{Z})^*$. *Further if* K *does not contain* p-power *roots of unity, the* p-part *of the Galois Brumer-Stark conjecture is also true for* K/k *and* S.

THEOREM 1.4.   *Assume* G *is isomorphic to* $Q_{2^{n+2}}$. *Then the* 2-part *of the Galois Brumer-Stark conjecture is true for* K/k *and* S.

The keys of the proofs of these theorems are [11, Propositions 5.3 and 5.8] (in fact we need slight refinements of [11, Propositions 5.3 and 5.8], which are Propositions 5.2 and 5.6 in §§5.1 and 5.2, respectively). These propositions were also needed to prove the "weak non-abelian Brumer-Stark conjecture" by Nickel under the same assumption as above [11, Theorems 5.1 and 5.6]. Therefore the author thinks it is a very interesting problem to find a relation between two conjectures, however, we do not study the problem in this paper.

NOTATION. For a finite group G, we write $\operatorname{Irr} G$ and $[G, G]$ for the set of **C**-valued irreducible characters of G and the commutator subgroup of G, respectively. If G is the Galois group of a CM-extension of number fields, we write $\operatorname{Irr}^- G$ for the subset of $\operatorname{Irr} G$ consisting of the characters χ such that $\chi(j) = -\chi(1)$ for the unique complex conjugation j of G.

For a number field F and a prime number p, we write $\mu(F)$, w_F, $w_{F,p}$ and $Cl(K)$ for the group of roots of unity in F, the order of $\mu(F)$, that of $\mu(F) \otimes \mathbf{Z}_p$ and the ideal class group of F, respectively. For a nonzero natural number n, we fix a primitive n-th root of unity ζ_n in a fixed algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$.

For a finite Galois extension K/k of number fields with Galois group G, we write S_∞, S_{ram} and K^{ab} for the set of all infinite places of k, the set of all finite places of k which ramify in K and the maximal abelian subextension of K/k i.e. $K^{ab} = K^{[G,G]}$, respectively.

2. Central conductors

Let G be a finite group. We set

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}, \quad \mathrm{pr}_\chi := \frac{|G|}{\chi(1)} e_\chi = \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1}, \quad \chi \in \operatorname{Irr} G.$$

Then e_χ is a central primitive idempotent of $\mathbf{C}[G]$ and pr_χ is the associated projector.

We choose a maximal $\mathbf{Z}$-order $\mathfrak{m}(G)$ in $\mathbf{Q}[G]$ which contains $\mathbf{Z}[G]$. Then the central conductor $\mathfrak{F}(G)$ of $\mathfrak{m}(G)$ over $\mathbf{Z}[G]$ is given by

$$\mathfrak{F}(G) := \{x \in \zeta(\mathbf{Z}[G]) \mid x\mathfrak{m}(G) \subset \mathbf{Z}[G]\},$$

where $\zeta(\mathbf{Z}[G])$ is the center of $\mathbf{Z}[G]$. By Jacobinski's central conductor formula ([7, Theorem 3] also see [3, §27]), we have

$$\mathfrak{F}(G) \cong \bigoplus_{\chi \in \operatorname{Irr} G/\sim} \frac{|G|}{\chi(1)} \mathfrak{D}^{-1}(\mathbf{Q}(\chi)/\mathbf{Q}) \tag{1}$$

where $\mathbf{Q}(\chi) = \mathbf{Q}(\chi(g); g \in G)$, $\mathfrak{D}^{-1}(\mathbf{Q}(\chi)/\mathbf{Q})$ is the inverse different of $\mathbf{Q}(\chi)$ over $\mathbf{Q}$ and the direct sum runs over the irreducible characters of G modulo $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-action. From this formula, we see that elements x in $\mathfrak{F}(G)$ are of the form

$$x = \sum_{\chi \in \operatorname{Irr} G/\sim} \sum_{\sigma \in \operatorname{Gal}(\mathbf{Q}(\chi)/\mathbf{Q})} x_\chi^\sigma \, \mathrm{pr}_{\chi^\sigma}, x_\chi \in \mathfrak{D}^{-1}(\mathbf{Q}(\chi)/\mathbf{Q}).$$

3. Stickelberger elements

Let K/k be a finite Galois extension of number fields with Galois group G and S a finite set of places of k which contains S_{ram} and S_∞. We define the Stickelberger element for K/k and S by

$$\theta_{K/k,S} := \sum_{\chi \in \mathrm{Irr}\,G} L_S(K/k, \check{\chi}, 0) e_\chi \in \zeta(\mathbf{C}[G]),$$

where $L_S(K/k, \check{\chi}, s)$ is the S-truncated Artin L-function attached to $\check{\chi}$ and $\check{\chi}$ is the contragredient character of χ. This element actually belongs to $\zeta(\mathbf{Q}[G])$ by the rank zero Stark conjecture, which is a theorem of Siegel and Klingen for abelian extensions, and is proved by Brauer induction for general extensions [13, Theorem 1.2]. If $S = S_{\mathrm{ram}} \cup S_\infty$, we abbreviate $\theta_{K/k,S}$ by $\theta_{K/k}$.

Let m_G be the least common multiple of the cardinalities of the conjugacy classes of G, and let s_G be the order of the commutator subgroup of G. We set $d_G := \mathrm{l.c.m}\{m_G, s_G\}$. Then in [4] Dejou and Roblot formulated the following conjecture concerning the integrality of Stickelberger elements.

CONJECTURE 3.1 (Int($K/k, S$)).

$$d_G \, \mathrm{Ann}_{\mathbf{Z}[G]}(\mu(K))\theta_{K/k,S} \subset \mathbf{Z}[G].$$

REMARK 3.2. In [10] Nickel formulated another integrality conjecture, which predicts that "T-modified" Stickelberger elements $\theta_{K/k,S}^T = \delta_T \theta_{K/k,S}$ belong to $\mathcal{I}(G)$. Here $\mathcal{I}(G)$ is the module over $\zeta(\mathbf{Z}[G])$ generated by the reduced norms of matrices over $\mathbf{Z}[G]$ and δ_T is an element in $\mathcal{I}(G)$. Let $\mathcal{H}(G)$ be the "denominator ideal" defined in [8] such that $\mathcal{H}(G)\mathcal{I}(G) \subset \zeta(\mathbf{Z}[\mathbf{G}])$. Then we have $\mathcal{H}(G)\delta_T \subset \mathrm{Ann}_{\zeta(\mathbf{Z}[G])}(\mu(K))$(cf. [10, Remark 1. ii)] and the proof of [4, Proposition A.1]). Therefore, we see that if Conjecture 3.1 is true, $d_G\mathcal{H}(G)\delta_T\theta_{K/k,S} \subset \zeta(\mathbf{Z}[G])$. However, it is not clear whether Conjecture 3.1 implies Nickel's integrality conjecture itself. Conversely, if Nickel's integrality conjecture is true, we have $\mathcal{H}(G)\delta_T\theta_{K/k,S} \subset \zeta(\mathbf{Z}[G])$ and hence $d_G\mathcal{H}(G)\delta_T\theta_{K/k,S} \subset \zeta(\mathbf{Z}[G])$. This gives stronger results than Conjecture 3.1 for the elements of the form $\mathcal{H}(G)\delta_T$. However, it seems that Nickel's integrality conjecture can not imply the full statement of Conjecture 3.1 since it is concerned only with central elements.

In [4] the above conjecture was proved in some cases, for example, the following was proved.

THEOREM 3.3 ([4], Theorem 7.1 and Proposition 7.3). *We set*

$$\theta_{K/k,S}^{>1} := \left(1 - \frac{1}{s_G}N_{[G,G]}\right)\theta_{K/k,S}$$

and assume G has an abelian normal subgroup H of prime index. We write S_H for the set of

places of K^H above S. Then we have

$$\theta_{K/k,S}^{>1} = \left(1 - \frac{1}{s_G}N_{[G,G]}\right)\theta_{K/K^H,S_H} \tag{2}$$

and $\mathrm{Int}(K/k, S)$ *is true for K/k and S.*

4. The Galois Brumer-Stark conjecture

In this section we review the formulation of the Galois Brumer-Stark conjecture. For the details, see the original paper [4].

Let K/k be a finite Galois extension with Galois group G. Then in [4] they introduce the notion "strong central extension", which is defined as follows.

DEFINITION 4.1 ([4]).   A finite field extension L/k is a strong central extension of K/k if L contains K, L/k is a Galois extension with Galois group Γ and $[\Gamma, \Gamma] \cap \mathrm{Gal}(L/K) = 1$.

Now we state the Galois Brumer-Stark conjecture.

CONJECTURE 4.2 ($BS_{\mathrm{Gal}}(K/k, S)$).   *Let K/k be a finite Galois extension and S a finite set of places which contains S_{ram} and S_∞ with $|S| \geq 2$. Then $\mathrm{Int}(K/k, S)$ is true, and for each fractional ideal $\mathfrak{A}$ of K, there exists α in K° such that $\mathfrak{A}^{d_G w_K \theta_{K/k,S}} = (\alpha)$ and $K(\alpha^{1/w_K})/k$ is a strong central extension of K/k.*

For each prime p, we get the p-part of the conjecture $BS_{\mathrm{Gal}}(K/k, S, p)$ by replacing w_k and $\mathfrak{A}$ by $w_{K,p}$ and $\mathfrak{A}$ whose class in $Cl(K)$ has p-power order, respectively. We see that $BS_{\mathrm{Gal}}(K/k, S)$ is true if and only if $BS_{\mathrm{Gal}}(K/k, S, p)$ is true for all primes p.

A peculiarity of the Galois Brumer-Stark conjecture is that the conjecture decomposes into the "abelian part" and the "non-abelian part". More precisely, in [4] they proved the following proposition.

THEOREM 4.3 ([4], Theorem 6.3).   *Assume that $\mathrm{Int}(K/k, S)$ and $BS(K^{\mathrm{ab}}/k, S)$ are true. Then $BS_{\mathrm{Gal}}(K/k, S)$ is true if and only if for any fractional ideal $\mathfrak{A}$ of K, $\mathfrak{A}^{d_G w_K \theta_{K/k,S}^{>1}} = (\beta)$ for some β in K° and $K(\beta^{1/w_K})/k$ is a strong central extension of K/k.*

By the same proof of the above result in [4], we get the following.

THEOREM 4.4.   *The same statement as Theorem 4.3 holds with $BS(K^{\mathrm{ab}}/k, S)$, $BS_{\mathrm{Gal}}(K/k, S)$ and β^{1/w_K} replaced by $BS(K^{\mathrm{ab}}/k, S, p)$, $BS_{\mathrm{Gal}}(K/k, S, p)$ and $\beta^{1/w_{K,p}}$, respectively.*

To prove our main results, we essentially use the above theorem.

5. Main results

In this section, we prove our main results Theorems 1.3 and 1.4.

5.1. Extensions with group D_{4p}. Let p be an odd prime. In this subsection, we prove our main result for a CM-extension whose Galois group G is isomorphic to the dihedral group D_{4p} of order $4p$. We use the presentation $D_{4p} = \langle x, y \mid x^{2p} = y^2 = 1, yxy^{-1} = x^{-1} \rangle$. Then the center of D_{4p} is $\{1, x^p\}$.

5.1.1. Calculation of d_G and Stickelberger elements. Conjugacy classes of D_{4p} are $\{1\}$, $\{x^p\}$, $\{x^{2k}y\}_{1 \le k \le p}$, $\{x^{2k-1}y\}_{1 \le k \le p}$ and $\{x^l, x^{-l}\}$, where $1 \le l \le p-1$. Then we have

$$\#\{1\} = \#\{x^p\} = 1, \quad \#\{x^l, x^{-l}\} = 2 \quad \text{and} \quad \#\{x^{2k}y\}_{1 \le k \le p} = \#\{x^{2k-1}y\}_{1 \le k \le p} = p.$$

Moreover, the commutator subgroup of D_{4p} is the cyclic group $\langle x^2 \rangle$ of order p. Combining these facts, we have

$$d_G = \text{l.c.m}\{1, 2, p\} = 2p.$$

LEMMA 5.1. *Let K/k be a finite Galois CM-extension of number fields whose Galois group is isomorphic to D_{4p} and let S be a finite set of places of k which contains S_{ram} and S_∞. We set $H := \langle x \rangle$. Then*

$$d_G w_K \theta_{K/k,S}^{>1} = \left(\sum_{\chi \in \mathrm{Irr}^- G, \, \chi(1)=2} \mathrm{pr}_\chi \right) w_K \theta_{K/K^H, S_H}.$$

PROOF. Regarding d_G as an element in $\mathbf{C}[G]$, we see that

$$d_G = 2p = \sum_{\chi \in \mathrm{Irr}\, G} 2p e_\chi$$

$$= \sum_{\chi \in \mathrm{Irr}\, G, \chi(1)=1} 2p \left(\frac{1}{4p} \mathrm{pr}_\chi \right) + \sum_{\chi \in \mathrm{Irr}\, G, \chi(1)=2} 2p \left(\frac{1}{2p} \mathrm{pr}_\chi \right)$$

$$= \sum_{\chi \in \mathrm{Irr}\, G, \chi(1)=1} \frac{1}{2} \mathrm{pr}_\chi + \sum_{\chi \in \mathrm{Irr}\, G, \chi(1)=2} \mathrm{pr}_\chi.$$

Then we have by Theorem 3.3

$$d_G w_K \theta_{K/k,S}^{>1} = \left(\sum_{\chi \in \mathrm{Irr}\, G, \chi(1)=1} \frac{1}{2} \mathrm{pr}_\chi + \sum_{\chi \in \mathrm{Irr}\, G, \chi(1)=2} \mathrm{pr}_\chi \right) w_K \left(1 - \frac{1}{s_G} N_{[G,G]} \right) \theta_{K/K^H, S_H}$$

$$= \left(\sum_{\chi \in \mathrm{Irr}\, G, \chi(1)=2} \mathrm{pr}_\chi \right) w_K \theta_{K/K^H, S_H}$$

$$= \left(\sum_{\chi \in \mathrm{Irr}^- G, \chi(1)=2} \mathrm{pr}_\chi \right) w_K \theta_{K/K^H, S_H}.$$

The last equality follows since we have

$$\mathrm{pr}_\chi w_K \theta_{K/K^H, S_H} = \mathrm{pr}_\chi d_G w_K \theta_{K/k,S}^{>1} = d_G w_K L_S(K/k, \check{\chi}, 0) \, \mathrm{pr}_\chi$$

for $\chi \in \mathrm{Irr}\, G$ with $\chi(1) = 2$ and $L_S(K/k, \check{\chi}, 0) = 0$ if χ does not belongs to $\mathrm{Irr}^-\, G$. $\qquad\square$

5.1.2. Main result. In this section we prove Theorem 5.4. In [11] the author proved the following result.

PROPOSITION 5.2 ([11], Proposition 5.3). *Suppose l coincides with p and $\zeta_p \notin K$ or l is an odd prime whose mod p class generates $(\mathbf{Z}/p\mathbf{Z})^*$. Take any element x in $\mathfrak{F}(D_{4p})$ of the form $x = \sum_{\chi \in \mathrm{Irr}^-\, G, \chi(1)=2} x_\chi \, \mathrm{pr}_\chi$. Then for each fractional ideal $\mathfrak{A}$ of K whose class in $Cl(K)$ has l-power order,*

- $\mathfrak{A}^{x w_K \theta_{K/K^H}} = (\alpha)$ *for some α in K°,*
- $K(\alpha^{1/w_{K,l}}) = K$,

where we set $H = \langle x \rangle$.

REMARK 5.3. The original statement of [11, Proposition 5.3] concerning the field $K(\alpha^{1/w_{K,l}})$ is that $K(\alpha^{1/w_{K,l}})/K^H$ is abelian. However, the author actually proved in [11] that $K(\alpha^{1/w_{K,l}}) = K$ for primes l as in Proposition 5.2.

Using this proposition, we prove the following result, which is our main result in this section.

THEOREM 5.4. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to D_{4p} and let S be a finite set of places of k which contains S_{ram} and S_∞. Then $BS(K/k, S, p)$ is true if $\zeta_p \notin K$ and $BS(K/k, S, l)$ is true for an odd prime l whose mod p class generates $(\mathbf{Z}/p\mathbf{Z})^*$.*

PROOF. Since H is an abelian normal subgroup of G with $[G : H] = 2$, Conjecture 3.1 is true by Theorem 3.3. Therefore, by Theorem 4.4, we have only to show that $BS(K^{\mathrm{ab}}/k, S, l)$ is true and

$$\mathfrak{A}^{d_G w_K \theta_{K/k,S}^{>1}} = (\alpha) \quad \text{for some } \alpha \in K^\circ \text{ and } K(\alpha^{1/w_{k,l}})/k \text{ is strong central.} \tag{3}$$

Since $\mathrm{Gal}(K^{\mathrm{ab}}/k)$ is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, $BS(K^{\mathrm{ab}}/k, S, l)$ is true by [12, Theorem 2.1]. By Lemma 5.1, we have

$$d_G w_K \theta_{K/k,S}^{>1} = \left(\sum_{\chi \in \mathrm{Irr}^-\, G, \; \chi(1)=2} \mathrm{pr}_\chi \right) \theta_{K/K^H, S_H} .$$

Since the element $\sum_{\chi \in \mathrm{Irr}^-\, G, \; \chi(1)=2} \mathrm{pr}_\chi$ is a central conductor as in Proposition 5.2, we obtain the statement (3) by Proposition 5.2. This completes the proof. $\qquad\square$

5.2. Extensions with group $Q_{2^{n+2}}$. Let n be a nonzero natural number. In this subsection, we prove our main result for a CM-extension whose Galois group G is isomorphic to $Q_{2^{n+2}}$. We use the presentation $Q_{2^{n+2}} = \langle x, y \mid x^{2^n} = y^2, x^{2^{n+1}} = 1, yxy^{-1} = x^{-1} \rangle$. Then the center of $Q_{2^{n+2}}$ is $\{1, x^{2^n}\}$.

5.2.1. Calculation of d_G and Stickelberger elements. Conjugacy classes of $Q_{2^{n+2}}$ are $\{1\}, \{x^{2^n}\}, \{x^{2k}y\}_{1 \le k \le 2^n}, \{x^{2k-1}y\}_{1 \le k \le 2^n}$ and $\{x^l, x^{-l}\}$, where $1 \le l \le 2^n - 1$. Then we have

$$\#\{1\} = \#\{x^{2^n}\} = 1, \#\{x^{2k}y\}_{1 \le k \le 2^n} = \#\{x^{2k-1}y\}_{1 \le k \le 2^n} = 2^n \text{ and } \#\{x^l, x^{-l}\} = 2 \,.$$

Moreover, the commutator subgroup of $Q_{2^{n+2}}$ is the cyclic group $\langle x^2 \rangle$ of order 2^n. Combining these facts, we have

$$d_G = \text{l.c.m}\{1, 2, 2^n\} = 2^n \,.$$

LEMMA 5.5.   *Let K/k be a finite Galois CM-extension of number fields whose Galois group is isomorphic to $Q_{2^{n+2}}$ and let S be a finite set of places of k which contains S_{ram} and S_∞. We set $H := \langle x^2 \rangle$. Then*

$$d_G w_K \theta_{K/k,S}^{>1} = \left(\sum_{\chi \in \text{Irr}^- G, \, \chi(1)=2} \text{pr}_\chi \right) \frac{1}{2} \theta_{K/K^H, S_H} \,.$$

PROOF.   Regarding d_G as an element in $\mathbf{C}[G]$, we see that

$$d_G = 2^n = \sum_{\chi \in \text{Irr } G} 2^n e_\chi$$

$$= \sum_{\chi \in \text{Irr } G, \chi(1)=1} 2^n \left(\frac{1}{2^{n+2}} \text{pr}_\chi \right) + \sum_{\chi \in \text{Irr } G, \chi(1)=2} 2^n \left(\frac{1}{2^{n+1}} \text{pr}_\chi \right)$$

$$= \sum_{\chi \in \text{Irr } G, \chi(1)=1} \frac{1}{2^2} \text{pr}_\chi + \sum_{\chi \in \text{Irr } G, \chi(1)=2} \frac{1}{2} \text{pr}_\chi \,.$$

Then by (2), we have

$$d_G w_K \theta_{K/k,S}^{>1} = \left(\sum_{\chi \in \text{Irr } G, \chi(1)=1} \frac{1}{2^2} \text{pr}_\chi + \sum_{\chi \in \text{Irr } G, \chi(1)=2} \frac{1}{2} \text{pr}_\chi \right) w_K \left(1 - \frac{1}{s_G} N_{[G,G]} \right) \theta_{K/K^H, S_H}$$

$$= \left(\sum_{\chi \in \text{Irr } G, \chi(1)=2} \frac{1}{2} \text{pr}_\chi \right) w_K \theta_{K/K^H, S_H}$$

$$= \left(\sum_{\chi \in \text{Irr}^- G, \chi(1)=2} \frac{1}{2} \text{pr}_\chi \right) w_K \theta_{K/K^H, S_H}$$

$$= \left(\sum_{\chi \in \text{Irr}^- G, \chi(1)=2} \text{pr}_\chi \right) w_K \frac{1}{2} \theta_{K/K^H, S_H} \,.$$

$\square$

5.2.2. Main result.   In this section we prove Theorem 5.8. We first prove the following, which is a slight refinement of [11, Proposition 5.8].

PROPOSITION 5.6. *Take any element x in $\mathfrak{F}(Q_{2^{n+2}})$ of the form $x = \sum_{\chi \in \mathrm{Irr}^- G, \chi(1)=2} x_\chi \, \mathrm{pr}_\chi$. Then for each fractional ideal $\mathfrak{A}$ of K whose class in $Cl(K)$ has 2-power order,*

- $\mathfrak{A}^{x w_K \frac{1}{2} \theta_{K/K^H}} = (\alpha)$ *for some α in K°,*
- $K(\alpha^{1/w_{K,2}}) = K$,

where we set $H = \langle x^2 \rangle$.

REMARK 5.7. The original statement of [11, Proposition 5.8] is

- $\mathfrak{A}^{x w_K \theta_{K/K^H}} = (\alpha)$ for some α in K°,
- $K(\alpha^{1/w_{K,2}})/K^H$ is abelian.

Note that the factor $\frac{1}{2}$ before θ_{K/K^H} does not exist in the original claim.

PROOF. We set $I_K^+ := \{\mathfrak{A} \mid \mathfrak{A} \text{ is an ideal of } K \text{ and } \mathfrak{A}^j = \mathfrak{A}\}$ and $A_K := \mathrm{Coker}(I_K^+ \to Cl(K)) \otimes \mathbf{Z}_2$. We write ϕ for a generator of the group of characters of $\langle x \rangle$. Then a key point of the proof of [11, Proposition 5.8] is to prove the following inequality, which is (5.8) in [11]:

$$|A_K| \leq \frac{4}{2^{2^n}} N_{\mathbf{Q}(\zeta_{2^{n+1}})/\mathbf{Q}} \left(\frac{L(K/K^H, \phi, 0)}{2} \right)$$
$$\leq N_{\mathbf{Q}(\zeta_{2^{n+1}})/\mathbf{Q}} \left(\frac{L(K/K^H, \phi, 0)}{2} \right).$$

However, in fact, the first line of the above inequalities implies the following stronger inequality:

$$|A_K| \leq N_{\mathbf{Q}(\zeta_{2^{n+1}})/\mathbf{Q}} \left(\frac{L(K/K^H, \phi, 0)}{4} \right).$$

Then the same argument of the proof of [11, Proposition 5.8] implies that for each fractional ideal $\mathfrak{A}$,

$$\mathfrak{A}^{\frac{1}{2} x \theta_{K/K^H}} = (\alpha')$$

for some anti-unit α'. Hence we have

$$\mathfrak{A}^{\frac{1}{2} x w_{K,2} \theta_{K/K^H}} = (\alpha'^{w_{K,2}}).$$

We set $\alpha := \alpha'^{w_{K,2}}$. Then we get the result as in Proposition 5.6. $\qquad \square$

Using this proposition, we prove the following, which is our main result in this section.

THEOREM 5.8. *Let K/k be a finite Galois CM-extension whose Galois group is isomorphic to $Q_{2^{n+2}}$ and let S be a finite set of places of k which contains S_{ram} and S_∞. Then $BS_{\mathrm{Gal}}(K/k, S, 2)$ is true.*

PROOF.   Since $H' = \langle x \rangle$ is an abelian normal subgroup of G with $[G : H'] = 2$, Conjecture 3.1 is true by Theorem 3.3. Therefore, by Theorem 4.4, we have only to show that $BS(K^{\mathrm{ab}}/k, S, 2)$ is true and

$$\mathfrak{A}^{d_G w_K \theta^{>1}_{K/k,S}} = (\alpha) \ \text{ for some } \ \alpha \in K^\circ \ \text{ and } \ K(\alpha^{1/w_{K,2}})/k \ \text{ is strong central.} \qquad (4)$$

Since $\mathrm{Gal}(K^{\mathrm{ab}}/k)$ is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, $BS(K^{\mathrm{ab}}/k, S, 2)$ is true by [12, Theorem 2.1]. By Lemma 5.5, we have

$$d_G w_K \theta^{>1}_{K/k,S} = \left(\sum_{\chi \in \mathrm{Irr}^- G, \ \chi(1)=2} \mathrm{pr}_\chi \right) \frac{1}{2} \theta_{K/K^H}, S_H \ .$$

Since the element $\sum_{\chi \in \mathrm{Irr}^- G, \ \chi(1)=2} \mathrm{pr}_\chi$ is a central conductor as in Proposition 5.6, we obtain the statement (4) by Proposition 5.6. This completes the proof.   □

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