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Positive Solutions for a Quasilinear Elliptic Problem Involving Sublinear and Superlinear Terms

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Abstract. We deal with the existence and non-existence of positive solutions for the problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = a(x)f(u) + \lambda b(x)g(u) \ \text{ in } \mathbf{R}^N, \\ \\ u > 0 \ \text{ in } \mathbf{R}^N, \quad u(x) \to 0 \text{ when } |x| \to \infty, \end{cases}$$

where Δ_p is the p-Laplacian operator, $1 < p < N$, $\lambda > 0$ is a real parameter, $f, g : (0, \infty) \to (0, \infty)$ and $m, a, b : \mathbf{R}^N \to [0, \infty)$; $a, b \neq 0$ are continuous functions. In this work we consider, for example, nonlinearities with combined effects of concave and convex terms, besides allowing the presence of singularities. For existence of solutions, we exploit the lower and upper solutions method, combined with a technique of monotone-regularization on the nonlinearities f and g and for non-existence we use a consequence of Picone identity.

1. Introduction

This paper deals with the existence and non-existence of solutions for the problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = a(x)f(u) + \lambda b(x)g(u) \ \text{ in } \mathbf{R}^N, \\ \\ u > 0 \ \text{ in } \mathbf{R}^N, \quad u(x) \to 0 \ \text{ when } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, is the p-Laplacian operator, $\lambda > 0$ is a real parameter, $f, g : (0, \infty) \to (0, \infty)$ and $m, a, b : \mathbf{R}^N \to [0, \infty)$; $a, b \neq 0$ are continuous functions.

By a solution of (1.1) we mean a function $u = u_\lambda \in C^1(\mathbf{R}^N)$ such that $u > 0$ in $\mathbf{R}^N$,

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with $u(x) \to 0$ when $|x| \to \infty$ and such that, for all $\phi \in C_0^\infty(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} [|\nabla u|^{p-2}\nabla u \nabla \phi + m(x)u^{p-1}\phi]dx = \int_{\mathbf{R}^N} [a(x)f(u) + \lambda b(x)g(u)]\phi dx\,.$$

In this paper, we say that an arbitrary function $h : (0, \infty) \to [0, \infty)$ is $(p-1)$-sublinear at 0 or at $+\infty$ respectively, if $\lim_{s\to 0} h(s)/s^{p-1} = \infty$ or $\lim_{s\to\infty} h(s)/s^{p-1} = 0$, $(p-1)$-superlinear at 0 or at $+\infty$ respectively, if $\lim_{s\to 0} h(s)/s^{p-1} = 0$ or $\lim_{s\to\infty} h(s)/s^{p-1} = \infty$ and $(p-1)$-asymptotically linear if there is a positive and finite number that corresponds to the values of these limits. In particular, a nonlinearity of the concave-convex type is $(p-1)$-sublinear at 0 and $(p-1)$-superlinear at infinity. Moreover, h is called singular at 0 if $\lim_{s\to 0^+} h(s) = \infty$.

Problems like (1.1) have been studied intensively in recent years including nonlinearities that behave like $(p-1)$-sublinear and $(p-1)$-superlinear at zero and/or infinity. Among others, in bounded domains we can cite [2, 14, 32] for the case $p = 2$, and [17, 15, 3] for the case $p \neq 2$. In general, there are no works in the literature dealing with p-Laplacian equations with singular terms and combined nonlinearities (i.e., with the combined effects of concave and convex terms). An exception is the recent work of Gasiński and Papageorgiou [19]. The propose of our work is to consider this type of nonlinearities.

We emphasize that our results do not require singularity of the functions f and g, but we are particularly interested in the case where f and g may have singularity at 0. For our readers' information, we note that problems including singular nonlinearities arise in various physical situations, present in electrical conductivity, in the theory of pseudoplastic fluids, in singular minimal surfaces, in reaction-diffusion processes, in obtaining various geophysical indexes and industrial processes, among others; see [6, 7, 16] for a detailed discussion.

From now on, we are going to denote the following

$$h^i := \lim_{s\to i} h(s)\,, \quad h_i := \lim_{s\to i} \frac{h(s)}{s^{p-1}} \quad \text{and} \quad h_{inf} = \inf_{s>0} \frac{h(s)}{s^{p-1}}\,,$$

for $i = 0$ or $i = \infty$, by $h^i, h_i, h_{inf} \in [0, \infty]$.

We define the function

$$\rho(x) := \min\{a(x), b(x)\}\,, \quad x \in \mathbf{R}^N\,, \tag{1.2}$$

and we suppose $\rho \neq 0$. Considering $\rho(x)$ restricted to a smooth bounded domain $\Omega \subset \mathbf{R}^N$, we denote by $\lambda_{1,\Omega}(m, \rho) > 0$ the first eigenvalue and by $\varphi_\Omega = \varphi_{1,\Omega} > 0$ the first eigenfunction of the problem

$$\begin{cases} -\Delta_p\varphi + m(x)\varphi^{p-1} = \lambda\rho(x)|\varphi|^{p-2}\varphi \ \text{ in } \Omega\,, \\ \qquad \varphi > 0 \ \text{ in } \Omega, \quad \varphi = 0 \ \text{ on } \partial\Omega\,. \end{cases} \tag{1.3}$$

Moreover, we are going to denote

$$\lambda_1(m, \rho) = \lim_{R\to\infty} \lambda_{1,B_R}(m, \rho) \geqslant 0\,,$$

where B_R is the ball centered at the origin of $\mathbf{R}^N$ with radius $R > 0$.

REMARK 1.1. The first eigenvalue of (1.3) is simple and positive. Its first associated eigenfunction, φ_Ω, is positive and belongs to $C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$ (see [31]).

We can show that if the number $\lambda_1(m, \rho)$ given above is equal to zero, there will be no solution to the problem (1.1).

THEOREM 1.1. *If* $\lambda_1(m, \rho) = 0$ *and* $f_i + g_i > 0$, $i = 0$ *and* $i = \infty$, *then there is no solution to the problem* (1.1), *for all* $\lambda > 0$. *In particular*, *if the terms a and b satisfy* $|x|^p a(x)$, $|x|^p b(x) \to \infty$ *when* $|x| \to \infty$ *and* $f_i + g_i > 0$, $i = 0$ *and* $i = \infty$, *then the problem*

$$\begin{cases} -\Delta_p u = a(x) f(u) + \lambda b(x) g(u) \quad \textit{in } \mathbf{R}^N , \\ \\ u > 0 \ \textit{ in } \mathbf{R}^N , \quad u(x) \to 0 \ \textit{ when } |x| \to \infty , \end{cases}$$

has no solution for all $\lambda > 0$.

The following conditions will be required in our results:

(M) there exists a solution $\omega_M \in C^1(\mathbf{R}^N)$ of

$$\begin{cases} -\Delta_p u + m(x) u^{p-1} = M(x) \ \text{ in } \mathbf{R}^N , \\ \\ u > 0 \ \text{ in } \mathbf{R}^N , \quad u(x) \to 0 \ \text{ when } |x| \to \infty , \end{cases} \tag{1.4}$$

where $M(x) := \max\{a(x), b(x)\}$, $x \in \mathbf{R}^N$;

(F) (F_0) $f_0 < 1/\|\omega_M\|^{p-1}_{L^\infty(\mathbf{R}^N)}$ or (F_∞) $f_\infty < 1/\|\omega_M\|^{p-1}_{L^\infty(\mathbf{R}^N)}$.

From now on, we are going to denote $\|\omega_M\|_{L^\infty(\mathbf{R}^N)}$ by $\|\omega_M\|_\infty$.

REMARK 1.2. We know that the number $\lambda_1(m, \rho)$ is non-negative and, if it is equal to zero, by Theorem 1.1 we have non-existence of solution for (1.1). Now, if we assume (M), then we have

$$\lambda_1(m, \rho) \geqslant \|\omega_M\|_\infty^{1-p} > 0 . \tag{1.5}$$

For details, see Lemma 2.2 in the next section.

REMARK 1.3. Conditions for the existence of solutions to problem (1.4) are considered in the next section. See Lemma 2.1.

Now, to state our main result, we let

$$\lambda_* := \begin{cases} 0, \ \text{ if } \ g_0 = 0 \quad \text{and} \quad f_0 > \lambda_1(m, \rho) , \\ \max\left\{0, \frac{\lambda_1(m,\rho) - f_0}{g_0}\right\} , \quad \text{if } \ 0 < g_0 < \infty , \\ 0 , \quad \text{if } \ g_0 = \infty . \end{cases} \tag{1.6}$$

Note that, with this definition, we are assuming $f_0 + g_0 > 0$.

THEOREM 1.2. *Assume that* (M) *and* (F) *hold. Then there exists* $0 < \lambda^* \leqslant \infty$ *such that problem* (1.1) *has*:

(a) *a solution for each* $\lambda_* < \lambda < \lambda^*$,
(b) *no solution if* $\lambda > \lambda^*$.

In addition,

$$\lambda^* \geqslant \max\left\{\frac{1}{g_0}\left(\frac{1}{\|\omega_M\|_\infty^{p-1}} - f_0\right), \frac{1}{g_\infty}\left(\frac{1}{\|\omega_M\|_\infty^{p-1}} - f_\infty\right)\right\}$$

and

$$f_{inf} + \lambda^* g_{inf} \leqslant \lambda_1(m, \rho)\,.$$

Moreover, *if*

$$(c)\ \ g_i, g_j > 0 \quad or \quad (d)\ \ g_i > 0,\, f_j > \lambda_1(m, \rho)$$

holds for $i, j \in \{0, \infty\},\ i \neq j$, *then* $\lambda^* < \infty$.

REMARK 1.4. About Theorem 1.2, it is important to note that

(i) If $f_0 > \lambda_1(m, \rho)$, the condition (F_0) cannot occur, due to (1.5).
(ii) Since in this result $0 \leqslant g_0 \leqslant \infty$ is permitted, it is allowed that our nonlinearities admit mixed behavior (such as concave-convex and/or singularities).

Now we present some tables that can contribute to a better understanding of our results. Here, we are considering $\lambda^* \geqslant 0$ as a divisor point, in the sense that the problem (1.1) has a solution before it but none afterwards (when it is possible). Under the assumptions of Theorem 1.2 and remembering that by (1.5) $\lambda_1(m, \rho) \geqslant \|\omega_M\|_\infty^{1-p}$, we have:

(I) If (F_0) holds, then:

<table>
<tr><td>$g_0 = 0$</td><td colspan="4">$\lambda^* = \infty$</td></tr>
<tr><td rowspan="4">$g_0 > 0$</td><td>$g_\infty > 0$</td><td colspan="3">$0 < \lambda^* < \infty$</td></tr>
<tr><td rowspan="3">$g_\infty = 0$</td><td>$f_\infty > \lambda_1(m, \rho)$</td><td colspan="2">$0 < \lambda^* < \infty$</td></tr>
<tr><td rowspan="2">$f_\infty \leq \lambda_1(m, \rho)$</td><td>$f_\infty < \|\omega_M\|_\infty^{1-p}$</td><td>$\lambda^* = \infty$</td></tr>
<tr><td>$\|\omega_M\|_\infty^{1-p} \leq f_\infty \leq \lambda_1(m, \rho)$</td><td>no available</td></tr>
</table>

(II) If (F_∞) holds, then:

<table>
<tr><td>$g_\infty = 0$</td><td colspan="4">$\lambda^* = \infty$</td></tr>
<tr><td rowspan="4">$g_\infty > 0$</td><td>$g_0 > 0$</td><td colspan="3">$0 < \lambda^* < \infty$</td></tr>
<tr><td rowspan="3">$g_0 = 0$</td><td>$f_0 > \lambda_1(m, \rho)$</td><td colspan="2">$0 < \lambda^* < \infty$</td></tr>
<tr><td rowspan="2">$f_0 \leq \lambda_1(m, \rho)$</td><td>$f_0 < \|\omega_M\|_\infty^{1-p}$</td><td>$\lambda^* = \infty$</td></tr>
<tr><td>$\|\omega_M\|_\infty^{1-p} \leq f_0 \leq \lambda_1(m, \rho)$</td><td>no available</td></tr>
</table>

(III) Concerning the classical problem

$$\begin{cases} -\Delta_p u = a(x)u^n + \lambda b(x)u^m \text{ in } \mathbf{R}^N , \\ u > 0 \text{ in } \mathbf{R}^N , \quad u(x) \to 0 \text{ when } |x| \to \infty , \end{cases} \tag{1.7}$$

which includes concave-convex nonlinearities, we have:

(III_1) If $|x|^p a(x)$, $|x|^p b(x) \to \infty$ when $|x| \to \infty$ and $-\infty < m \leq p-1 \leq n$ or $-\infty < n \leq p-1 \leq m$, then there is no solution to (1.7), for all $\lambda > 0$.

(III_2) If a, b are functions such that hypothesis (M) is satisfied, we can observe that: if $-\infty < n < p-1$, then $f_0 = \infty$ and $f_\infty = 0$, which implies that (F_∞) holds. Now, if $n > p-1$, then $f_0 = 0$ and $f_\infty = \infty$, which implies that (F_0) holds.

We can conclude:

	$-\infty < n < p-1$	$n = p-1,\ \|\omega_M\|_\infty < 1$	$n > p-1$
$-\infty < m < p-1$	$\lambda^* = \infty$	$\lambda^* = \infty$	$0 < \lambda^* < \infty$
$m = p-1$	$0 < \lambda^* < \infty$	$0 < \lambda^* < \infty$	$0 < \lambda^* < \infty$
$m > p-1$	$0 < \lambda^* < \infty$	$\lambda^* = \infty$	no available

To motivate our work, we relate that problems like (1.1) have been studied intensively in recent years. In the semilinear case $p = 2$, we begin by citing the work of Lair and Shaker [25], who in 1996 determined the existence of a unique classical solution to the problem

$$\begin{cases} -\Delta u = a(x)u^{-\gamma} \text{ in } \mathbf{R}^N , \\ u > 0 \text{ in } \mathbf{R}^N , \quad u(x) \to 0 \text{ when } |x| \to \infty , \end{cases} \tag{1.8}$$

where $N \geqslant 3$, $\gamma > 0$, $a(x) > 0$ and

$$\int_0^\infty t\phi(t)dt < \infty , \quad \text{with } \phi(t) = \max_{|x|=t} a(x) . \tag{1.9}$$

In 1997, the same authors [26] extended this result considering the more general problem

$$\begin{cases} -\Delta u = a(x)f(u) \text{ in } \mathbf{R}^N, \\ u > 0 \text{ in } \mathbf{R}^N, \quad u(x) \to 0 \text{ when } |x| \to \infty, \end{cases} \tag{1.10}$$

where $a(x) \geqslant 0, a \neq 0$ satisfies the hypothesis (1.9) above, f is singular at $s = 0$ with $f(s) > 0$ if $s > 0$ and $f'(s) \leqslant 0$.

To insert these problems in the context of our work, in (1.8) we identified $m \equiv 0, \lambda = 0$ and $f(s) = s^{-\gamma}$, which implies that $f_0 = \infty$ and $f_\infty = 0$. With this, the hypothesis (F_∞) holds. Moreover, by [21] we can see that (1.9) implies that (M) holds. In a similar analysis for (1.10), we can see that Theorem 1.2 resolves both problems.

With the objective of studying the singular effect of the sign-changing of f' on the structure of the ground state solution set in (1.10), in 2003 Yijing and Shujie [35] considered nonlinearity as the sum of a singular and a sublinear term, thus establishing the existence of the classical solution to the problem

$$\begin{cases} -\Delta u = a(x)u^{-\gamma} + b(x)u^{\alpha} \text{ in } \mathbf{R}^N, \\ u > 0 \text{ in } \mathbf{R}^N, \ u(x) \to 0 \text{ when } |x| \to \infty, \end{cases}$$

where $0 < \gamma, \alpha < 1,\ a, b \in C^{\alpha}_{loc}(\mathbf{R}^N)$ are nonnegative functions such that $a(x) + b(x) \neq 0$ for all $x \in \mathbf{R}^N$, and they satisfy (1.9). In this case, by identifying $f(s) = s^{-\gamma}$ and $g(s) = s^{\alpha}$, we obtain $f_0 = g_0 = \infty$, $f_\infty = g_\infty = 0$, which implies that (F_∞) holds and $\lambda^* = \infty$. With this, the problem above is solved by Theorem 1.2, for all $\lambda > 0$.

We also cite the work of Ahmed Mohammed [30], who in 2009 established the existence of solution in $C^{2,\alpha}_{loc}(\mathbf{R}^N)$ to the problem

$$\begin{cases} -\Delta u = \gamma a(x)f(u) + \eta b(x)g(u) \text{ in } \mathbf{R}^N, \\ u > 0 \text{ in } \mathbf{R}^N, \ u(x) \to 0 \text{ when } |x| \to \infty, \end{cases}$$

where a, b are functions in $C^{\alpha}_{loc}(\mathbf{R}^N, [0, \infty))$, $a \neq 0$, η, γ are positive real parameters and f, g in $C^1((0, \infty), (0, \infty))$ satisfy g is bounded near 0, $f(t)/t$ is bounded on $[\epsilon, \infty)$, for every $\epsilon > 0$ and $\lim_{t \to 0^+} f(t)/t = \infty$. Furthermore, it was assumed that there is a positive solution to the problem

$$\begin{cases} -\Delta \omega = a(x) + b(x) \text{ in } \mathbf{R}^N, \\ \omega(x) \to 0, \text{ when } |x| \to \infty. \end{cases}$$

In this work, Mohammed generalized the results in [22] and [35], allowing far more general nonlinearities and weakening the conditions on a and b. Here, we have $f_0 = \infty,\ f_\infty < \infty$

and, by the conditions in g, we obtain $g_0 = \infty$ or $g_0 < \infty$. Since $f_\infty < \infty$, the hypothesis (F) occurs only to sufficiently small parameter γ.

In the quasilinear case $p \neq 2$, in 2004 Gonçalves and Santos [21] studied the problem (1.10) with $a \geqslant 0$ being continuous and radially symmetric and f singular at $s = 0$, $f(s)/s^{p-1}$ nonincreasing in $(0, \infty)$, $\liminf_{s\to 0} f(s) > 0$ and $f_\infty = 0$. Assuming $1 < p < N$ and

$$\begin{aligned} &\text{(i)} \quad \int_0^\infty r^{\frac{1}{p-1}} a(r)^{\frac{1}{p-1}}\, dr < \infty\,, \quad \text{if } 1 < p \leqslant 2 \quad \text{and} \\ &\text{(ii)} \quad \int_0^\infty r^{\frac{(p-2)N+1}{p-1}} a(r)\, dr < \infty\,, \quad \text{if } 2 \leqslant p < \infty\,, \end{aligned} \tag{1.11}$$

the authors used fixed point arguments, the shooting method and a lower-upper solutions argument to determine the existence and non-existence of radially symmetric solutions to the problem. In 2005, Covei [11] extended their results to the case where a is nonradial but positive and locally Hölder continuous. Replacing $a(r)$ in (1.11) by $\max_{|x|=r} a(x)$, he showed the existence of solution to the same problem.

In these works, we can identify $\lambda = 0$, $f_0 = \infty$ and $f_\infty = 0$, which show that hypothesis (F_∞) holds. Since (1.11) with the cited replacement implies that (M) holds (see [21]), Theorem 1.2 resolves these problems.

Recently, using the concept of generalized solutions that are not subject to any decay at infinity, Carl and Perera [8] used perturbation and comparison arguments as well as regularity results for the p-Laplacian to obtain solutions to the singular p-Laplacian problem. For other works, we refer the reader to [9, 10, 29, 33] and their references.

2. Preliminaries

In this section, our first result has dealt with the existence of solutions of Problem (1.4). That is, we provide a situation whose hypothesis (M) holds.

LEMMA 2.1. *Let $m : \mathbf{R}^N \to [0, \infty)$ a continuous function and M be as given in (M). Moreover, suppose that*

$$\int_0^\infty \left[s^{1-N} \int_0^s t^{N-1} \widehat{M}(t) dt \right]^{\frac{1}{p-1}} ds < \infty$$

holds, where $\widehat{M}(t) = \sup_{|x|=t} M(x)$, $t > 0$ and $1 < p < N$. Then there exists at least one $u \in C^1(\mathbf{R}^N)$ solution of (1.4).

PROOF. First, consider the problem

$$\begin{cases} -(r^{N-1}|v'(r)|^{p-2}v'(r))' = r^{N-1}\widehat{M}(r)\,, & r > 0\,, \\ v > 0 \in \mathbf{R}^N\,, \quad v(r) \to 0 \text{ when } r \to \infty\,. \end{cases} \tag{2.1}$$

Defining

$$v(r) = \int_r^\infty \left[s^{1-N} \int_0^s t^{N-1} \widehat{M}(t) dt \right]^{\frac{1}{p-1}} ds\,,$$

we can show that $v \in C^1(0,\infty)$ is a solution of the problem (2.1).

Now, set $w(x) = v(|x|)$. Since $m \geqslant 0$, we obtain that $w \in C^1(\mathbf{R}^N)$ is an upper solution of (1.4), i.e.,

$$\int_{\mathbf{R}^N} [|\nabla w|^{p-2} \nabla w \nabla \phi + m(x) w^{p-1} \phi] \geqslant \int_{\mathbf{R}^N} M(x)\phi, \;\; \phi \in C_0^\infty(\mathbf{R}^N).$$

Now, consider the following problem

$$\begin{cases} -\Delta_p u + m(x) u^{p-1} = M(x) \text{ in } B_R, \\ \\ u > 0 \text{ in } B_R, \; u(x) = 0 \text{ on } \partial B_R, \end{cases} \tag{2.2}$$

where B_R is the ball whose center is the origin of $\mathbf{R}^N$ and which has radius R.

By Pezzo and Bonder [5] and using the regularity results of Lieberman [28], we have that there exists $u_R \in C^{1,\beta}(\overline{B_R})$, $0 < \beta < 1$, satisfying (2.2). Consider the function $\underline{u}_R$ defined by $\underline{u}_R(x) = u_R(x)$, if $x \in B_R$, and $\underline{u}_R(x) = 0$, if $|x| \geqslant R$. We claim that

$$\underline{u}_1(x) \leqslant \underline{u}_2(x) \leqslant \cdots \leqslant \underline{u}_R(x) \leqslant \underline{u}_{R+1}(x) \leqslant \cdots \leqslant w(x)\,, \quad x \in \mathbf{R}^N\,. \tag{2.3}$$

Indeed, for each $R \geqslant 1$, note that both $\underline{u}_R$ and $\underline{u}_{R+1}$ satisfy the equation given in (2.2) in the ball B_R. Besides this, $\underline{u}_R(x) = 0 < \underline{u}_{R+1}(x)$, for $x \in \partial B_R$. Then, by the Comparison Principle of Tolksdorf [34] we have $\underline{u}_R \leqslant \underline{u}_{R+1}$in B_R. Moreover, $\underline{u}_R = 0 < \underline{u}_{R+1}$ in $B_{R+1} \backslash B_R$. For the last part, note that $\underline{u}_R(x) = 0 < w(x)$, $x \in B_R$. So, again by Tolksdorf [34], $\underline{u}_R \leqslant w$ in B_R for each $R \geqslant 1$. This completes the proof of statement (2.3).

Finally, letting $u(x) = \lim\limits_{R\to\infty} \underline{u}_R(x)$ and using compactness arguments we have that $u \in C^1(\mathbf{R}^N)$. Furthermore, it satisfies (1.4). □

Our next result will be necessary to prove that λ^*, announced in Theorem 1.2, is finite. Moreover, the inequality (1.5) also is showed. Its proof uses Picone's identity [1] and density arguments.

LEMMA 2.2. *Assume that given* $\lambda \in \mathbf{R}$ *there exists a* $0 < v = v_\lambda \in W^{1,p}_{loc}(\mathbf{R}^N)$ *satisfying* $-\Delta_p v + m v^{p-1} \geqslant \lambda \rho v^{p-1}$ *in the distributional sense, where* $m : \mathbf{R}^N \to [0,\infty)$ *is a continuous function and* ρ *is given by* (1.2). *Then* $\lambda \leqslant \lambda_1(m,\rho)$. *In particular, if* (M) *holds, then* $\lambda_1(m,\rho) \geqslant \|\omega_M\|_\infty^{1-p}$.

PROOF. Firstly, for each $R \geqslant 1$, we consider the functions v and ω_M restricted to the ball B_R. They will be denoted by $(v)_R$ and $(\omega_M)_R$.

Now, we pick $\{\phi_n\}_{n\in\mathbf{N}} \subset C_0^\infty(B_R)$ with $\phi_n \geqslant 0$ and $\phi_n \to \varphi_{B_R}$ in $W_0^{1,p}(B_R)$, where $\varphi_{B_R} > 0$ is the first eigenfunction of the problem (1.3) associated to its first eigenvalue $\lambda_{1,B_R}(m,\rho)$. So, applying Picone's identity (see [1]) and density arguments, we have

$$0 \leqslant \int_{B_R} |\nabla\phi_n|^p - \int_{B_R} |\nabla(v)_R|^{p-2}\nabla(v)_R\nabla\left(\frac{\phi_n^p}{(v)_R^{p-1}}\right)$$

$$\leqslant \int_{B_R} |\nabla\phi_n|^p - \int_{B_R} \lambda\rho(x)\phi_n^p + \int_{B_R} m(x)\phi_n^p .$$

Now, making $n \to \infty$, we obtain

$$\lambda_{1,B_R}(m,\rho)\int_{B_R} \rho(x)\varphi_{B_R}^p = \int_{B_R} [|\nabla\varphi_{B_R}|^p + m(x)\varphi_{B_R}^p] \geqslant \lambda \int_{B_R} \rho(x)\varphi_{B_R}^p ,$$

that is, $\lambda \leqslant \lambda_{1,B_R}(m,\rho)$ because ρ is non-negative and not identically zero. When $R \to \infty$, we get $\lambda \leqslant \lambda_1(m,\rho)$.

To finish our proof, we define for each $\tau > 0$, $v(x) = v_\tau(x) = \tau\|(\omega_M)_R\|_{L^\infty(B_R)}^{-1}(\omega_M)_R(x)$, $x \in B_R$. So, we have that $0 < v \leqslant \tau$ and

$$\begin{aligned}
&\int_{B_R} |\nabla v|^{p-2}\nabla v\nabla\phi + \int_\Omega m(x)v^{p-1}\phi \\
&\quad= \frac{\tau^{p-1}}{\|(\omega_M)_R\|_{L^\infty(B_R)}^{p-1}}\left[\int_{B_R} |\nabla(\omega_M)_R|^{p-2}\nabla(\omega_M)_R\nabla\phi + \int_\Omega m(x)(\omega_M)_R^{p-1}\phi\right] \\
&\quad= \frac{1}{\|(\omega_M)_R\|_{L^\infty(B_R)}^{p-1}}\int_{B_R} M(x)\tau^{p-1}\phi \geqslant \frac{1}{\|(\omega_M)_R\|_{L^\infty(B_R)}^{p-1}}\int_{B_R} \rho(x)v^{p-1}\phi ,
\end{aligned}$$

for all $\phi \in C_0^\infty(B_R)$, $\phi \geqslant 0$. Applying the first part of the lemma, this shows that

$$\lambda_{1,B_R}(m,\rho) \geqslant \frac{1}{\|(\omega_M)_R\|_{L^\infty(B_R)}^{p-1}}, \quad \text{for each} \quad R \geqslant 1 .$$

As we have that $\|(\omega_M)_R\|_{L^\infty(B_R)} \leqslant \|\omega_M\|_\infty$, when $R \to \infty$ we obtain

$$\lambda_1(m,\rho) \geqslant \frac{1}{\|\omega_M\|_\infty^{p-1}},$$

which ends the proof. □

Now, we consider the equation

$$-\Delta_p u + m(x)u^{p-1} = h(x,u), \quad u > 0 \text{ in } \mathbf{R}^N , \tag{2.4}$$

where $h : \mathbf{R}^N \times (0,\infty) \to \mathbf{R}$ is a continuous function.

A function $u \in C^1(\mathbf{R}^N)$ is called a solution to (2.4) in the distributional sense, if $u > 0$ in $\mathbf{R}^N$ and, for all $\phi \in C_0^\infty(\mathbf{R}^N)$, we have:

$$\int_{\mathbf{R}^N} [|\nabla u|^{p-2}\nabla u \nabla\phi + m(x)u^{p-1}\phi]dx = \int_{\mathbf{R}^N} h(x,u)\phi dx.$$

Moreover, a function $u \in C^1(\mathbf{R}^N)$ is called a lower solution (upper solution) to (2.4) in the distributional sense, if $u > 0$ in $\mathbf{R}^N$ and, for all nonnegative functions $\phi \in C_0^\infty(\mathbf{R}^N)$, holds

$$\int_{\mathbf{R}^N} [|\nabla u|^{p-2}\nabla u \nabla\phi + m(x)u^{p-1}\phi]dx \leqslant (\geqslant) \int_{\mathbf{R}^N} h(x,u)\phi dx\,. \tag{2.5}$$

In this sense, we prove the following result for possibly singular problems in $\mathbf{R}^N$, which will be used in the proof of Theorem 1.2:

THEOREM 2.1. *Suppose that there exist a lower solution $\underline{u}$ and an upper solution $\overline{u}$ of* (2.4) *such that $\underline{u} \leqslant \overline{u}$ in $\mathbf{R}^N$. Then Problem* (2.4) *has a solution $u \in [\underline{u}, \overline{u}]$.*

PROOF (Based on arguments of [27]). We know that $\mathbf{R}^N = \cup_{R=1}^\infty B_R$, where B_R is the ball centered at the origin of the $\mathbf{R}^N$ with radius R. For each $R \geqslant 1$, we consider the family of problems

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = h(x,u) \text{ in } B_R\,, \\ u = \underline{u} \text{ on } \partial B_R, \end{cases} \tag{2.6}$$

where h is the function given by (2.4) restricted to $B_R \times (0,\infty)$ and $\underline{u}$ is the lower solution defined in (2.5) restricted to B_R.

Since that $\underline{u}, \overline{u} \in C^1(\mathbf{R}^N)$ and $\underline{u}, \overline{u} > 0$ in $\mathbf{R}^N$, the numbers

$$\overline{a}_R = \max_{\overline{B}_R} \overline{u} \quad \text{and} \quad \underline{a}_R = \min_{\overline{B}_R} \underline{u}$$

are positive.

Now, we define the function $\tilde{h} : B_R \times \mathbf{R} \to \mathbf{R}$ by

$$\tilde{h}(x,s) := \begin{cases} h(x,\underline{a}_R)\,, & \text{if } s < \underline{a}_R \\ h(x,\overline{a}_R)\,, & \text{if } s > \overline{a}_R \\ h(x,s)\,, & \text{if } \underline{a}_R \leqslant s \leqslant \overline{a}_R\,. \end{cases} \tag{2.7}$$

It follows from of the continuity of the function h and of the definition (2.7) that $\tilde{h}$ is a bounded function in $B_R \times \mathbf{R}$. Then, there is a function $K \in L^{p'}(B_R)$ such that

$$|\tilde{h}(x,s)| \leqslant |K(x)|\,, \quad \text{a.e. } x \in B_R\,.$$

Now, since m is a continuous function in $\mathbf{R}^N$ and $s \in \mathbf{R}$, we have

$$|\tilde{h}(x,s) - m(x)s^{p-1}| \leqslant |K(x)| + r(|s|)\,, \quad \text{a.e. } x \in B_R\,,$$

for all $s > 0$, where $r(s) = \|m\|_{L^\infty(B_R)} s^{p-1}$, $s > 0$ is a nondecreasing function such that $r(|\varphi|) \in L^{p'}(B_R)$, for $\varphi \in L^p(B_R)$.

Besides this, we have that $\underline{u}$ and $\overline{u}$ are, respectively, lower solution and upper solution (in the sense of [24]) of the problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = \tilde{h}(x,u) \text{ in } B_R\,, \\ u = \underline{u} \text{ on } \partial B_R\,. \end{cases} \tag{2.8}$$

With these, we can apply a theorem of lower and upper solution by [24] to conclude that the problem (2.8) (and also (2.6)) has a solution u_R such that $\underline{u} \leqslant u_R \leqslant \overline{u}$ a.e. in B_R.

Now, using a diagonal argument on R and a standard regularity theory, we get a $u \in C^1(\mathbf{R}^N)$ such that $\underline{u} \leqslant u \leqslant \overline{u},\ u > 0$ in $\mathbf{R}^N$ and, for each $\phi \in C_0^\infty(\mathbf{R}^N)$, we have

$$\int_{\mathbf{R}^N} [|\nabla u|^{p-2}\nabla u \nabla\phi + m(x)u^{p-1}\phi]dx = \int_{\mathbf{R}^N} h(x,u)\phi dx\,.$$

The proof of Theorem 2.1 is finished. □

3. Proof of Theorem 1.1

PROOF. Suppose, by contradiction, that there exists a $\lambda > 0$ such that problem (1.1) has solution $u = u_\lambda$.

With this, we claim that there exists $c = c_\lambda > 0$ such that

$$\eta_\lambda(s) := \frac{f(s)}{s^{p-1}} + \lambda\frac{g(s)}{s^{p-1}} \geqslant c\,, \quad \text{for all } s > 0\,. \tag{3.1}$$

Note that the hypothesis $f_i + g_i > 0,\ i = 0$ and $i = \infty$, ensures the validity of (3.1), since it implies that $\eta_\lambda(s) > 0,$ for all $s > 0$.

Then, using (3.1) and the fact that $\rho \leqslant a, b$, we have

$$\begin{aligned} -\Delta_p u + m(x)u^{p-1} - c\rho(x)u^{p-1} &\geqslant -\Delta_p u + m(x)u^{p-1} - \eta_\lambda(u)\rho(x)u^{p-1} \\ &= -\Delta_p u + m(x)u^{p-1} - \left[\frac{f(u)}{u^{p-1}} + \lambda\frac{g(u)}{u^{p-1}}\right]\rho(x)u^{p-1} \\ &\geqslant -\Delta_p u + m(x)u^{p-1} - [a(x)f(u) + \lambda b(x)g(u)] = 0\,. \end{aligned}$$

By Lemma 2.2 and by hypothesis,

$$0 < c \leqslant \lambda_1(m,\rho) = 0\,,$$

what is an absurd. Therefore, problem (1.1) has no solution for any $\lambda > 0$.

Now, we will prove the particular case. Note that $|x|^p a(x), |x|^p b(x) \to \infty$ when $|x| \to \infty$ implies in $|x|^p\rho(x) \to \infty$ when $|x| \to \infty$, since $\rho(x) := \min\{a(x), b(x)\}$. We want to show that $\lambda_1(0,\rho) = 0$.

Initially, we claim that the assumption $|x|^p\rho(x) \to \infty$ when $|x| \to \infty$ implies in $|x|^p\hat{\rho}(|x|) \to \infty$ when $|x| \to \infty$, where $\hat{\rho}(|x|) = \hat{\rho}(r) := \min_{|x|=r}\rho(x), r > 0$. Indeed, suppose that there exists $r_n \to \infty$ such that $\liminf r_n^p\hat{\rho}(r_n) < \infty$. We can see that $\hat{\rho}(r_n) = \rho(x_n)$, with $|x_n| = r_n$. So, $\lim |x_n|^p\rho(x_n) = \lim r_n^p\hat{\rho}(r_n) < \infty$, when $n \to \infty$, what is an absurd, since $|x_n|^p\rho(x_n) \to \infty$.

Now we will prove that $\lambda_1(0,\rho) = 0$. Suppose, by contradiction, $\lambda_1(0,\rho) > 0$. By [20], the inequality $\hat{\rho} \leqslant \rho$ implies in $\lambda_1(0,\hat{\rho}) \geqslant \lambda_1(0,\rho) > 0$. We take $\lambda := \lambda_1(0,\hat{\rho})$ and consider the radially symmetric solution $u \in C^2((0,\infty)) \cap C^1([0,\infty))$ of the equation

$$-\Delta_p u = \lambda\hat{\rho}(x)u^{p-1} \quad \text{in } \mathbf{R}^N,$$

or equivalently, the solution of the initial value problem

$$\begin{cases} -(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1}\hat{\rho}(r)u^{p-1}, & \text{in } (0,\infty) \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

Note that $u \neq 0$ in $\mathbf{R}^N$, since $u(0) = 1$. Moreover, $|x|^p\hat{\rho}(x) \to \infty$ when $|x| \to \infty$ implies that the problem above does not have positive entire solution (see [4]). So, there is $R > 0$ such that $u(R) = 0$. With this, u satisfies

$$\begin{cases} -\Delta_p u = \lambda\hat{\rho}(x)u^{p-1} \text{ in } B_R(0) \\ u > 0 \text{ in } B_R(0), \quad u = 0 \text{ on } \partial B_R(0). \end{cases}$$

By [20] we have

$$\lambda := \lambda_1(0,\hat{\rho}) = \lambda_{1,B_R(0)}(0,\hat{\rho}) > \lambda_1(0,\hat{\rho}).$$

This contradiction means $\lambda_1(0,\hat{\rho}) = 0$, which shows that also $\lambda_1(0,\rho) = 0$. □

4. Some Auxiliary Functions

One of our main purposes in this paper is to consider, in the Problem (1.1), not only $(p-1)$-sublinear terms f and g but also $(p-1)$-superlinear and $(p-1)$-asymptotically linear terms. To prove Theorem 1.2 with such nonlinearities, we improved a technique of regularization-motonicity used, among others, by Fen and Liu [13], Zhang [36] and Mohammed [29].

Observing that for none monotonicity will be required of our nonlinearities, we introduce a truncation of the terms f and g through a real parameter $\gamma > 0$ and, from it, we build some auxiliary functions which allow us to obtain not only the monotonicity but also the necessary regularity for the proof of our results. In this way, given $\gamma > 0$, we define the continuous functions

$$\zeta_{f,\gamma}(s) := \begin{cases} f(s), & \text{if } 0 < s \leqslant \gamma \\ I_f(\gamma)s^{p-1}, & \text{if } s \geqslant \gamma, \end{cases}$$

and

$$\zeta_{g,\gamma}(s) := \begin{cases} g(s), & \text{if } 0 < s \leqslant \gamma \\ I_g(\gamma)s^{p-1}, & \text{if } s \geqslant \gamma, \end{cases}$$

where

$$I_f(\gamma) := \frac{f(\gamma)}{\gamma^{p-1}} \quad \text{and} \quad I_g(\gamma) := \frac{g(\gamma)}{\gamma^{p-1}}.$$

For each $s > 0$, consider the continuous and monotonous functions

$$\hat{\zeta}_{f,\gamma}(s) := \sup\left\{\frac{\zeta_{f,\gamma}(t)}{t^{p-1}},\ t > s\right\}, \quad \hat{\zeta}_{g,\gamma}(s) := \sup\left\{\frac{\zeta_{g,\gamma}(t)}{t^{p-1}},\ t > s\right\} \tag{4.1}$$

and

$$\hat{\zeta}_{\lambda,\gamma}(s) = s^{p-1}\hat{\zeta}_{f,\gamma}(s) + \lambda s^{p-1}\hat{\zeta}_{g,\gamma}(s), \quad \text{for each } \lambda \geqslant 0. \tag{4.2}$$

It follows from the above definitions that

(i) $\dfrac{\hat{\zeta}_{\lambda,\gamma}(s)}{s^{p-1}}$ is non-increasing in $s > 0$; (ii) $\hat{\zeta}_{\lambda,\gamma}(s) \geqslant \zeta_{f,\gamma}(s) + \lambda\zeta_{g,\gamma}(s),\ s > 0$;

(iii) $\displaystyle\lim_{s\to\infty} \frac{\hat{\zeta}_{\lambda,\gamma}(s)}{s^{p-1}} = I_f(\gamma) + \lambda I_g(\gamma)$.

The function $\hat{\zeta}_{\lambda,\gamma}(s)/s^{p-1}$ already has monotonicity, but does not have the enough regularity. So, defining

$$H_{\lambda,\gamma}(s) = \frac{s^2}{\displaystyle\int_0^s \frac{t}{\hat{\zeta}_{\lambda,\gamma}(t)^{\frac{1}{p-1}}}dt}, \quad s > 0,$$

and using (i)–(iii) above, we have

LEMMA 4.1. *The function H satisfies*:

(i) $H_{\lambda,\gamma} \in C^1((0,\infty),(0,\infty))$; (ii) $\hat{\zeta}_{\lambda,\gamma}(s) \leqslant [H_{\lambda,\gamma}(s)]^{p-1},\ s > 0$;

(iii) $\dfrac{H_{\lambda,\gamma}(s)}{s}$ *is non-increasing in* $s > 0$; (iv) $\displaystyle\lim_{s\to\infty}\frac{H_{\lambda,\gamma}(s)}{s} = (I_f(\gamma)+\lambda I_g(\gamma))^{\frac{1}{p-1}}$.

Depending on $H_{\lambda,\gamma}$, we define the γ-function

$$\Gamma_\lambda(\gamma) = \frac{1}{\gamma}\int_0^\gamma \frac{t}{H_{\lambda,\gamma}(t)}dt,\ \gamma > 0.$$

Using the previously defined functions and their properties, we have

LEMMA 4.2. *Assume that (M) and (F) hold. Then*:

(i) $\lim_{\gamma\to\infty} \Gamma_\lambda(\gamma) = \dfrac{1}{(f_\infty + \lambda g_\infty)^{\frac{1}{p-1}}}$, *for each* $\lambda \geqslant 0$;

(ii) $\lim_{\gamma\to 0} \Gamma_\lambda(\gamma) = \dfrac{1}{(f_0 + \lambda g_0)^{\frac{1}{p-1}}}$, *for each* $\lambda \geqslant 0$;

(iii) Γ_λ *is decreasing in* $\lambda > 0$, *for each* $\gamma > 0$;

(iv) *there exists a* $\tilde{\gamma} > 0$ *such that* $\Gamma_0(\tilde{\gamma}) > \|\omega_M\|_\infty$.

In the appendix of this paper, we prove item (i). The proof of part (ii) is similar. The monotonicity of the function Γ_λ follows from the definitions of the functions involved, while (iv) is a consequence of the assumption (F) together with (i)–(ii).

By above lemma, we can define the nonempty set

$$\mathcal{A}_{\mathbf{R}^N} := \{\gamma \in (0,\infty) \;/\; \Gamma_0(\gamma) > \|\omega_M\|_\infty\}\,.$$

Now, as a consequence of $\lim_{\lambda\to\infty} \Gamma_\lambda(\gamma) = 0$, $\lim_{\lambda\to 0} \Gamma_\lambda(\gamma) = \Gamma_0(\gamma)$ for each $\gamma > 0$, and from the last lemma, the function $\Lambda^* : \mathcal{A}_{\mathbf{R}^N} \to (0,\infty)$ that associate, for each $\gamma \in \mathcal{A}_{\mathbf{R}^N}$, the unique positive real number $\Lambda^*(\gamma)$ such that

$$\Gamma_{\Lambda^*(\gamma)}(\gamma) = \|\omega_M\|_\infty\,, \tag{4.3}$$

is well defined.

Thus, we can define the positive number

$$\Lambda_S^* = \Lambda_{S,\mathbf{R}^N}^* := \sup\{\Lambda^*(\gamma)\;;\; \gamma \in \mathcal{A}_{\mathbf{R}^N}\}\,. \tag{4.4}$$

After these, we have that

LEMMA 4.3. *Suppose* (M) *and* (F) *hold. Then*

$$\Lambda_S^* \geqslant \max\left\{\frac{1}{g_0}\left(\frac{1}{\|\omega_M\|_\infty^{p-1}} - f_0\right), \frac{1}{g_\infty}\left(\frac{1}{\|\omega_M\|_\infty^{p-1}} - f_\infty\right)\right\} = \max\{\lambda^0, \lambda^\infty\}\,,$$

where $\lambda^i := \dfrac{1}{g_i}\left(\dfrac{1}{\|\omega_M\|_\infty^{p-1}} - f_i\right)$, $i \in \{0,\infty\}$.

PROOF. If (F_0) occurs and $g_0 < \infty$, then given $0 < \delta < \lambda^0$, it follows from Lemma 4.2(ii), that

$$\lim_{\gamma\to 0}(\Gamma_\delta(\gamma) - \|\omega_M\|_\infty) = \frac{1}{(f_0 + \delta g_0)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty > \frac{1}{(f_0 + \lambda^0 g_0)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty = 0\,.$$

Now, if (F_∞) occurs and $g_\infty < \infty$, using Lemma 4.2(i), we have

$$\lim_{\gamma\to \infty}(\Gamma_\delta(\gamma) - \|\omega_M\|_\infty) = \frac{1}{(f_\infty + \delta g_\infty)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty > \frac{1}{(f_\infty + \lambda^\infty g_\infty)^{\frac{1}{p-1}}} - \|\omega_M\|_\infty = 0\,,$$

for each $0 < \delta < \lambda^\infty$ given.

So, in both cases, there exists a $\gamma_0 = \gamma_0(\delta) > 0$ such that $\Gamma_\delta(\gamma_0) > \|\omega_M\|_\infty$. By Lemma 4.2(iii), we have that $\gamma_0 \in \mathcal{A}_{\mathbf{R}^N}$, since that $\Gamma_0(\gamma_0) > \Gamma_\delta(\gamma_0) > \|\omega_M\|_\infty$. From (4.3), $\Gamma_{\Lambda^*(\gamma_0)}(\gamma_0) = \|\omega_M\|_\infty$. Thus, we have $\Gamma_{\Lambda^*(\gamma_0)}(\gamma_0) < \Gamma_\delta(\gamma_0)$, which implies in $\Lambda^*(\gamma_0) > \delta$, by Lemma 4.2(iii) again. So, by the arbitrariness of δ, we have the claimed. Lemma 4.3 is proved. □

Now, for each $\gamma, \lambda > 0$, we define

$$\eta_\lambda(s) = \frac{1}{\gamma}\int_0^s \frac{t}{H_{\lambda,\gamma}(t)}dt\,, \quad s > 0\,. \tag{4.5}$$

By Lemma 4.2(iii), definition of $\mathcal{A}_{\mathbf{R}^N}$ and (4.3), it follows that

$$\eta_\lambda(\gamma) = \Gamma_\lambda(\gamma) > \Gamma_{\Lambda^*(\gamma)}(\gamma) = \|\omega_M\|_\infty\,, \tag{4.6}$$

for each $\gamma \in \mathcal{A}_{\mathbf{R}^N}$ and $0 < \lambda < \Lambda^*(\gamma)$ given.

Besides this, using the auxiliaries functions constructed in this section and their properties, we can check the following result.

LEMMA 4.4. *Suppose* (M) *and* (F) *hold. Then, for each* $0 < \lambda < \Lambda^*_S$ *given*:

(i) $[0, \|\omega_M\|_\infty] \subset Im(\eta_\lambda)$;
(ii) $\eta_\lambda \in C^2((0,\infty), Im(\eta_\lambda))$ *is increasing in* $s > 0$;
(iii) $\eta_\lambda^{-1} := \psi_\lambda \in C^2(Im(\eta_\lambda)\backslash\{0\}, (0,\infty))$ *is increasing in* $s > 0$;
(iv) $\psi_\lambda'(s) = \dfrac{\gamma H_{\lambda,\gamma}(\psi_\lambda(s))}{\psi_\lambda(s)}$, $s > 0$;
(v) $\psi_\lambda''(s) \leqslant 0$, $s > 0$;
(vi) η_λ *is decreasing in* λ.

5. An upper solution for the perturbed problem

Due to the possible of singularity under the functions f and g we construct, in the next result, a bounded positive upper solution for the perturbed problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = a(x)f(u+\epsilon) + \lambda b(x)g(u+\epsilon) \ \text{ in } \mathbf{R}^N \\ u > 0 \ \text{ in } \mathbf{R}^N\,, \quad u(x) \to 0 \ \text{ when } |x| \to \infty\,, \end{cases} \tag{5.1}$$

for each $\epsilon > 0$ small enough.

LEMMA 5.1. *Under the assumptions* (F) *and* (M), *given* $\epsilon > 0$ *small enough, there exists* $v = v_\lambda \in C^1(\mathbf{R}^N)$, *independent of* ϵ, *upper solution of* (5.1), *for each* $0 < \lambda < \Lambda^*_S$, *where* Λ^*_S *is defined in* (4.4).

PROOF. Given $0 < \lambda < \Lambda^*_S$, since $\omega_M \in C^1(\mathbf{R}^N)$ and $\psi_\lambda \in C^2(Im(\eta_\lambda)\backslash\{0\}, (0,\infty))$, we can define the function $v = v_\lambda \in C^1(\mathbf{R}^N)$ by

$$v(x) := \psi_\lambda(\omega_M(x))\,, \quad x \in \mathbf{R}^N\,. \tag{5.2}$$

We will show that v is an upper solution of (5.1). By (4.5), (4.6), Lemma 4.4 and (M), there is a $\gamma_0 = \gamma_0(\lambda) \in \mathcal{A}_{\mathbf{R}^N}$ such that $0 < v(x) < \gamma_0$, for all $x \in \mathbf{R}^N$, and $\lim_{|x|\to\infty} v(x) = 0$.

So, exists $\epsilon > 0$ small enough such that

$$\|v\|_{L^\infty(\mathbf{R}^N)} < \gamma_0 - \epsilon\,. \tag{5.3}$$

Now, given $\phi \in C_0^\infty(\mathbf{R}^N)$, $\phi \geqslant 0$, it follows from (5.2) that

$$\begin{aligned}
&\int_{\mathbf{R}^N} [|\nabla v|^{p-2}\nabla v \nabla\phi + m(x)v^{p-1}\phi]dx \\
&\quad = \int_{\mathbf{R}^N} [\psi_\lambda'(\omega_M)]^{p-1}|\nabla\omega_M|^{p-2}\nabla\omega_M\nabla\phi dx + \int_{\mathbf{R}^N} m(x)v^{p-1}\phi dx \\
&\quad = \int_{\mathbf{R}^N} |\nabla\omega_M|^{p-2}\nabla\omega_M\nabla([\psi_\lambda'(\omega_M)]^{p-1}\phi)dx \qquad (5.4)\\
&\qquad - (p-1)\int_{\mathbf{R}^N} |\nabla\omega_M|^p[\psi_\lambda'(\omega_M)]^{p-2}\psi_\lambda''(\omega_M)\phi dx + \int_{\mathbf{R}^N} m(x)[\psi_\lambda(\omega_M)]^{p-1}\phi dx
\end{aligned}$$

In the appendix of this paper, we prove that

$$\omega_M\psi_\lambda'(\omega_M) \leqslant \psi_\lambda(\omega_M)\,, \quad x \text{ in } \mathbf{R}^N\,. \tag{5.5}$$

Since $\psi_\lambda''(s) \leqslant 0$, $s > 0$, by (5.5), (1.4) and Lemma 4.4(iv), we can rewrite (5.4) as

$$\begin{aligned}
\int_{\mathbf{R}^N} |\nabla v|^{p-2}\nabla v\nabla\phi dx + \int_{\mathbf{R}^N} m(x)v^{p-1}\phi dx &\geqslant \int_{\mathbf{R}^N} M(x)[\psi_\lambda'(\omega_M)]^{p-1}\phi dx \\
&= \int_{\mathbf{R}^N} M(x)\gamma_0^{p-1}\left[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M)}\right]^{p-1}\phi dx\,. \qquad (5.6)
\end{aligned}$$

Now, it follows from the definitions, properties of the functions involved and (5.3), that

$$\begin{aligned}
\int_{\mathbf{R}^N} M(x)\gamma_0^{p-1}\left[\frac{H_{\lambda,\gamma_0}(\psi_\lambda(\omega_M))}{\psi_\lambda(\omega_M)}\right]^{p-1}\phi dx &\geqslant \int_{\mathbf{R}^N} M(x)\gamma_0^{p-1}\frac{\hat{\zeta}_{\lambda,\gamma_0}(v+\epsilon)}{(v+\epsilon)^{p-1}}\phi dx \\
&\geqslant \int_{\mathbf{R}^N} M(x)\gamma_0^{p-1}\frac{\hat{\zeta}_{\lambda,\gamma_0}(v+\epsilon)}{(\gamma_0)^{p-1}}\phi dx \\
&\geqslant \int_{\mathbf{R}^N} [a(x)f(v+\epsilon) + \lambda b(x)g(v+\epsilon)]\phi dx\,.
\end{aligned}$$

This ends the proof of proposition 5.1. □

6. Problem (1.1) in bounded domains

In this section, our objective is to study the problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = a(x)f(u) + \lambda b(x)g(u) \text{ in } \Omega\,, \\ u > 0 \text{ in } \Omega,\ \ u = 0 \text{ on } \partial\Omega\,, \end{cases} \tag{6.1}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain and we consider the same functions m, a, b and ρ previously given restricted to Ω.

As a particular case of a result in [5] and by [28], we have that there exists a unique $\omega = \omega_{M,\Omega} \in C^1(\overline{\Omega})$ solution of the problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = M(x) \text{ in } \Omega\,, \\ u > 0 \text{ in } \Omega, \;\; u = 0 \text{ on } \partial\Omega\,. \end{cases} \tag{6.2}$$

By a comparison principle in [18], we can conclude that $\omega_{M,\Omega} \leqslant \omega_M$, where ω_M is the function given in (1.4) restricted to Ω. With this, $\|\omega_{M,\Omega}\|_{L^\infty(\Omega)} \leqslant \|\omega_M\|_{L^\infty(\mathbf{R}^N)}$. So, (F_0) and (F_∞) imply in

$$f_0 < 1/\|\omega_{M,\Omega}\|^{p-1}_{L^\infty(\Omega)} \quad \text{and} \quad f_\infty < 1/\|\omega_{M,\Omega}\|^{p-1}_{L^\infty(\Omega)}\,,$$

respectively. Similarly as done before, we define

$$\lambda_{*,\Omega} := \begin{cases} 0\,, \;\; \text{if } g_0 = 0 \;\; \text{and} \;\; f_0 > \lambda_1(m,\rho)\,, \\ \max\left\{0, \frac{\lambda_{1,\Omega}(m,\rho) - f_0}{g_0}\right\}, \;\; \text{if } 0 < g_0 < \infty\,, \\ 0\,, \;\; \text{if } g_0 = \infty\,, \end{cases} \tag{6.3}$$

and we state our result:

THEOREM 6.1. *Assume that* (F) *holds. Then there exists* $0 < \lambda^*_\Omega \leqslant \infty$ *such that problem* (6.1) *has*:

(a) *a solution, for each* $\lambda_{*,\Omega} < \lambda < \lambda^*_\Omega$,
(b) *no solutions if* $\lambda > \lambda^*_\Omega$.

In addition,

$$\lambda^*_\Omega \geqslant \max\left\{\frac{1}{g_0}\left(\frac{1}{\|\omega_{M,\Omega}\|^{p-1}_{L^\infty(\Omega)}} - f_0\right), \frac{1}{g_\infty}\left(\frac{1}{\|\omega_{M,\Omega}\|^{p-1}_{L^\infty(\Omega)}} - f_\infty\right)\right\}.$$

PROOF. Analogously to the case done after Lemma 4.2, we can define the nonempty set $\mathcal{A}_\Omega = \{\gamma \in (0,\infty) \,/\, \Gamma_0(\gamma) > \|\omega_{M,\Omega}\|_{L^\infty(\Omega)}\}$, the function $\Lambda^* : \mathcal{A}_\Omega \to (0,\infty)$ such that $\Gamma_{\Lambda^*(\gamma)}(\gamma) = \|\omega_{M,\Omega}\|_{L^\infty(\Omega)}$ and the positive number

$$\Lambda^*_S = \Lambda^*_{S,\Omega} = \sup\{\Lambda^*(\gamma) \,/\, \gamma \in \mathcal{A}_\Omega\}\,.$$

Given $0 < \lambda < \Lambda^*_S$, as in (4.6) we can conclude that exists $\overline{\sigma} = \overline{\sigma}(\lambda,\gamma) > 0$ small enough such that

$$\eta_\lambda(\gamma) > \|\omega_{M,\Omega}\|_\infty + \bar{\sigma}\,, \quad \text{for each } \gamma \in \mathcal{A}_\Omega\,.$$

With this, the function η_λ satisfies $[\bar{\sigma}, \|\omega_{M,\Omega}\|_\infty + \bar{\sigma}] \subset \mathrm{Im}(\eta_\lambda)$ and the others items of Lemma 4.4. This allows us to define, for each $\sigma \in (0,\bar{\sigma}]$, the function $v = v_\sigma \in C^1(\overline{\Omega})$, increasing in σ, by

$$v_\sigma(x) = v_{\sigma,\lambda}(x) := \psi_\lambda(\omega_{M,\Omega}(x) + \sigma)\,, \quad x \in \overline{\Omega}\,, \tag{6.4}$$

where $\omega_{M,\Omega}$ is the unique solution of (6.2). As in Lemma 5.1, there are $\gamma_0 = \gamma_0(\lambda) \in \mathcal{A}_\Omega$ and $\epsilon > 0$ sufficiently small such that (5.3) holds.

This function v_σ is an upper solution to the perturbed problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = a(x)f(u+\epsilon) + \lambda b(x)g(u+\epsilon) \ \text{ in } \ \Omega \\ u > 0 \ \text{ in } \ \Omega\,, \quad u = 0 \ \text{ on } \ \partial\Omega\,. \end{cases} \tag{6.5}$$

Indeed, since $\omega_{M,\Omega} = 0$ on $\partial\Omega$, the introduction of the parameter $\sigma > 0$ has the objective of ensuring the regularity required by theorem of lower–upper solution in [15] for the function v_σ on $\partial\Omega$.

Similarly to the proof of Lemma 5.1, for each $0 < \sigma < \overline{\sigma}$ and $\phi \in C_0^\infty(\Omega)$, we can show that

$$\int_\Omega [|\nabla v_\sigma|^{p-2}\nabla v_\sigma \nabla\phi + m(x)v_\sigma^{p-1}\phi]dx \geqslant \int_\Omega [a(x)f(v_\sigma+\epsilon) + \lambda b(x)g(v_\sigma+\epsilon)]\phi dx\,.$$

Moreover, by (6.4) follows that $v_\sigma(x) = \psi_\lambda(\sigma) > 0$ on $\partial\Omega$ and, since ψ_λ is increasing in $s > 0$ and $\omega_{M,\Omega} > 0$ in Ω, we have also $v_\sigma(x) := \psi_\lambda(\omega_M(x)+\sigma) > \psi_\lambda(\sigma) > 0$ in Ω.

Now, we will show that for some appropriate constant $C = C_\Omega > 0$, the function $(C\varphi_\Omega)$ will be a lower solution to (6.5), where $\varphi_\Omega > 0$ is an eigenfunction associated with the first eigenvalue $\lambda_{1,\Omega}(m,\rho) > 0$ of the problem (1.3).

Given $\lambda > \lambda_{*,\Omega}$, by (6.3) there exists an $\epsilon_1 \in (0, \gamma_0)$, such that

$$\frac{f(s)}{s^{p-1}} + \lambda\frac{g(s)}{s^{p-1}} \geqslant \lambda_{1,\Omega}(m,\rho)\,, \quad \text{for any } \ 0 < s \leqslant \epsilon_1\,. \tag{6.6}$$

Choose $C = C(\Omega,\epsilon_1) > 0$ such that $C\|\varphi_\Omega\|_{L^\infty(\Omega)} = \epsilon_1/2$. So, for each $0 < \epsilon < \epsilon_1/2$, we have

$$C\|\varphi_\Omega\|_{L^\infty(\Omega)} + \epsilon < C\|\varphi_\Omega\|_{L^\infty(\Omega)} + \epsilon_1/2 = \epsilon_1\,. \tag{6.7}$$

Then, for each $\phi \in C_0^\infty(\Omega)$, $\phi \geqslant 0$, and $0 < \epsilon < \epsilon_1/2$, it follows from (1.3) and (6.6) that

$$\begin{aligned} &\int_\Omega [|\nabla(C\varphi_\Omega)|^{p-2}\nabla(C\varphi_\Omega)\nabla\phi + m(x)(C\varphi_\Omega)^{p-1}\phi]dx \\ &\qquad \leqslant \lambda_{1,\Omega}(m,\rho)\int_\Omega \rho(x)(C\varphi_\Omega+\epsilon)^{p-1}\phi dx \\ &\qquad \leqslant \int_\Omega [f(C\varphi_\Omega+\epsilon) + \lambda g(C\varphi_\Omega+\epsilon)]\rho(x)\phi dx\,. \end{aligned}$$

We claim that

$$C\varphi_\Omega(x) \leqslant v_\Omega(x)\,, \quad \text{for all } \ x \in \overline{\Omega}\,. \tag{6.8}$$

In fact, from (4.1)–(4.2), we have

$$\begin{aligned}
-\Delta_p(C\varphi_\Omega) + m(x)(C\varphi_\Omega)^{p-1} &\leqslant \gamma_0^{p-1} a(x) \frac{f(C\varphi_\Omega + \epsilon)}{\gamma_0^{p-1}} + \gamma_0^{p-1} \lambda b(x) \frac{g(C\varphi_\Omega + \epsilon)}{\gamma_0^{p-1}} \\
&\leqslant \gamma_0^{p-1} a(x) \frac{\zeta_{f,\gamma_0}(C\varphi_\Omega + \epsilon)}{(C\varphi_\Omega + \epsilon)^{p-1}} + \gamma_0^{p-1} \lambda b(x) \frac{\zeta_{g,\gamma_0}(C\varphi_\Omega + \epsilon)}{(C\varphi_\Omega + \epsilon)^{p-1}} \\
&\leqslant M(x)\gamma_0^{p-1}[\hat{\zeta}_{f,\gamma_0}(C\varphi_\Omega + \epsilon) + \lambda \hat{\zeta}_{g,\gamma_0}(C\varphi_\Omega + \epsilon)] \\
&= M(x)\gamma_0^{p-1} \frac{\hat{\zeta}_{\lambda,\gamma_0}(C\varphi_\Omega + \epsilon)}{(C\varphi_\Omega + \epsilon)^{p-1}} \\
&\leqslant M(x)\gamma_0^{p-1} \frac{\hat{\zeta}_{\lambda,\gamma_0}(C\varphi_\Omega)}{(C\varphi_\Omega)^{p-1}}\,, \qquad (6.9)
\end{aligned}$$

where such inequalities are considered in the distributional sense. Moreover, by analogous relationship to (5.6), we have

$$-\Delta_p v_\sigma + m(x) v_\sigma^{p-1} \geqslant M(x)\gamma_0^{p-1} \frac{\hat{\zeta}_{\lambda,\gamma_0}(v_\sigma)}{v_\sigma^{p-1}}\,. \qquad (6.10)$$

Note that $\hat{\zeta}_{\lambda,\gamma_0}(s)/s^{p-1}$ and $-s^{p-1}$ are non-increasing in $s > 0$, $v_\sigma, C\varphi_\Omega \in W^{1,p}(\Omega)$ and $C\varphi_\Omega = 0 < v_\sigma$ on $\partial\Omega$. So, from (6.9) and (6.10), we can apply a comparison principle for weak solutions to quasilinear equations which is due to Tolksdorf [34] to obtain (6.8).

Define $\hat{F}_\epsilon : \Omega \times [0, \infty) \to [0, \infty)$ by

$$\hat{F}_\epsilon(x, s) := \begin{cases} a(x) f(s + \epsilon) + \lambda b(x) g(s + \epsilon)\,, & s \leqslant v_\sigma \\ a(x) f(v_\sigma + \epsilon) + \lambda b(x) g(v_\sigma + \epsilon)\,, & s \geqslant v_\sigma\,, \end{cases}$$

and consider the auxiliary problem

$$\begin{cases} -\Delta_p u + m(x) u^{p-1} = \hat{F}_\epsilon(x, u) \ \text{ in } \ \Omega \\ u > 0 \ \text{ in } \ \Omega \ \text{ and } \ u = 0 \ \text{ on } \ \partial\Omega\,. \end{cases} \qquad (6.11)$$

It follows that $\hat{F}_\epsilon$ satisfies the Carathéodory conditions, v_σ is an upper solution and $C\varphi_\Omega$ is a lower solution for (6.11). In addition, for each $s_0 > 0$ there exists a constant A such that

$$|\hat{F}_\epsilon(x, s) - m(x)s^{p-1}| \leqslant A\,, \quad (x, s) \in \Omega \times [-s_0, s_0]\,.$$

By considering the above, we mainly apply a theorem of lower–upper solution, due to [15] to conclude that there exists a $u_{\sigma,\epsilon} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $0 < C\varphi \leqslant u_{\sigma,\epsilon} \leqslant v_\sigma$, satisfying (6.11) and consequently (6.5).

Now, we using a standard diagonal argument and we can show that there is a function $u_\lambda \in C^1(\Omega) \cap C(\overline{\Omega})$ solution of (6.1), for each $\lambda_{*,\Omega} < \lambda < \Lambda_S^*$ given, with $0 < C\varphi_\Omega \leqslant u_\lambda \leqslant v < \gamma_0$ in Ω.

Finally, let

$$\lambda^*_\Omega := \sup\{\lambda > 0 : (6.1) \text{ has solution}\}\,.$$

It is clear that $\Lambda^*_S = \Lambda^*_{S,\Omega} \leqslant \lambda^*_\Omega$. Taking $\lambda < \lambda^*_\Omega$, there exists $\tilde\lambda \in (\lambda, \lambda^*_\Omega)$ such that (6.1) has solution, namely, there exists $u_{\tilde\lambda}$ such that

$$\begin{aligned}-\Delta_p u_{\tilde\lambda} + m(x)u^{p-1} &= a(x)f(u_{\tilde\lambda}) + \tilde\lambda b(x)g(u_{\tilde\lambda})\\ &\geqslant a(x)f(u_{\tilde\lambda}) + \lambda b(x)g(u_{\tilde\lambda}) \ \text{ in } \Omega\,.\end{aligned}$$

So, $u_{\tilde\lambda}$ is an upper solution and $C\varphi_\Omega$ is a lower solution for (6.1). By [12], we claim that $C\varphi_\Omega \leqslant u_{\tilde\lambda}$ in Ω. Moreover, we can apply a lower and upper solution theorem for singular problems from [23] and ensure that (6.1) has solution u with $C\varphi_\Omega \leqslant u \leqslant u_{\tilde\lambda}$, a.e. $x \in \Omega$. This ends the proof of Theorem 6.1. □

7. Proof of Theorem 1.2

PROOF. At first, it follows from Lemma 2.2(iv) that $\mathcal{A}_{\mathbf{R}^N} \subset \mathcal{A}_{B_R}$, for all $R \geqslant 1$, because if $\gamma \in \mathcal{A}_{\mathbf{R}^N}$ then $\Gamma_0(\gamma) > \|\omega_M\|_{L^\infty(\mathbf{R}^N)} \geqslant \|\omega_{M,B_R}\|_{L^\infty(B_R)}$, for all $R \geqslant 1$, i.e., $\gamma \in \mathcal{A}_{B_R}$. So, we have that $\Lambda^*_S = \Lambda^*_{S,\mathbf{R}^N} := \sup\{\Lambda^*(\gamma) \,/\, \gamma \in \mathcal{A}_{\mathbf{R}^N}\} \leqslant \sup\{\Lambda^*(\gamma) \,/\, \gamma \in \mathcal{A}_{B_R}\} := \Lambda^*_{S,B_R}$, for all $R \geqslant 1$, where Λ^*_S is as in (4.4) and the functions ω_M and ω_{M,B_R} was given in (1.4) and (6.2) respectively.

So, given $\lambda_* < \lambda < \Lambda^*_S$ and taking $v_R = v_{|B_R}$ as an upper solution, where v is given by Lemma 5.1, there exists, by Theorem 6.1 and its demonstration, a $u_R \in C^1(B_R) \cap C(\overline{B}_R)$ satisfying

$$\begin{cases} -\Delta_p u_R + m(x)u_R^{p-1} = a(x)f(u_R) + \lambda b(x)g(u_R) \ \text{ in } \ B_R \\ u_R > 0 \ \text{ in } \ B_R\,, \quad u_R = 0 \ \text{ on } \ \partial B_R\,, \end{cases} \tag{7.1}$$

for each $R \geqslant 1$ given with

$$0 < C_R\varphi_R \leqslant u_R \leqslant v_R \ \text{ in } B_R\,.$$

Besides this, from $\lambda > \lambda_*$, $\lambda_1(m,\rho) = \lim_{R\to\infty} \lambda_{1,B_R}(m,\rho)$ and (1.6), it follows that there exists an $L_0 > 1$ such that $\lambda_{1,B_{L_0}}(m,\rho) < \lambda g_0 + f_0$. That is, from the monotonicity of the first eigenvalue in relation to the domain, there exists one $\delta = \delta(L_0) > 0$ such that

$$f(s) + \lambda g(s) > \lambda_{1,B_R}(m,\rho)s^{p-1}\,, \quad \text{for all } s \in (0,\delta) \text{ and } R \geqslant L_0\,. \tag{7.2}$$

Now, we choose a constant $C = C(\delta) \in (0, C_{L_0})$ small enough such that

$$0 < C \parallel \varphi_{L_0} \parallel_{L^\infty(B_{L_0})} < \delta\,, \tag{7.3}$$

where C_{L_0} is the constant of the lower solution of (6.1) with $R = L_0$ defined in (6.7).

Now, we claim that

$$C\varphi_{L_0}(x) \leqslant u_R(x)\,, \quad x \in B_{L_0}\,, \quad \text{for all } \; R > L_0\,. \tag{7.4}$$

To see this, suppose by contradiction that there are $x_0 \in B_{L_0}$ and $R_0 > L_0$ such that $C\varphi_{L_0}(x_0) > u_{R_0}(x_0)$. Thus, the open set

$$A_{R_0,L_0} := \{x \in B_{L_0} \;/\; C\varphi_{L_0}(x) > u_{R_0}(x)\}$$

is nonempty.

Recalling that $C\varphi_{L_0}$ and u_R satisfy (1.3) and (7.1), respectively, it follows from (7.2), (7.3) and Díaz and Saá's inequality [12], that

$$\begin{aligned}
0 &\leqslant \int_{A_{R_0,L_0}} \left[\frac{-\Delta_p(C\varphi_{L_0})}{(C\varphi_{L_0})^{p-1}} + \frac{\Delta_p u_{R_0}}{(u_{R_0})^{p-1}}\right][(C\varphi_{L_0})^p - (u_{R_0})^p]dx \\
&\leqslant \int_{A_{R_0,L_0}} \left[\lambda_{1,B_{L_0}}(\rho)\rho(x) - a(x)\frac{f(u_{R_0})}{(u_{R_0})^{p-1}} - \lambda b(x)\frac{g(u_{R_0})}{(u_{R_0})^{p-1}}\right][(C\varphi_{L_0})^p - (u_{R_0})^p]dx \\
&\leqslant \int_{A_{R_0,L_0}} \rho(x)\left[\lambda_{1,B_{L_0}}(\rho) - \frac{f(u_{R_0})}{(u_{R_0})^{p-1}} - \lambda\frac{g(u_{R_0})}{(u_{R_0})^{p-1}}\right][(C\varphi_{L_0})^p - (u_{R_0})^p]dx \leqslant 0\,.
\end{aligned}$$

That is, $C\varphi_{L_0} = du_{R_0}$, for some $d > 0$. By definition of A_{R_0,L_0}, it follows that $C\varphi_{L_0} = u_{R_0}$. This is impossible. Therefore, $A_{R_0,L_0} = \emptyset$ and (7.4) is verified.

Noting that $\mathbf{R}^N = \bigcup_{R=1}^{\infty} B_R(0)$ and proceeding as at the end of proof of Theorem 6.1, we finish the proof of existence, for each $\lambda_* < \lambda < \Lambda_S^*$ given.

Now, defining

$$\lambda^* := \sup\{\lambda > 0 : (1.1) \text{ has solution}\}\,,$$

we have that $\Lambda_S^* \leqslant \lambda^*$.

We can define the functions

$$\hat{f}(t) = t^{p-1}\inf\left\{\frac{f(s)}{s^{p-1}}, 0 < s \leqslant t\right\} \quad \text{and} \quad \hat{g}(t) = t^{p-1}\inf\left\{\frac{g(s)}{s^{p-1}}, 0 < s \leqslant t\right\}\,.$$

Note that $\hat{f} + \hat{g}$ is positive, non-increasing, and $\hat{f}(t) + \hat{g}(t) \leqslant f(t) + g(t)$, for all $t > 0$. Moreover, $\hat{f}_0 = f_0$ and $\hat{g}_0 = g_0$.

Given $\lambda_* < \lambda < \lambda^*$, there exists $\tilde{\lambda} \in (\lambda, \lambda^*)$ such that (1.1) has a solution $u_{\tilde{\lambda}}$. Then,

$$\begin{cases}
-\Delta_p u_{\tilde{\lambda}} + m(x)u_{\tilde{\lambda}}^{p-1} = a(x)f(u_{\tilde{\lambda}}) + \tilde{\lambda}b(x)g(u_{\tilde{\lambda}}) \\
\qquad\qquad\qquad\quad \geqslant a(x)f(u_{\tilde{\lambda}}) + \lambda b(x)g(u_{\tilde{\lambda}}) \\
\qquad\qquad\qquad\quad \geqslant a(x)\hat{f}(u_{\tilde{\lambda}}) + \lambda b(x)\hat{g}(u_{\tilde{\lambda}}) \quad \text{in } \mathbf{R}^N \\
u_{\tilde{\lambda}} > 0 \quad \text{in } \mathbf{R}^N\,, \quad u_{\tilde{\lambda}}(x) \to 0 \text{ when } |x| \to \infty\,.
\end{cases} \tag{7.5}$$

By Theorem 6.1, there is a function u_R, for each $R \geqslant 1$, satisfying the problem

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = a(x)\hat{f}(u) + \lambda b(x)\hat{g}(u) \ \text{ in } \ B_R\,, \\ u > 0 \text{ in } B_R\,, \quad u = 0 \ \text{ on } \ \partial B_R\,. \end{cases}$$

Now, using the monotonicity of $\hat{f}(t)/t^{p-1}$ and $\hat{g}(t)/t^{p-1}$ and Díaz and Saá's inequality [12], we can show that $u_R \leqslant u_{\tilde{\lambda}}$.

As made in (7.2), (7.3) and (7.4), there exists an $L_0 > 1$ and $C \in (0, C_{L_0})$ such that

$$0 < C\varphi_{L_0}(x) \leqslant u_R(x) \leqslant u_{\tilde{\lambda}}(x)\,, \quad \text{for all } \ x \in B_{L_0}\,, \ \ R > L_0\,.$$

Defining $z(x) = \lim_{R\to\infty} u_R(x),\ x \in \mathbf{R}^N$, and using a diagonal argument, we can show that z satisfies

$$\begin{cases} -\Delta_p z + m(x)z^{p-1} = a(x)\hat{g}(z) \leqslant a(x)f(z) + \lambda b(x)g(z) \quad \text{in } \ \mathbf{R}^N \\ z > 0 \ \text{ in } \ \mathbf{R}^N,\ z(x) \to 0 \ \text{ when } \ |x| \to \infty\,. \end{cases} \tag{7.6}$$

By (7.5) and (7.6), follow that $u_{\tilde{\lambda}}$ and z are, respectively, upper and lower solution to the problem (1.1). Moreover, $z \leqslant u_{\tilde{\lambda}}$ in $\mathbf{R}^N$.

By Theorem 2.1, the problem (1.1) has a solution $u \in C^1(\mathbf{R}^N)$ with $0 < u \leqslant u_{\tilde{\lambda}}$. Therefore, for any $\lambda_* < \lambda < \lambda^*$, it follows that (1.1) has a solution.

Since that $\Lambda_S^* \leqslant \lambda^*$, the Lemma 4.3 proved that

$$\lambda^* \geqslant \max\left\{\frac{1}{g_0}\left(\frac{1}{\|\omega_M\|_\infty^{p-1}} - f_0\right), \frac{1}{g_\infty}\left(\frac{1}{\|\omega_M\|_\infty^{p-1}} - f_\infty\right)\right\}.$$

To see the other inequality, i.e.,

$$f_{inf} + \lambda^* g_{inf} \leqslant \lambda_1(m, \rho)\,,$$

let

$$\eta_\lambda(s) := \frac{f(s)}{s^{p-1}} + \lambda\frac{g(s)}{s^{p-1}}\,, \quad s,\ \ \lambda > 0\,. \tag{7.7}$$

Of course, $\eta_\lambda(s) \geqslant f_{inf} + \lambda g_{inf}$, for all $\lambda > 0$. Given $0 < \lambda < \lambda^*$, it follows from Theorem 1.2 that there exists a function $u = u_\lambda$ solution of the problem (1.1). Then

$$\begin{aligned} -\Delta_p u + m(x)u^{p-1} - (f_{inf} + \lambda g_{inf})\rho(x)u^{p-1} &\geqslant -\Delta_p u + m(x)u^{p-1} - \eta_\lambda(u)\rho(x)u^{p-1} \\ &= -\Delta_p u + m(x)u^{p-1} - \left[\frac{f(u)}{u^{p-1}} + \lambda\frac{g(u)}{u^{p-1}}\right]\rho(x)u^{p-1} \\ &\geqslant -\Delta_p u + m(x)u^{p-1} - [a(x)f(u) + \lambda b(x)g(u)] = 0\,. \end{aligned}$$

By Lemma 2.2 and by arbitrarity of $0 < \lambda < \lambda^*$, it follows that $f_{inf} + \lambda^* g_{inf} \leqslant \lambda_1(m, \rho)$.

Now, remains only for us to show that, under the conditions (c) or (d) given in Theorem 1.2, we have $\lambda^* < \infty$.

Let η_λ as in (7.7). Consider the sequence $(\lambda_n) \subset (0, +\infty)$ such that $\lambda_n \to \infty$ when $n \to \infty$.

We claim that there exist $n_0 \in \mathbf{N}$ large arbitrarily and $\theta = \theta(\lambda_{n_0}) > 0$ such that

$$\eta_{\lambda_n}(s) > \theta > \lambda_1(m, \rho)\,, \quad \text{for all } s > 0\,, \quad n \geqslant n_0\,. \tag{7.8}$$

To prove (7.8), given an arbitrary $n > 0$, we consider $\eta_{\lambda_n}(s)$, for $s \in [1/j, j]$, $j > 0$. Of course, there is $s_j = s_{j,n} \in [1/j, j]$ such that $\eta_{\lambda_n}(s_j) \leqslant \eta_{\lambda_n}(s)$, for all $s \in [1/j, j]$. Making $j \to \infty$, note that might happen, less of subsequence:

(i) $s_j \to s_{0,n} \in (0, \infty)$, what implies that $\eta_{\lambda_n}(s) \geqslant f(s_{0,n})/s_{0,n}^{p-1} + \lambda_n g(s_{0,n})/s_{0,n}^{p-1}$;
(ii) $s_j \to 0$, resulting in $\eta_{\lambda_n}(s) \geqslant f_0 + \lambda_n g_0$;
(iii) $s_j \to \infty$, and then $\eta_{\lambda_n}(s) \geqslant f_\infty + \lambda_n g_\infty$.

In case (i) above, we need to repeat this analysis, because when $n \to \infty$, may happen that: $s_{0,n} \to s_0 \in (0, \infty)$, $s_{0,n} \to 0$ or $s_{0,n} \to \infty$.

Note that in any of these cases, by the hypotheses given, we have

$$\lim_{n\to\infty} \eta_{\lambda_n}(s) > \lambda_1(m, \rho)\,,$$

what shows (7.8).

Let us assume that (1.1) has solution $u = u_{\lambda_0}$, $\lambda_0 = \lambda_{n_0}$. By (7.8), (7.7) and (1.1), it follows that

$$\begin{aligned}
&-\Delta_p u_{\lambda_0} + m(x) u_{\lambda_0}^{p-1} - \theta\rho(x) u_{\lambda_0}^{p-1} \\
&\qquad \geqslant -\Delta_p u_{\lambda_0} + m(x) u_{\lambda_0}^{p-1} - \eta_{\lambda_0}(u_{\lambda_0})\rho(x) u_{\lambda_0}^{p-1} \\
&\qquad = -\Delta_p u_{\lambda_0} + m(x) u_{\lambda_0}^{p-1} - \left[\frac{f(u_{\lambda_0})}{u_{\lambda_0}^{p-1}} + \lambda_0 \frac{g(u_{\lambda_0})}{u_{\lambda_0}^{p-1}}\right]\rho(x) u_{\lambda_0}^{p-1} \\
&\qquad \geqslant -\Delta_p u_{\lambda_0} + m(x) u_{\lambda_0}^{p-1} - [a(x) f(u_{\lambda_0}) + \lambda_0 b(x) g(u_{\lambda_0})] = 0\,.
\end{aligned}$$

By considering the above, we apply Lemma 2.2 to conclude that $\theta \leqslant \lambda_1(m, \rho)$, which contradicts (7.8). So, u_{λ_0} cannot be a solution of the problem (1.1). Therefore, $\lambda^* < \infty$. This ends the proof of Theorem 1.2. □

8. Appendix

PROOF OF LEMMA 4.2(i):

PROOF. Given $\beta \in (0, 1)$ by (4.2)(i) and Lemma (4.1)(iii), we have that

$$\begin{aligned}
\lim_{\gamma\to\infty} \Gamma_\lambda(\gamma) &\geq \lim_{\gamma\to\infty} \left(\frac{1}{\gamma}\int_{\beta\gamma}^{\gamma} \frac{t}{H_{\lambda,\gamma}(t)} dt\right) \geq \lim_{\gamma\to\infty} \left(\frac{1}{\gamma}\frac{\beta\gamma}{H_{\lambda,\gamma}(\beta\gamma)}\gamma(1-\beta)\right) \\
&= \lim_{\gamma\to\infty} \frac{(1-\beta)}{\beta\gamma}\int_0^{\beta\gamma} \frac{t}{\hat{\zeta}_{\lambda,\gamma}(t)^{\frac{1}{p-1}}} dt \geq \lim_{\gamma\to\infty} \frac{(1-\beta)}{\beta\gamma}\int_{\beta^2\gamma}^{\beta\gamma} \frac{t}{\hat{\zeta}_{\lambda,\gamma}(t)^{\frac{1}{p-1}}} dt
\end{aligned}$$

$$\geq \lim_{\gamma\to\infty}\left[\frac{(1-\beta)}{\beta\gamma}\frac{\beta^2\gamma}{[\hat{\zeta}_{\lambda,\gamma}(\beta^2\gamma)]^{\frac{1}{p-1}}}\beta\gamma(1-\beta)\right].$$

By the properties of the functions involved, it follows that

$$\begin{aligned}\lim_{\gamma\to\infty}\Gamma_\lambda(\gamma) &\geq \lim_{\gamma\to\infty}\frac{(1-\beta)^2\beta^2\gamma}{[(\beta^2\gamma)^{p-1}\hat{\zeta}_{f,\gamma}(\beta^2\gamma)+\lambda(\beta^2\gamma)^{p-1}\hat{\zeta}_{g,\gamma}(\beta^2\gamma)]^{\frac{1}{p-1}}}\\ &= \lim_{\gamma\to\infty}\frac{(1-\beta)^2}{\left[\sup\left\{\frac{\zeta_{f,\gamma}(t)}{t^{p-1}},\ \ t>\beta^2\gamma\right\}+\lambda\sup\left\{\frac{\zeta_{g,\gamma}(t)}{t^{p-1}},\ \ t>\beta^2\gamma\right\}\right]^{\frac{1}{p-1}}}\\ &= \frac{(1-\beta)^2}{(f_\infty+\lambda g_\infty)^{\frac{1}{p-1}}}\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lim_{\gamma\to\infty}\Gamma_\lambda(\gamma) &\leq \lim_{\gamma\to\infty}\frac{1}{\gamma}\frac{\gamma}{H_{\lambda,\gamma}(\gamma)}\gamma = \lim_{\gamma\to\infty}\frac{1}{\gamma}\int_0^\gamma\frac{t}{\hat{\zeta}_{\lambda,\gamma}(t)^{\frac{1}{p-1}}}dt\\ &\leq \lim_{\gamma\to\infty}\frac{\gamma}{\gamma\hat{\zeta}_{f,\gamma}(\gamma)^{\frac{1}{p-1}}+\lambda^{\frac{1}{p-1}}\gamma\hat{\zeta}_{g,\gamma}(\gamma)^{\frac{1}{p-1}}}\\ &= \lim_{\gamma\to\infty}\frac{1}{\left[\sup\left\{\frac{\zeta_{f,\gamma}(t)}{t^{p-1}},\ \ t>\gamma\right\}+\lambda\sup\left\{\frac{\zeta_{g,\gamma}(t)}{t^{p-1}},\ \ t>\gamma\right\}\right]^{\frac{1}{p-1}}}\\ &= \frac{1}{(f_\infty+\lambda g_\infty)^{\frac{1}{p-1}}}\,.\end{aligned}$$

Then,

$$\frac{(1-\beta)^2}{(f_\infty+\lambda g_\infty)^{\frac{1}{p-1}}}\leq\lim_{\gamma\to\infty}\Gamma_\lambda(\gamma)\leq\frac{1}{(f_\infty+\lambda g_\infty)^{\frac{1}{p-1}}}\,.$$

When $\beta\to 0$, we have the claimed. □

PROOF OF (5.5):

PROOF. Firstly, we show that

$$\gamma_0\eta_\lambda(t)\frac{H_{\lambda,\gamma_0}(t)}{t}\leqslant t\,,\quad \text{for all } t>0\,. \tag{8.1}$$

In fact, by (4.5) and Lemma 4.1(iii) we have

$$\gamma_0\eta_\lambda(t)\frac{H_{\lambda,\gamma_0}(t)}{t}=\gamma_0\left(\frac{1}{\gamma_0}\int_0^t\frac{s}{H_{\lambda,\gamma_0}(s)}ds\right)\frac{H_{\lambda,\gamma_0}(t)}{t}$$

$$\leqslant \frac{t}{H_{\lambda,\gamma_0}(t)}\, t\, \frac{H_{\lambda,\gamma_0}(t)}{t} = t\,.$$

Then, using Lemma 4.4(iii) and (iv), we obtain

$$\omega_\lambda'(\eta_\lambda(t)) = \gamma_0 \frac{H_{\lambda,\gamma_0}(\psi_\lambda(\eta_\lambda(t)))}{\psi_\lambda(\eta_\lambda(t))} = \gamma_0 \frac{H_{\lambda,\gamma_0}(t)}{t}\,,$$

what implies, by (8.1), for all $t > 0$, that

$$\eta_\lambda(t)\psi_\lambda'(\eta_\lambda(t)) = \gamma_0 \eta_\lambda(t) \frac{H_{\lambda,\gamma_0}(t)}{t} \leqslant t = \psi_\lambda(\eta_\lambda(t))\,. \tag{8.2}$$

Now, by Lemma 4.4(i), we have $\omega_M(x) \in \mathrm{Im}(\eta_\lambda)$, for all $x \in \overline{\Omega}$, i.e., given $x \in \overline{\Omega}$, there exists $t_x > 0$ such that $\eta_\lambda(t_x) = \omega_M(x)$, what allows us to write (8.2) as

$$\omega_M(x)\psi_\lambda'(\omega_M(x)) \leqslant \psi_\lambda(\omega_M(x))\,, \quad \text{for all } x \in \overline{\Omega}\,.$$

□

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