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A Cohomological Splitting Criterion for Rank 2 Vector Bundles on Hirzebruch Surfaces

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Abstract. In this note, we give a cohomological characterization of all rank 2 split vector bundles on Hirzebruch surfaces.

1. Introduction

Throughout this paper we work over an algebraically closed field k. A vector bundle on a smooth projective variety is called *split* if it is decomposed into a direct sum of line bundles. Recently, Fulger and Marchitan ([2]) obtained a cohomological characterization of some rank 2 split vector bundles on Hirzebruch surfaces over the complex number field, by using Buchdahl's Beilinson type spectral sequence ([1]). However, it seems difficult to apply their argument to general cases. The purpose of this paper is to give a simple characterization of all rank 2 split vector bundles on Hirzebruch surfaces by cohomological informations in arbitrary characteristic.

THEOREM 1. *Let $\mathcal{E}$ be a* rank 2 *vector bundle on a Hirzebruch surface $\mathbf{F}_n = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$. Let $\mathcal{F}$ be a split* rank 2 *vector bundle on $\mathbf{F}_n$. If* $\dim_k H^i(\mathcal{E} \otimes \mathcal{L}) = \dim_k H^i(\mathcal{F} \otimes \mathcal{L})$ *for any $0 \leqslant i \leqslant 2$ and any line bundle $\mathcal{L}$ on $\mathbf{F}_n$, then $\mathcal{E}$ is isomorphic to $\mathcal{F}$.*

It seems that our assumption of Theorem 1 is stronger than the one of Theorem in [2]. However, if we know the Chern classes of $\mathcal{E}$, we need only a few assumptions (see Lemma 2). In the cases treated in [2], the assumption of Lemma 2 is essentially equivalent to the one of the Theorem of [2].

Notation

For a smooth projective variety X and a vector bundle $\mathcal{E}$ on X, let $\mathbf{P}_X(\mathcal{E})$ be the projectivization of $\mathcal{E}$ in the sense of Grothendieck. We denote $\dim_k H^i(\mathcal{E})$ by $h^i(\mathcal{E})$. We put $\mathrm{CH}_0(X)$

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the Chow group of 0-cycles of X. We denote the Hirzebruch surface $\mathbf{P}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$ by $\mathbf{F}_n$, the natural projection $\mathbf{F}_n \to \mathbf{P}^1$ by π, the minimal section on $\mathbf{F}_n$ by σ (i.e. $\sigma \cong \mathbf{P}^1$, $\sigma^2 = -n$) and a fiber of π by f. It is well-known that $\mathrm{Pic}(\mathbf{F}_n) = \mathbf{Z}\sigma \oplus \mathbf{Z}f$.

2. Proof of Theorem 1

To give the proof of Theorem 1, we show the following lemma.

LEMMA 2. *Let $\mathcal{E}$ be a* rank 2 *vector bundle on $\mathbf{F}_n$. Assume that $h^0(\mathcal{E}(-\sigma)) = h^0(\mathcal{E}(-f)) = 0$, $h^0(\mathcal{E}) \geqslant 1$ and $c_2(\mathcal{E}) = 0$.*

1. *Assume that $c_1(\mathcal{E}) = -a\sigma - bf$, where a and b are nonnegative integers, and that $h^0(\mathcal{E}(a\sigma + bf)) \geqslant 1 + h^0(\mathcal{O}_{\mathbf{F}_n}(a\sigma + bf))$. Then $\mathcal{E}$ is isomorphic to $\mathcal{O}_{\mathbf{F}_n} \oplus \mathcal{O}_{\mathbf{F}_n}(-a\sigma - bf)$.*
2. *Assume that $c_1(\mathcal{E}) = a\sigma - bf$, where a and b are integers such that $ab > 0$, and that $h^0(\mathcal{E}(-a\sigma + bf)) \geqslant 1 + h^0(\mathcal{O}_{\mathbf{F}_n}(-a\sigma + bf))$ Then $\mathcal{E}$ is isomorphic to $\mathcal{O}_{\mathbf{F}_n} \oplus \mathcal{O}_{\mathbf{F}_n}(a\sigma - bf)$.*

PROOF. Put $\mathcal{L} = \mathcal{O}_{\mathbf{F}_n}(-a\sigma - bf)$ in Case 1 and $\mathcal{L} = \mathcal{O}_{\mathbf{F}_n}(a\sigma - bf)$ in Case 2. Since $h^0(\mathcal{E}) \neq 0$, we can take a nonzero section $0 \neq s \in \mathrm{H}^0(\mathcal{E})$. Put $Z := (s = 0)$. If s takes zero on some nonzero effective divisor $D > 0$, we have a nonzero section of $\mathrm{H}^0(\mathcal{E}(-D))$. This is a contradiction because $h^0(\mathcal{E}(-D)) \leqslant \max\{h^0(\mathcal{E}(-\sigma)), h^0(\mathcal{E}(-f))\} = 0$. Therefore, Z is of codimension 2. Since $c_2(\mathcal{E}) = 0$, we have $Z = \emptyset$. Hence we obtain an exact sequence,

$$0 \to \mathcal{O}_{\mathbf{F}_n} \xrightarrow{s} \mathcal{E} \to \mathcal{L} \to 0\,,$$

since $c_1(\mathcal{E}) = -a\sigma - bf$ in Case 1 and $c_1(\mathcal{E}) = a\sigma - bf$ in Case 2. We show that the above exact sequence splits. Consider the long exact sequence ;

$$0 \to \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathbf{F}_n}) \to \mathrm{Hom}(\mathcal{L}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{L}, \mathcal{L}) \to \mathrm{Ext}^1(\mathcal{L}, \mathcal{O}_{\mathbf{F}_n}) \to \dots .$$

By the assumption, we have

$$\begin{aligned} &\dim_k \mathrm{Hom}(\mathcal{L}, \mathcal{E}) = h^0(\mathcal{E} \otimes \mathcal{L}^\vee) \\ &\geqslant h^0(\mathcal{L}^\vee) + h^0(\mathcal{O}_{\mathbf{F}_n}) = \dim_k \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{\mathbf{F}_n}) + \dim_k \mathrm{Hom}(\mathcal{L}, \mathcal{L})\,. \end{aligned}$$

Therefore, the homomorphism $\mathrm{Hom}(\mathcal{L}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{L}, \mathcal{L})$ is surjective. Hence $\mathcal{E}$ is isomorphic to $\mathcal{O}_{\mathbf{F}_n} \oplus \mathcal{L}$. □

REMARK 3. Let $\mathcal{E}$ be a split rank 2 vector bundle on $\mathbf{F}_n$. It is readily seen that there exists a line bundle $\mathcal{L}$ on $\mathbf{F}_n$ such that $c_1(\mathcal{E} \otimes \mathcal{L}) = \alpha\sigma + \beta f$ with $\alpha, \beta \leq 0$ or $\alpha\beta < 0$ and that $c_2(\mathcal{E} \otimes \mathcal{L}) = 0$. Therefore, by Lemma 2, we can characterize all split rank 2 vector bundles on $\mathbf{F}_n$.

From now on, we give a proof of Theorem 1. We will begin with a proof of the following Claim.

CLAIM 4. *Under the assumptions of* Theorem 1, *we have* $\det(\mathcal{E}) = \det(\mathcal{F})$ *in* $\mathrm{Pic}(\mathbf{F}_n)$ *and* $c_2(\mathcal{E}) = c_2(\mathcal{F})$ *in* $\mathrm{CH}_0(\mathbf{F}_n)$.

PROOF. Take arbitrarily a very ample divisor D on $\mathbf{F}_n$. We may assume that D is smooth. By the assumptions, we have $\chi(\mathcal{E}) = \chi(\mathcal{F})$ and $\chi(\mathcal{E}(-D)) = \chi(\mathcal{F}(-D))$. Therefore, we obtain $\chi(\mathcal{E}|_D) = \chi(\mathcal{F}|_D)$. By Riemann-Roch theorem on the smooth curve D, we have $c_1(\mathcal{E}) \cdot D = c_1(\mathcal{F}) \cdot D$ (cf. [3], Example 15.2.1.). Hence $c_1(\mathcal{E})$ is numerically equivalent to $c_1(\mathcal{F})$. Because $\mathbf{F}_n$ is rational, we get $\det(\mathcal{E}) = \det(\mathcal{F})$. We also have $c_2(\mathcal{E}) = c_2(\mathcal{F})$ from the Riemann-Roch theorem on $\mathbf{F}_n$ (cf. [3], Example 15.2.2.). □

Now we conclude the proof of Theorem 1. By Remark 3, we may assume that $\mathcal{F}$ is isomorphic to $\mathcal{O} \oplus \mathcal{L}$, where $\mathcal{L} = \mathcal{O}_{\mathbf{F}_n}(-a\sigma - bf)$ with nonnegative integers $a, b \geqslant 0$ or $\mathcal{L} = \mathcal{O}_{\mathbf{F}_n}(a\sigma - bf)$ with positive integers $a, b > 0$.

By Claim 4, we have $c_1(\mathcal{E}) = c_1(\mathcal{F}) = c_1(\mathcal{L})$ and $c_2(\mathcal{E}) = 0$. By the assumptions of Theorem 1, we also have $h^0(\mathcal{E}(-\sigma)) = h^0(\mathcal{F}(-\sigma)) = 0$, $h^0(\mathcal{E}(-f)) = h^0(\mathcal{F}(-f)) = 0$, $h^0(\mathcal{E}) = h^0(\mathcal{F}) \geqslant 1$ and $h^0(\mathcal{E} \otimes \mathcal{L}^\vee) = h^0(\mathcal{F} \otimes \mathcal{L}^\vee) = 1 + h^0(\mathcal{L}^\vee)$. Then, by Lemma 2, we obtain the result of Theorem 1.

Similar arguments of the proof of Theorem 1 imply the following theorem.

THEOREM 5. *Let S be a smooth projective surface with the Picard group* $\mathrm{Pic}(S) \cong \mathbf{Z}$. *Let $\mathcal{E}$ be a* rank 2 *vector bundle on S. Let $\mathcal{F}$ be a split* rank 2 *vector bundle on S. If $h^i(\mathcal{E} \otimes \mathcal{L}) = h^i(\mathcal{F} \otimes \mathcal{L})$ for any $0 \leqslant i \leqslant 2$ and any line bundle $\mathcal{L}$ on S, then $\mathcal{E}$ is isomorphic to $\mathcal{F}$.*

PROOF. We may assume that $\mathcal{F} \cong \mathcal{O}_S \oplus \mathcal{M}$ where $\mathcal{M}$ is a line bundle on S such that $\deg \mathcal{M} \leqslant 0$. In the same manner as in Claim 4, we can verify that $\det(\mathcal{E}) = \det(\mathcal{F})$ in $\mathrm{Pic}(S)$ since $\mathrm{Pic}(S)$ is isomorphic to $\mathbf{Z}$. Moreover we also have $c_2(\mathcal{E}) = c_2(\mathcal{F})$ in $\mathrm{CH}_0(S)$. Therefore, we have an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{M} \to 0\,.$$

Moreover, we can show that $\mathcal{E} \cong \mathcal{O}_S \oplus \mathcal{M}$ in the same way as in the proof of Theorem 1. □

REMARK 6. There are many surfaces having the Picard group $\mathrm{Pic}(S) \cong \mathbf{Z}$. In fact, on the complex number field, it is known that a very general surface $S \subseteq \mathbf{P}^3$ of degree $d \geqslant 4$ has the Picard group $\mathrm{Pic}(S) \cong \mathbf{Z}$. (cf. [4], Theorem)

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