Maximal Diameter Sphere Theorem for Manifolds with
Nonconstant Radial Curvature

Nathaphon BOONNAM

Tokai University
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Abstract. We generalize Toponogov’s maximal diameter sphere theorem from the radial curvature geometry’s standpoint. As a corollary to our main theorem, we prove that for a complete connected Riemannian \( n \)-manifold \( M \) having radial sectional curvature at a point bounded from below by the radial curvature function of an ellipsoid of prolate type, the diameter of \( M \) does not exceed the diameter of the ellipsoid. Furthermore if the diameter of such an \( M \) equals that of the ellipsoid, then \( M \) is isometric to the \( n \)-dimensional ellipsoid of revolution.

1. Introduction

The maximal diameter sphere theorem proved by Toponogov says as follows:

THEOREM 1.1 ([T]). Let \( M \) be a complete connected Riemannian manifold whose sectional curvature is bounded from below by a positive constant \( H \). Then the diameter of \( M \) does not exceed \( \pi / \sqrt{H} \). Furthermore if the diameter of \( M \) equals \( \pi / \sqrt{H} \), then \( M \) is isometric to the sphere with radius \( \sqrt{H} \).

This theorem was generalized by Cheng [Ch] for a complete connected Riemannian manifold whose Ricci curvature is bounded from below by a positive constant \( H \).

A natural extension of the maximal diameter sphere theorem by the radial curvature would be that for a complete connected Riemannian manifold \( M \) whose radial sectional curvature at a point \( p \in M \) is not less than a positive constant \( H \),

(A) is the diameter of \( M \) at most \( \pi / \sqrt{H} \)?

(B) Furthermore, if the diameter of \( M \) equals \( \pi / \sqrt{H} \), is \( M \) isometric to the sphere with the radius \( \sqrt{H} \)?

Notice that the problem (A) can be affirmatively solved. It is an easy consequence from Theorem ?? (or the Main theorem in [SST]). Here, we define the radial plane and radial curvature from a point \( p \) of a complete connected Riemannian manifold \( M \). For each point

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\( q \in M \) distinct from the point \( p \), a 2-dimensional linear subspace \( \sigma \) of \( T_qM \) is called a radial plane at \( q \) if there exists a unit speed minimal geodesic segment \( \gamma : [0, d(p, q)] \to M \) satisfying \( \gamma'(d(p, q)) \in \sigma \). The sectional curvature \( K(\sigma) \) of a radial plane \( \sigma \subset T_qM \) at \( q \) is called a radial curvature at \( q \).

The problem (B) is still open, but one can generalize the maximal diameter sphere theorem for a manifold which has radial curvature at a point bounded from below by the radial curvature function of a 2-sphere of revolution, which will be defined later, if the 2-sphere of revolution belongs to a certain class.

For introducing this class of a 2-sphere of revolution, we start to define a 2-sphere of revolution. Let \( \tilde{M} \) denote a complete Riemannian manifold homeomorphic to a 2-sphere. \( \tilde{M} \) is called a 2-sphere of revolution if \( \tilde{M} \) admits a point \( \tilde{p} \) such that for any two points \( \tilde{q}_1, \tilde{q}_2 \) on \( \tilde{M} \) with \( d(\tilde{p}, \tilde{q}_1) = d(\tilde{p}, \tilde{q}_2) \), where \( d(\cdot, \cdot) \) denotes the Riemannian distance function, there exists an isometry \( f \) on \( \tilde{M} \) satisfying \( f(\tilde{q}_1) = \tilde{q}_2 \) and \( f(\tilde{p}) = \tilde{p} \). The point \( \tilde{p} \) is called a pole of \( \tilde{M} \). It is proved in [ST] that \( \tilde{M} \) has another pole \( \tilde{q} \) and the Riemannian metric \( g \) of \( \tilde{M} \) is expressed as
\[
g = dr^2 + m(r)^2 d\theta^2 \text{ on } \tilde{M} \setminus \{\tilde{p}, \tilde{q}\},
\]
where \( (r, \theta) \) denote geodesic polar coordinates around \( \tilde{p} \) and
\[
m(r(x)) := \sqrt{g \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right)}.
\]

Hence \( \tilde{M} \) has a pair of poles \( \tilde{p} \) and \( \tilde{q} \). In what follows, \( \tilde{p} \) denotes a pole of \( \tilde{M} \) and we fix it. Each unit speed geodesic emanating from \( \tilde{p} \) is called a meridian. It is observed in [ST] that each meridian \( \mu : [0, 4a] \to \tilde{M} \), where \( a := \frac{1}{2}d(\tilde{p}, \tilde{q}) \), passes through \( \tilde{q} \) and is periodic, hence, \( \mu(0) = \mu(4a) = \tilde{p}, \mu'(0) = \mu'(4a) \). The function \( G \circ \mu : [0, 2a] \to R \) is called the radial curvature function of \( \tilde{M} \), where \( G \) denotes the Gaussian curvature of \( \tilde{M} \).

A 2-sphere of revolution \( \tilde{M} \) with a pair of poles \( \tilde{p} \) and \( \tilde{q} \) is called a model surface if \( \tilde{M} \) satisfies the following two properties:

1. (1.1) \( \tilde{M} \) has a reflective symmetry with respect to the equator, \( r = a = \frac{1}{2}d(\tilde{p}, \tilde{q}) \).
2. (1.2) The Gaussian curvature \( G \) of \( \tilde{M} \) is strictly decreasing along a meridian from the point \( \tilde{p} \) to the point on the equator.

A typical example of a model surface is an ellipsoid of prolate type, i.e., the surface defined by
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1, \quad b > a > 0.
\]
The points \( (0, 0, \pm b) \) are a pair of poles and \( z = 0 \) is the equator.

The fact that the Gaussian curvature of a model surface is not always positive everywhere is the worthy of note. In [ST], an interesting model surface was introduced. The surface
generated by the \((x, z)\)-plane curve \((m(t), 0, z(t))\) is a model surface, where

\[
m(t) := \frac{\sqrt{3}}{10} \left( 9 \sin \frac{\sqrt{3}}{9} t + 7 \sin \frac{\sqrt{3}}{3} t \right), \quad z(t) := \int_0^t \sqrt{1 - m'(t)^2} \, dt.
\]

It is easy to see that the Gaussian curvature of the equator \(r = 3\sqrt{3}\pi/2\) is \(-1\).

Let \(M\) be a complete connected \(n\)-dimensional Riemannian manifold with a base point \(p\). \(M\) is said to have radial sectional curvature at \(p\) bounded from below by that of a model surface \(\tilde{M}\) if for any point \(q(\neq p)\) and any radial plane \(\sigma \subset T_qM\) at \(q\), the sectional curvature \(K(\sigma)\) of \(M\) satisfies \(K(\sigma) \geq G \circ \mu(d(p, q))\).

For each 2-dimensional model \(\tilde{M}\) with a Riemannian metric \(dr^2 + m(r)^2 d\theta^2\), we define an \(n\)-dimensional model \(\tilde{M}^n\) homeomorphic to an \(n\)-sphere \(S^n\) with a Riemannian metric

\[
g^n = dr^2 + m(r)^2 d\theta^2,
\]

where \(d\theta^2\) denotes the Riemannian metric of the \((n - 1)\)-dimensional unit sphere \(S^{n-1}(1)\). For example, the \(n\)-dimensional model of the ellipsoid above is the \(n\)-dimensional ellipsoid defined by

\[
\sum_{i=1}^n \frac{x_i^2}{a_i^2} + \frac{x_{n+1}^2}{b^2} = 1.
\]

In this paper, we generalize the maximal diameter sphere theorem as follows:

**Main Theorem.** Let \(M\) be a complete connected \(n\)-dimensional Riemannian manifold with a base point \(p \in M\) whose radial sectional curvature at \(p\) bounded from below by that of a model surface \(\tilde{M}\). Then, the diameter of \(M\) does not exceed the diameter of \(\tilde{M}\). Furthermore if the diameter of \(M\) equals that of \(\tilde{M}\), then \(M\) is isometric to the \(n\)-dimensional model \(\tilde{M}^n\).

As a corollary, we get an interesting result:

**Corollary to Main Theorem.** For any complete connected \(n\)-dimensional Riemannian manifold \(M\) having radial sectional curvature at a point \(p\) bounded from below by that of the ellipsoid \(\tilde{M}\) defined by

\[
\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad b > a > 0,
\]

the diameter of \(M\) does not exceed the diameter of \(\tilde{M}\). Furthermore if the diameter of such an \(M\) equals that of \(\tilde{M}\), then \(M\) is isometric to the \(n\)-dimensional ellipsoid \(\sum_{i=1}^n \frac{x_i^2}{a_i^2} + \frac{x_{n+1}^2}{b^2} = 1\).

We refer to [CE] for basic tools in Riemannian Geometry, and [SST] for some properties of geodesics on a surface of revolution.

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2. Preliminaries

Here, we review the notion of a cut point and a cut locus. Let $M$ be a complete Riemannian manifold with a base point $p$. Let $\gamma : [0, a] \to M$ denote a unit speed minimal geodesic segment emanating from $p = \gamma(0)$ on $M$. If any extended geodesic segment $\tilde{\gamma} : [0, b] \to M$ of $\gamma$, where $b > a$, is not minimizing arc joining $p$ to $\tilde{\gamma}(b)$ anymore, then the endpoint $\gamma(a)$ of the geodesic segment is called a cut point of $p$ along $\gamma$. For each point $p$ on $M$, the cut locus $C_p$ is defined by the set of all cut points along the minimal geodesic segments emanating from $p$.

REMARK 2.1. It is known (for example see [SST]) that the cut locus has a local tree structure for 2-dimensional Riemannian manifolds.

We need the following two theorems, which was proved by Sinclair and Tanaka [ST].

THEOREM 2.2 ([ST]). Let $M$ be a 2-sphere of revolution with a pair of poles $p, q$ satisfying the following two properties,

(i) $M$ is symmetric with respect to the reflection fixing $r = a$, where $2a$ denotes the distance between $p$ and $q$.

(ii) The Gaussian curvature $G$ of $M$ is monotone along a meridian from the point $p$ to the point on $r = a$.

Then the cut locus of a point $x \in M \setminus \{p, q\}$ with $\theta(x) = 0$ is a single point or a subarc of the opposite half meridian $\theta = \pi$ (resp. the parallel $r = 2a - r(x)$) when $G$ is decreasing (resp. increasing) along a meridian from $p$ to the point on $r = a$. Furthermore, if the cut locus of a point $x \in M \setminus \{p, q\}$ is a single point, then the Gaussian curvature is constant.

THEOREM 2.3 ([ST]). Let $M$ be a complete connected $n$-dimensional Riemannian manifold with a base point $p$ such that $M$ has radial sectional curvature at $p$ bounded from below by the radial curvature function of a 2-sphere of revolution $\tilde{M}$ with a pair of poles $\tilde{p}, \tilde{q}$. Suppose that the cut locus of any point on $\tilde{M}$ distinct from its two poles is a subset of the half meridian opposite to the point. Then for each geodesic triangle $\triangle(pxy)$ in $M$, there exists a geodesic triangle $\tilde{\triangle}(\tilde{p}\tilde{x}\tilde{y}) := \triangle(\tilde{\tilde{p}}\tilde{x}\tilde{y})$ in $\tilde{M}$ such that

$$d(p, x) = d(\tilde{p}, \tilde{x}), \quad d(p, y) = d(\tilde{p}, \tilde{y}), \quad d(x, y) = d(\tilde{x}, \tilde{y}),$$

(2.1)

and such that

$$\angle(pxy) \geq \angle(\tilde{p}\tilde{x}\tilde{y}), \quad \angle(pyx) \geq \angle(\tilde{p}\tilde{y}\tilde{x}), \quad \angle(xpy) \geq \angle(\tilde{x}\tilde{p}\tilde{y}).$$

(2.2)

Here, $\angle(pxy)$ denotes the angle at the vertex $x$ of the geodesic triangle $\triangle(pxy)$.

3. Proof of Main Theorem

Let $M$ be a complete connected $n$-dimensional Riemannian manifold with a base point $p$ and $\tilde{M}$ a 2-sphere of revolution with a pair of poles $\tilde{p}, \tilde{q}$ satisfying (1.1) and (1.2) in the
introduction, i.e., a model surface.

From now on, we assume that $M$ has radial sectional curvature at $p$ bounded from below by that of $\tilde{M}$. By scaling the Riemannian metrics of $M$ and $\tilde{M}$, we may assume that $2a = \pi$.

**Lemma 3.1.** The perimeter of any geodesic triangle $\tilde{\Delta}(pxy)$ of $\tilde{M}$ does not exceed $2\pi$, i.e.,

$$d(\tilde{p}, \tilde{x}) + d(\tilde{p}, \tilde{y}) + d(\tilde{x}, \tilde{y}) \leq 2\pi.$$  

**Proof.** Since $d(\tilde{\rho}, \tilde{\varrho}) = 2a = \pi$, it follows from the triangle inequality that

$$d(\tilde{x}, \tilde{y}) \leq d(\tilde{\varrho}, \tilde{x}) + d(\tilde{\varrho}, \tilde{y})$$

$$= (\pi - d(\tilde{\rho}, \tilde{x})) + (\pi - d(\tilde{\rho}, \tilde{y}))$$

$$= 2\pi - d(\tilde{\rho}, \tilde{x}) - d(\tilde{\rho}, \tilde{y}).$$

Therefore, the inequality (3.1) holds. □

**Lemma 3.2.** The perimeter of a geodesic triangle $\Delta(px)$ of $M$ does not exceed $2\pi$.

**Proof.** Let $\Delta(px)$ be any geodesic triangle of $M$. From Theorem ??, we get a geodesic triangle $\tilde{\Delta}(p xy)$ of $\tilde{M}$ satisfying (2.1). Hence, by Lemma 3.1, the perimeter of $\Delta(px)$ does not exceed $2\pi$. □

**Lemma 3.3.** The diameter of $\tilde{M}$ equals $\pi$, where the diameter $\text{diam} \tilde{M}$ of $\tilde{M}$ is defined by

$$\text{diam} \tilde{M} := \max\{d(\tilde{x}, \tilde{y})|\tilde{x}, \tilde{y} \in \tilde{M}\}.$$ 

**Proof.** Choose any points $\tilde{x}, \tilde{y}$ on $\tilde{M}$. By the triangle inequality,

$$d(\tilde{x}, \tilde{y}) \leq d(\tilde{\varrho}, \tilde{x}) + d(\tilde{\varrho}, \tilde{y}).$$  

(3.2)

Thus, by combining (3.1) and (3.2), we obtain

$$d(\tilde{x}, \tilde{y}) \leq \pi = d(\tilde{\rho}, \tilde{\varrho})$$

for any $\tilde{x}, \tilde{y}$ on $\tilde{M}$. □

**Lemma 3.4.** The diameter $\text{diam} M$ of $M$ does not exceed the diameter of $\tilde{M}$.

**Proof.** Choose a pair of points $x, y \in M$ satisfying $d(x, y) = \text{diam} M$. We first consider the case where $x = p$ or $y = p$. By the Rauch comparison theorem, there does not exist a minimal geodesic segment emanating from $p$ whose length exceeds $\pi$, since the manifold $M$ has radial curvature at $p$ bounded from below by the radial curvature function of the model surface $\tilde{M}$. Thus, $\text{diam} M = d(x, y) \leq \pi$. Hence we assume $x \neq p$ and $y \neq p$. Then, for the geodesic triangle $\Delta(pxy)$ in $M$, there exists a geodesic triangle $\tilde{\Delta}(p xy)$ in $\tilde{M}$ satisfying (2.1). Therefore, we obtain $\text{diam} M = d(\tilde{x}, \tilde{y}) \leq \text{diam} \tilde{M}$. □
LEMMA 3.5. If \( \text{diam} M = \text{diam} \tilde{M} \), then there exists a point \( q \in M \) with \( d(p, q) = \text{diam} \tilde{M} \).

PROOF. Let \( x, y \in M \) be points satisfying \( \pi = \text{diam} M = d(x, y) \). Supposing that \( x \neq p \) and \( y \neq p \), we will get a contradiction. Then, there exists a geodesic triangle \( \Delta(pxy) \) with \( d(x, y) = \pi \). It follows from Theorem ?? that there exists a geodesic triangle \( \tilde{\Delta}(pxy) \) corresponding to \( \Delta(pxy) \) satisfying \( d(\tilde{x}, \tilde{y}) = d(x, y) = \pi \). By the triangle inequality, \( d(\tilde{p}, \tilde{x}) + d(\tilde{p}, \tilde{y}) \geq d(\tilde{x}, \tilde{y}) = \pi \), and Lemma 3.1, we get
\[
d(\tilde{p}, \tilde{x}) + d(\tilde{p}, \tilde{y}) = \pi = d(\tilde{x}, \tilde{y}).
\]
This means that \( \angle(\tilde{p}\tilde{x}\tilde{y}) = \pi \) so that the subarc \( \alpha \) (passing through \( \tilde{p} \)) of the meridian joining \( \tilde{x} \) to \( \tilde{y} \) is minimal. Hence the complementary subarc of \( \alpha \) in the meridian is also a minimal geodesic segment joining \( \tilde{x} \) to \( \tilde{y} \), since the length of each meridian is \( 2\pi \). Therefore, by Theorem ??, \( \tilde{y} \) is a unique cut point of \( \tilde{x} \) and hence, the Gaussian curvature \( G \) of \( \tilde{M} \) is constant. We get a contradiction since \( G \) is strictly decreasing along a meridian from \( p \) to the point on the equator. This implies the existence of the point \( q \).

\[\square\]

LEMMA 3.6. If there exists a point \( q \in M \) with \( d(p, q) = \text{diam} M \), then \( q \) is a unique cut point of \( p \), and
\[K(\sigma) = G \circ \mu(d(p, x))\]
holds for any point \( x \in M \setminus \{p\} \) and any radial plane \( \sigma \) at \( x \).

PROOF. It follows from Lemma 3.4 that the point \( q \) is the farthest point from \( p \). Hence \( q \in C_p \). Choose any point \( x \in M \setminus \{p, q\} \). By the triangle inequality,
\[d(p, x) + d(x, q) \geq d(p, q) = \pi\]
and by Lemma 3.2,
\[d(p, x) + d(x, q) + d(p, q) \leq 2\pi.
\]
Hence, we get
\[d(p, x) + d(x, q) = d(p, q) = \pi\]
and it is easy to see that \( q \) is a unique cut point of \( p \) because \( \angle(pxq) = \pi \).

Next, we will prove that \( K(\sigma) = G \circ \mu(d(p, x)) \) for any \( x \in M \setminus \{p, q\} \) and any radial plane \( \sigma \) at \( x \). Suppose that there exist a point \( x \in M \setminus \{p, q\} \) and a radial plane \( \sigma \) at \( x \) such that \( K(\sigma) > G \circ \mu(d(p, x)) \). Let \( \gamma : [0, \pi] \to M \) denote the minimal geodesic segment emanating from \( p \) passing through \( x \). Choose a unit tangent vector \( v \in \sigma \subset T_xM \) orthogonal to \( \gamma'(d(p, x)) \). Let \( Y(t) \) denote the Jacobi field along \( \gamma(t) \) satisfying \( Y(0) = 0 \) and \( Y(d(p, x)) = v \), and hence \( \sigma \) is spanned by \( Y(d(p, x)) \) and \( \gamma'(d(p, x)) \). By the Rauch comparison theorem, there exists a conjugate point \( \gamma(t_1) \) of \( p \) along \( \gamma \) for some \( t_1 \in (0, \pi) \), since \( K(\sigma) > G \circ \mu(d(p, x)) \) and the sectional curvature of the radial plane spanned by
\( Y(t) \) and \( \gamma'(t) \) is not less than \( G \circ \mu(t) \) for each \( t \in (0, \pi) \). This contradicts the fact that the geodesic segment \( \gamma \) is minimal. 

**Proof of Main Theorem.** The first claim is clear from Lemma 3.4. Assume \( \text{diam} M = \text{diam} \tilde{M} \). By Lemmas 3.5 and 3.6, \( K(\sigma) = G \circ \mu(d(p, x)) \) for any point \( x \in M \setminus \{p\} \) and any radial plane \( \sigma \) at \( x \). Thus, it follows from Lemma 1 and Theorem 3 in [KK] that \( M \) is isometric to the \( n \)-dimensional model of \( \tilde{M} \). Incidentally, the explicit isometry \( \varphi \) between \( M \) and the \( n \)-dimensional model of \( \tilde{M} \) is given by

\[
\varphi(x) := \begin{cases} 
\exp_p \circ I \circ \exp_p^{-1}(x) & \text{if } x \neq q \\
\tilde{q} & \text{if } x = q,
\end{cases}
\]

where \( I : T_p M \to T_\tilde{p} \tilde{M} \) denotes a linear isometry and \( q \) denotes the unique cut point of \( p \).

**References**


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**Present Address:**

DEPARTMENT OF MATHEMATICS,

TOKAI UNIVERSITY,

HIRATSUKA CITY, KANAGAWA 259 – 1292, JAPAN.

e-mail: nut4297nb@gmail.com