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A Criterion for Dualizing Modules

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Abstract. We establish a characterization of dualizing modules among semidualizing modules. Let R be a finite dimensional commutative Noetherian ring with identity and C a semidualizing R-module. We show that C is a dualizing R-module if and only if $\mathrm{Tor}_i^R(E, E')$ is C-injective for all C-injective R-modules E and E' and all $i \geq 0$.

1. Introduction

Throughout this paper, R will denote a commutative Noetherian ring with non-zero identity. The injective envelope of an R-module M is denoted by $\mathrm{E}_R(M)$.

A finitely generated R-module C is called *semidualizing* if the homothety map $R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. Immediate examples of such modules are free R-modules of rank one. A semidualizing R-module C with finite injective dimension is called *dualizing*. Although R always possesses a semidualizing module, it does not possess a dualizing module in general. Keeping [BH, Theorem 3.3.6] in mind, it is straightforward to see that the ring R possesses a dualizing module if and only if it is Cohen-Macaulay and it is homomorphic image of a finite dimensional Gorenstein ring.

Let $(R, \mathfrak{m}, k)$ be a local ring. There are several characterizations in the literature for a semidualizing R-module C to be dualizing. For instance, Christensen [C, Proposition 8.4] has shown that a semidualizing R-module C is dualizing if and only if the Gorenstein dimension of k with respect to C is finite. Also, Takahashi et al. [TYY, Theorem 1.3] proved that a semidualizing R-module C is dualizing if and only if every finitely generated R-module can be embedded in an R-module of finite C-dimension. Our aim in this paper is to give a new characterization for a semidualizing R-module C to be dualizing.

Let C be a semidualizing R-module. An R-module M is said to be *C-projective* (respectively *C-flat*) if it has the form $C \otimes_R U$ for some projective (respectively flat) R-module U. Also, a *C-free* R-module is defined as a direct sum of copies of C. We can see that every

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C-projective R-module is a direct summand of a C-free R-module and over a local ring every finitely generated C-flat R-module is C-free. Also, an R-module M is said to be *C-injective* if it has the form $\mathrm{Hom}_R(C, I)$ for some injective R-module I.

Yoneda raised a question of whether the tensor product of injective modules is injective. Ishikawa in [I, Theorem 2.4] showed that if $\mathrm{E}_R(R)$ is flat, then $E \otimes_R E'$ is injective for all injective R-modules E and E'. Further, Enochs and Jenda [EJ, Theorem 4.1] proved that R is Gorenstein if and only if for every injective R-modules E and E' and any $i \geq 0$, $\mathrm{Tor}_i^R(E, E')$ is injective. We extend this result in terms of a semidualizing R-module. More precisely, for a semidualizing R-module C, we show that the following are equivalent (see Theorem 2.7):

(i) $C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

(ii) For any prime ideal $\mathfrak{p}$ of R and any $i \geq 0$,

$$\mathrm{Tor}_i^R(\mathrm{E}_C(R/\mathfrak{p}), \mathrm{E}_C(R/\mathfrak{p})) = \begin{cases} 0 & \text{if } i \neq \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}} \\ \mathrm{E}_C(R/\mathfrak{p}) & \text{if } i = \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}}\,, \end{cases}$$

where $\mathrm{E}_C(R/\mathfrak{p}) := \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))$.

(iii) For any C-injective R-modules E and E' and any $i \geq 0$, $\mathrm{Tor}_i^R(E, E')$ is C-injective.

2. The Results

Let $\mathfrak{p}$ be a prime ideal of R. Recall that an R-module M is said to have property $t(\mathfrak{p})$ if for each $r \in R - \mathfrak{p}$, the map $M \xrightarrow{r} M$ is an isomorphism and if for each $x \in M$ we have $\mathfrak{p}^m x = 0$ for some $m \geq 1$. If an R-module M has $t(\mathfrak{p})$-property, then it has the structure as an $R_{\mathfrak{p}}$-module. It is known that $\mathrm{E}_R(R/\mathfrak{p})$ has $t(\mathfrak{p})$-property.

To prove Theorem 2.7, which is our main result, we shall need the following five preliminary lemmas.

LEMMA 2.1. *Let C be a semidualizing R-module. Then the following statements hold true.*

(i) $\mathrm{E}_C(R/\mathfrak{p}) := \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))$ *has $t(\mathfrak{p})$-property for each* $\mathfrak{p} \in \mathrm{Spec}\, R$.

(ii) *If $\mathfrak{p}$ and $\mathfrak{q}$ are two distinct prime ideals of R, then* $\mathrm{Tor}_i^R(\mathrm{E}_C(R/\mathfrak{p}), \mathrm{E}_C(R/\mathfrak{q})) = 0$ *for all $i \geq 0$.*

PROOF. (i) As $\mathrm{E}_R(R/\mathfrak{p})$ has t($\mathfrak{p}$)-property, one can easily check that for any finitely generated R-module M, the R-module $\mathrm{Hom}_R(M, \mathrm{E}_R(R/\mathfrak{p}))$ has $t(\mathfrak{p})$-property.

(ii) By (i) $\mathrm{E}_C(R/\mathfrak{p})$ has $t(\mathfrak{p})$-property and $\mathrm{E}_C(R/\mathfrak{q})$ has $t(\mathfrak{q})$-property. So, [EH, 5] implies that

$$\mathrm{Tor}_i^R(\mathrm{E}_C(R/\mathfrak{p}), \mathrm{E}_C(R/\mathfrak{q})) = 0$$

for all $i \geq 0$. □

LEMMA 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring, C a semidualizing R-module and I an Artinian C-injective R-module. Then $\operatorname{Hom}_R(I, \mathrm{E}_R(k))$ is a finitely generated $\widehat{C}$-free $\widehat{R}$-module.*

PROOF. Denote the functor $\operatorname{Hom}_R(-, \mathrm{E}_R(k))$ by $(-)^\vee$. We have $I = \operatorname{Hom}_R(C, I')$ for some injective R-module I'. Clearly, $C \otimes_R I$ is also an Artinian R-module. Since

$$C \otimes_R I \cong C \otimes_R \operatorname{Hom}_R(C, I') \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, C), I') \cong I',$$

we deduce that I' is also Artinian. So, $I' \cong \overset{n}{\oplus} \mathrm{E}_R(k)$ for some nonnegative integer n.

Now, one has

$$I^\vee = \operatorname{Hom}_R(C, I')^\vee \cong C \otimes_R I'^\vee \cong \overset{n}{\oplus} \widehat{C}\,,$$

and so $I^\vee$ is a finitely generated $\widehat{C}$-free $\widehat{R}$-module. □

In the next result, we collect some useful known properties of semidualizing modules. We may use them without any further comments.

LEMMA 2.3. *Let C be a semidualizing R-module and $\underline{r} := r_1, \ldots, r_n$ a sequence of elements of R. The following statements hold.*

(i) $\operatorname{Supp}_R C = \operatorname{Spec} R$, *and so* $\dim_R C = \dim R$.

(ii) *If R is local, then $\widehat{C}$ is a semidualizing $\widehat{R}$-module.*

(iii) *$\underline{r}$ is a regular R-sequence if and only if $\underline{r}$ is a regular C-sequence.*

(iv) *If $\underline{r}$ is a regular R-sequence, then $C/(\underline{r})C$ is a semidualizing $R/(\underline{r})$-module.*

(v) *If R is local and $\underline{r}$ is a regular R-sequence, then C is a dualizing R-module if and only if $C/(\underline{r})C$ is a dualizing $R/(\underline{r})$-module.*

PROOF. (i) and (ii) follow easily by the definition of a semidualizing module.

(iii) and (iv) are hold by [S, Corollary 3.3.3].

(v) Assume that R is local and $\underline{r}$ is a regular R-sequence. Then by (iv), $C/(\underline{r})C$ is a semidualizing $R/(\underline{r})$-module. On the other hand, [BH, Corollary 3.1.15] yields that

$$\operatorname{id}_{\frac{R}{(\underline{r})}} \frac{C}{(\underline{r})C} = \operatorname{id}_R C - n\,.$$

This implies the conclusion. □

In the proof of the following result, $R \ltimes C$ will denote the trivial extension of R by C. For any $R \ltimes C$-module X, its Gorenstein injective dimension will be denoted by $\operatorname{Gid}_{R \ltimes C} X$. Also, we recall that for a module M over a local ring $(R, \mathfrak{m}, k)$, the width of M is defined by $\operatorname{width}_R M := \inf\{i \in \mathbf{N}_0 | \operatorname{Tor}_i^R(k, M) \neq 0\}$.

LEMMA 2.4. *Let $(R, \mathfrak{m}, k)$ be a local ring and C a semidualizing R-module. Then $\mathrm{E}_C(k) \otimes_R \mathrm{E}_C(k)$ is a non-zero C-injective R-module if and only if C is a dualizing R-module of dimension* 0.

PROOF. Suppose that $\mathrm{E}_C(k) \otimes_R \mathrm{E}_C(k)$ is a non-zero C-injective R-module. As $\mathrm{E}_C(k)$ is Artinian, by [KLS, Corollary 3.9] the length of $\mathrm{E}_C(k) \otimes_R \mathrm{E}_C(k)$ is finite. So, also, $(\mathrm{E}_C(k) \otimes_R \mathrm{E}_C(k))^\vee$ has finite length. Since

$$\mathrm{Hom}_R(\mathrm{E}_C(k), \widehat{C}) \cong (\mathrm{E}_C(k) \otimes_R \mathrm{E}_C(k))^\vee ,$$

by Lemma 2.2, we deduce that $\mathrm{Hom}_R(\mathrm{E}_C(k), \widehat{C})$ is isomorphic to a direct sum of finitely many copies of $\widehat{C}$. This, in particular, implies that $\widehat{C}$ has finite length. Thus Lemma 2.3 yields that

$$\dim R = \dim_R C = \dim_{\widehat{R}} \widehat{C} = 0,$$

and so, in particular, R is complete. Next, one has

$$\begin{aligned} \mathrm{Hom}_R(\mathrm{E}_C(k), R) &\cong \mathrm{Hom}_R(\mathrm{E}_C(k), \mathrm{Hom}_R(C, C)) \\ &\cong \mathrm{Hom}_R(C, \mathrm{Hom}_R(\mathrm{E}_C(k), C)) \\ &\cong \overset{n}{\oplus} \mathrm{Hom}_R(C, C) \\ &\cong R^n \end{aligned}$$

for some $n > 0$. This, in particular, implies that

$$\mathrm{Ann}_R(\mathrm{Hom}_R(\mathrm{E}_C(k), R)) = \mathrm{Ann}_R R .$$

Since R is Artinian, $\mathfrak{m}^t = 0$ and $\mathfrak{m}^{t-1} \neq 0$ for some $t > 0$. If for every $f \in \mathrm{Hom}_R(\mathrm{E}_C(k), R)$, $\mathrm{im}\, f \subseteq \mathfrak{m}$, then $\mathfrak{m}^{t-1} f = 0$ so $\mathfrak{m}^{t-1} \mathrm{Hom}_R(\mathrm{E}_C(k), R) = 0$ a contradiction. Thus there is an epimorphism $\mathrm{E}_C(k) \to R \to 0$, and so R is a direct summand of $\mathrm{E}_C(k)$. Next, [HJ1, Lemma 2.6] implies that R is a Gorenstein injective $R \ltimes C$-module. This yields that C is a dualizing R-module, because by [HJ2, Proposition 4.5], one has

$$\mathrm{id}_R C \leq \mathrm{Gid}_{R \ltimes C} R + \mathrm{width}_R R .$$

Conversely, if C is a dualizing R-module of dimension 0, then $\dim R = 0$ by Lemma 2.3 (i). Hence, $\mathrm{E}_R(k)$ is a dualizing R-module, and then by [BH, Theorem 3.3.4 (b)] we have $C \cong \mathrm{E}_R(k)$. Thus

$$\begin{aligned} \mathrm{E}_C(k) \otimes_R \mathrm{E}_C(k) &\cong \mathrm{Hom}_R(\mathrm{E}_R(k), \mathrm{E}_R(k)) \otimes_R \mathrm{Hom}_R(\mathrm{E}_R(k), \mathrm{E}_R(k)) \\ &\cong R \otimes_R R \\ &\cong R \\ &\cong \mathrm{Hom}_R(C, \mathrm{E}_R(k)) , \end{aligned}$$

which is a non-zero C-injective R-module. □

REMARK 2.5 (See [B, (2.5)]). Let M be an R-module and let $r \in R$ be a non-unit which is a non-zero divisor of both R and M. Let $0 \to M \to I^0 \xrightarrow{d^0} I^1 \to \cdots$ be a minimal injective resolution of M. Then there is a natural $R/(r)$-isomorphism $M/(r)M \cong \mathrm{Hom}_R(R/(r), \mathrm{im}\, d^0)$ and

$$0 \to \mathrm{Hom}_R(R/(r), I^1) \to \mathrm{Hom}_R(R/(r), I^2) \to \cdots$$

is a minimal injective resolution of the $R/(r)$-module $M/(r)M$.

Next, we recall the definition of the notion of co-regular sequences. Let X be an R-module. An element r of R is said to be *co-regular* on X if the map $X \xrightarrow{r} X$ is surjective. A sequence $r_1, \ldots, r_n$ of elements of R is said to be a *co-regular sequence* on X if r_i is co-regular on $(0 :_M (r_1, \ldots, r_{i-1}))$ for all $i = 1, \ldots, n$.

The following result plays a crucial role in the proof of Theorem 2.7.

LEMMA 2.6. *Let $(R, \mathfrak{m}, k)$ be a local ring and C a semidualizing R-module. Let $r \in \mathfrak{m}$ be a non-zero divisor of R. Assume that r is co-regular on* $\mathrm{Tor}_i^R(\mathrm{E}_C(k), \mathrm{E}_C(k))$ *for all i. Then for any $i \geq 0$, we have a natural $\bar{R}$-isomorphism*

$$\mathrm{Tor}_{i-1}^{\bar{R}}(\mathrm{E}_{\bar{C}}(k), \mathrm{E}_{\bar{C}}(k)) \cong \mathrm{Hom}_R(\bar{R}, \mathrm{Tor}_i^R(\mathrm{E}_C(k), \mathrm{E}_C(k))),$$

where $\bar{R} := R/(r)$, $\bar{C} := C/(r)C$, $\mathrm{E}_C(k) := \mathrm{Hom}_R(C, \mathrm{E}_R(k))$ *and* $\mathrm{E}_{\bar{C}}(k) := \mathrm{Hom}_{\bar{R}}(\bar{C}, \mathrm{E}_{\bar{R}}(k))$.

PROOF. Let $0 \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of C. Then

$$\cdots \to \mathrm{Hom}_R(I^1, \mathrm{E}_R(k)) \to \mathrm{Hom}_R(I^0, \mathrm{E}_R(k)) \to 0$$

is a flat resolution of $\mathrm{E}_C(k)$. Applying $\mathrm{E}_C(k) \otimes_R -$, we get the complex

$$\cdots \to \mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^1, \mathrm{E}_R(k)) \to \mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^0, \mathrm{E}_R(k)) \to 0\,.$$

We will denote $\mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k))$ by X_i and set

$$X_\bullet := \cdots \longrightarrow X_i \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0\,.$$

Then for each $i \geq 0$, we have $H_i(X_\bullet) = \mathrm{Tor}_i^R(\mathrm{E}_C(k), \mathrm{E}_C(k))$.

By Remark 2.5,

$$0 \to \mathrm{Hom}_R(\bar{R}, I^1) \to \mathrm{Hom}_R(\bar{R}, I^2) \to \cdots$$

is a minimal injective resolution of $\bar{C}$ as an $\bar{R}$-module. So,

$$\cdots \to \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, I^2), \mathrm{E}_{\bar{R}}(k)) \to \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, I^1), \mathrm{E}_{\bar{R}}(k)) \to 0$$

is a flat resolution of $\mathrm{E}_{\bar{C}}(k)$ as an $\bar{R}$-module. Thus for each $i \geq 1$, the $\bar{R}$-module $\mathrm{Tor}_{i-1}^{\bar{R}}(\mathrm{E}_{\bar{C}}(k), \mathrm{E}_{\bar{C}}(k))$ is isomorphic to the ith homology of the following complex

$$(\star) \cdots \longrightarrow \mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, I^2), \mathrm{E}_{\bar{R}}(k))$$

$$\longrightarrow \mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, I^1), \mathrm{E}_{\bar{R}}(k)) \to 0\,.$$

We shall show that the later complex is isomorphic to the complex $Y_\bullet := \mathrm{Hom}_R(\bar{R}, X_\bullet)$.

Noting that $\mathrm{E}_{\bar{R}}(k) \cong \mathrm{Hom}_R(\bar{R}, \mathrm{E}_R(k))$ and using Adjointness yields that

$$\mathrm{E}_{\bar{C}}(k) = \mathrm{Hom}_{\bar{R}}(\bar{C}, \mathrm{E}_{\bar{R}}(k)) \cong \mathrm{Hom}_R(\bar{R}, \mathrm{E}_C(k)).$$

Hence for each $i \geq 0$, by using Adjointness, Hom-evaluation and Tensor-evaluation, one has the following natural $\bar{R}$-isomorphisms:

$$\begin{aligned}
\mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, I^i), \mathrm{E}_{\bar{R}}(k)) &\cong \mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, I^i), \mathrm{Hom}_R(\bar{R}, \mathrm{E}_R(k)))\\
&\cong \mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{Hom}_R(\mathrm{Hom}_R(\bar{R}, I^i), \mathrm{E}_R(k))\\
&\cong \mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} (\bar{R} \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k)))\\
&\cong \mathrm{Hom}_R(\bar{R}, \mathrm{E}_C(k)) \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k))\\
&\cong \mathrm{Hom}_R(\bar{R}, \mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k)))\\
&\cong Y_i\,.
\end{aligned}$$

Note that $\mathrm{Hom}_R(I^i, \mathrm{E}_R(k))$ is a flat R-module. As r is a non-zero divisor of R, it is also a non-zero divisor of C. This implies that r is a non-zero divisor of I^0, and so $\mathrm{Hom}_R(\bar{R}, I^0) = 0$. Thus

$$Y_0 \cong \mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{Hom}_{\bar{R}}(\mathrm{Hom}_R(\bar{R}, I^0), \mathrm{E}_{\bar{R}}(k)) = 0\,.$$

Therefore, the two complexes $(\star)$ and $Y_\bullet$ are isomorphic, and so we deduce that $\mathrm{Tor}^{\bar{R}}_{i-1}(\mathrm{E}_{\bar{C}}(k), \mathrm{E}_{\bar{C}}(k)) = H_i(Y_\bullet)$ for all $i \geq 0$.

Since r is a non-zero divisor of C, it is co-regular on $\mathrm{E}_C(k)$, and so it is co-regular on X_i for all i. Thus, we can deduce the following exact sequence of complexes

$$0 \longrightarrow Y_\bullet \longrightarrow X_\bullet \xrightarrow{r} X_\bullet \longrightarrow 0\,.$$

It yields the following exact sequences of modules

$$\begin{aligned}
\cdots \longrightarrow \mathrm{Tor}^R_{i+1}(\mathrm{E}_C(k), \mathrm{E}_C(k)) \xrightarrow{r} \mathrm{Tor}^R_{i+1}(\mathrm{E}_C(k), \mathrm{E}_C(k)) \longrightarrow \mathrm{Tor}^{\bar{R}}_{i-1}(\mathrm{E}_{\bar{C}}(k), \mathrm{E}_{\bar{C}}(k))\\
\xrightarrow{f_i} \mathrm{Tor}^R_i(\mathrm{E}_C(k), \mathrm{E}_C(k)) \xrightarrow{r} \mathrm{Tor}^R_i(\mathrm{E}_C(k), \mathrm{E}_C(k)) \longrightarrow \cdots\,.
\end{aligned}$$

As r is a co-regular element on $\mathrm{Tor}^R_i(\mathrm{E}_C(k), \mathrm{E}_C(k))$ for all i, we deduce that f_i is a monomorphism for all i. This implies our desired isomorphisms. □

THEOREM 2.7. *Let C be a semidualizing R-module. The following are equivalent*:

(i) *$C_{\mathfrak{p}}$ is a dualizing $R_{\mathfrak{p}}$-module for all* $\mathfrak{p} \in \mathrm{Spec}\, R$.

(ii) *For any prime ideal* $\mathfrak{p}$ *of R and any* $i \geq 0$,

$$\mathrm{Tor}^R_i(\mathrm{E}_C(R/\mathfrak{p}), \mathrm{E}_C(R/\mathfrak{p})) = \begin{cases} 0 & \text{if } i \neq \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}} \\ \mathrm{E}_C(R/\mathfrak{p}) & \text{if } i = \dim_{R_{\mathfrak{p}}} C_{\mathfrak{p}}\,, \end{cases}$$

where $\mathrm{E}_C(R/\mathfrak{p}) := \mathrm{Hom}_R(C, \mathrm{E}_R(R/\mathfrak{p}))$.

(iii) *For any* C*-injective* R*-modules* E *and* E' *and any* $i \geq 0$, $\mathrm{Tor}_i^R(E, E')$ *is* C*-injective.*

PROOF. (i) $\Rightarrow$ (ii) Let $\mathfrak{p}$ be a prime ideal of R. There are natural $R_\mathfrak{p}$-isomorphisms $\mathrm{E}_C(R/\mathfrak{p}) \cong \mathrm{E}_{C_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ and

$$\mathrm{Tor}_i^R(\mathrm{E}_C(R/\mathfrak{p}), \mathrm{E}_C(R/\mathfrak{p})) \cong \mathrm{Tor}_i^{R_\mathfrak{p}}(\mathrm{E}_{C_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}), \mathrm{E}_{C_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$$

for all $i \geq 0$. Hence, we can complete the proof of this part by showing that if C is a dualizing module of a local ring $(R, \mathfrak{m}, k)$, then

$$\mathrm{Tor}_i^R(\mathrm{E}_C(k), \mathrm{E}_C(k)) = \begin{cases} 0 & i \neq \dim_R C \\ \mathrm{E}_C(k) & i = \dim_R C\,. \end{cases}$$

Set $d := \dim_R C$. As C is a dualizing R-module, [BH, Theorem 3.3.10] implies that for any prime ideal $\mathfrak{p}$, one has

$$\mu^i(\mathfrak{p}, C) = \begin{cases} 0 & i \neq \mathrm{ht}\,\mathfrak{p} \\ 1 & i = \mathrm{ht}\,\mathfrak{p}\,. \end{cases}$$

So, if $I^\bullet = 0 \to I^0 \to I^1 \to \cdots$ is a minimal injective resolution of C, then $I^d \cong \mathrm{E}_R(k)$ and for any $i \neq d$, $\mathrm{E}_R(k)$ is not a direct summand of I^i. In particular, $\mathrm{Hom}_R(R/\mathfrak{m}, I^i) = 0$ for all $i \neq d$. Now, $\mathrm{Hom}_R(I^\bullet, \mathrm{E}_R(k))$ is a flat resolution of $\mathrm{E}_C(k)$. Clearly, one has

$$\mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^d, \mathrm{E}_R(k)) \cong \mathrm{E}_C(k) \otimes_R \widehat{R} \cong \mathrm{E}_C(k)\,.$$

Next, let $i \neq d$. Since $\mathrm{Hom}_R(I^i, \mathrm{E}_R(k))$ is a flat R-module, [M, Theorem 23.2 (ii)] implies that

$$\mathrm{Ass}_R(\mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k))) = \mathrm{Ass}_R(R/\mathfrak{m} \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k)))\,.$$

But,

$$R/\mathfrak{m} \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k)) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{m}, I^i), \mathrm{E}_R(k)) = 0\,,$$

and so $\mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^i, \mathrm{E}_R(k)) = 0$. Therefore, it follows that the complex $\mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^\bullet, \mathrm{E}_R(k))$ has $\mathrm{E}_C(k)$ in its d-place and 0 in its other places. Thus, we deduce that

$$\mathrm{Tor}_i^R(\mathrm{E}_C(k), \mathrm{E}_C(k)) = H_i(\mathrm{E}_C(k) \otimes_R \mathrm{Hom}_R(I^\bullet, \mathrm{E}(k))) = \begin{cases} 0 & i \neq d \\ \mathrm{E}_C(k) & i = d\,. \end{cases}$$

(ii) $\Rightarrow$ (iii) Let E be an injective R-module. Since $E \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}\, R} \mathrm{E}_R(R/\mathfrak{p})^{\mu^0(\mathfrak{p}, E)}$ and C

is finitely generated, we have

$$\operatorname{Hom}_R(C, E) \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} \mathrm{E}_C(R/\mathfrak{p})^{\mu^0(\mathfrak{p}, E)}.$$

As R is Noetherian, clearly any direct sum of C-injective R-modules is again C-injective, and so (ii) yields (iii) by Lemma 2.1 (ii).

(iii) $\Rightarrow$ (i) It is easy to check that a given $R_{\mathfrak{p}}$-module M is $C_{\mathfrak{p}}$-injective if and only if it is the localization at $\mathfrak{p}$ of a C-injective R-module. Thus, it is enough to show that if C is a semidualizing module of a local ring $(R, \mathfrak{m}, k)$ such that $\operatorname{Tor}_i^R(E, E')$ is C-injective for all C-injective R-modules E and E' and all $i \geq 0$, then C is dualizing.

Let $\underline{r} = r_1, \ldots, r_d \in \mathfrak{m}$ be a maximal regular R-sequence. Then $\underline{r}$ is also a regular C-sequence. It is easy to verify that $\underline{r}$ is a co-regular sequence on any C-injective R-module, and consequently $\underline{r}$ is a co-regular sequence on $\operatorname{Tor}_i^R(\mathrm{E}_C(k), \mathrm{E}_C(k))$ for all $i \geq 0$. Letting $\bar{R} := R/(\underline{r})$ and $\bar{C} := C/(\underline{r})C$, by Lemma 2.3 (iv), it turns out that $\bar{C}$ is a semidualizing $\bar{R}$-module. Making repeated use of Lemma 2.6, we can establish the following natural $\bar{R}$-isomorphism

$$\mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{E}_{\bar{C}}(k) \cong \operatorname{Hom}_R(\bar{R}, \operatorname{Tor}_d^R(\mathrm{E}_C(k), \mathrm{E}_C(k))).$$

So, $\mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{E}_{\bar{C}}(k)$ is a $\bar{C}$-injective $\bar{R}$-module. Lemma 2.3 implies that

$$\operatorname{depth}_{\widehat{\bar{R}}} \widehat{\bar{C}} = \operatorname{depth}_{\bar{R}} \bar{C} = \operatorname{depth}_{\bar{R}} \bar{R} = 0,$$

and so there are natural inclusion maps $k \overset{i}{\hookrightarrow} \bar{C}$ and $k \overset{j}{\hookrightarrow} \widehat{\bar{C}}$. By applying the functor $\operatorname{Hom}_{\bar{R}}(-, \mathrm{E}_{\bar{R}}(k))$ on i, we get an epimorphism $\mathrm{E}_{\bar{C}}(k) \twoheadrightarrow k$. Next, by applying the functor $\operatorname{Hom}_{\bar{R}}(-, \widehat{\bar{C}})$ on the later map, we see that

$$\operatorname{Hom}_{\bar{R}}(\mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{E}_{\bar{C}}(k), \mathrm{E}_{\bar{R}}(k)) \cong \operatorname{Hom}_{\bar{R}}(\mathrm{E}_{\bar{C}}(k), \widehat{\bar{C}}) \neq 0\,.$$

Hence, $\mathrm{E}_{\bar{C}}(k) \otimes_{\bar{R}} \mathrm{E}_{\bar{C}}(k)$ is a non-zero $\bar{C}$-injective $\bar{R}$-module, and so Lemma 2.4 yields that $\bar{C}$ is a dualizing $\bar{R}$-module. Now, by Lemma 2.3 (v), we deduce that C is a dualizing R-module. □

We end the paper with the following immediate corollary.

COROLLARY 2.8. *Let R be a finite dimensional ring and C a semidualizing R-module. Then C is a dualizing R-module if and only if* $\operatorname{Tor}_i^R(E, E')$ *is C-injective for all C-injective R-modules E and E' and all $i \geq 0$.*

References

[B] H. BASS, On the ubiquity of Gorenstein rings, Math. Z. **82** (1963), 8–28.

[BH] W. BRUNS and J. HERZOG, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[C] L.W. CHRISTENSEN, Semi-dualizing complexes and their Auslander categories, Trans. Amer. Math. Soc. **353**(5), (2001), 1839–1883.

[EH] E. E. ENOCHS and Z. HUANG, Canonical filtrations of Gorenstein injective modules, Proc. Amer. Math. Soc. **139**(7), (2011), 2415–2421.

[EJ] E. E. ENOCHS and O. M. G. JENDA, Tensor and torsion products of injective modules, J. Pure Appl. Algebra **76**(2), (1991), 143–149.

[HJ1] H. HOLM and P. JØRGENSEN, Semi-dualizing modules and related Gorenstein homological dimensions, J. Pure Appl. Algebra **205**(2), (2006), 423–445.

[HJ2] H. HOLM and P. JØRGENSEN, Cohen-Macaulay homological dimensions, Rend. Semin. Mat. Univ. Padova **117** (2007), 87–112.

[I] T. ISHIKAWA, On injective modules and flat modules, J. Math. Soc. Japan **17**(3), (1965), 291–292.

[KLS] B. KUBIK, M. J. LEAMER and S. SATHER-WAGSTAFF, Homology of Artinian and Matlis reflexive modules, I, J. Pure Appl. Algebra **215**(10), (2011), 2486–2503.

[M] H. MATSUMURA, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1986.

[S] S. SATHER-WAGSTAFF, Semidualizing modules, 2009, preprint.

[TYY] R. TAKAHASHI, S. YASSEMI and Y. YOSHINO, On the existence of embeddings into modules of finite homological dimensions, Proc. Amer. Math. Soc. **138**(7), (2010), 2265–2268.

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