TOKYO J. MATH.
VOL. 38, NO. 1, 2015

Effective Nonvanishing of Pluri-adjoint Line Bundles

Tomoki ARAKAWA

Sophia University

(Communicated by H. Tahara)

Abstract. Let L be an ample line bundle over a smooth complex projective n-fold X such that $K_X + L$ is nef. We prove that $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$ holds for every $m \geq n(n+1)/2 + 2$.

1. Introduction

Let X be a smooth projective variety over $\mathbf{C}$ and L an ample line bundle over X. Then the pair (X, L) is called a *polarized manifold*.

In the classification theory of polarized manifolds, it is important to study a condition on the integer m for which $|K_X + mL|$ is free. Fujita's freeness conjecture ([6]) predicts that $|K_X + mL|$ is free for any $m \geq \dim X + 1$. In [11], Y. Kawamata gave an affirmative answer to this conjecture in the case of $\dim X \leq 4$. In higher dimensional case, H. Tsuji ([17]) proved that $|K_X + mL|$ is free for any $m \geq \dim X(\dim X + 1)/2 + 1$ (see also [2]).

On the other hand when $K_X + L$ is nef, by virtue of the nonvanishing theorem due to V. Shokurov ([16]), $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$ holds for $m \gg 0$. Then it is important to find an integer m with $H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$. Concerning this, F. Ambro ([1]) and Y. Kawamata ([12]) proposed the following conjecture:

CONJECTURE 1.1. *Let X be a normal projective variety and let B be a $\mathbf{Q}$-effective divisor on X such that (X, B) is a KLT pair. Let D be a nef Cartier divisor on X such that $D - (K_X + B)$ is nef and big. Then $H^0(X, \mathcal{O}_X(D)) \neq 0$ holds.*

We note that if X is smooth, $B = 0$ and $D := K_X + L$ is nef, this implies that $H^0(X, \mathcal{O}_X(K_X + L)) \neq 0$ holds for every polarized manifold (X, L) with $K_X + L$ nef. In [12], Kawamata solved the conjecture above when X is 2-dimensional and when X is a minimal 3-fold. In [10], A. Hoering solved it when X is a normal projective 3-fold with at most $\mathbf{Q}$-factorial canonical singularities, $B = 0$, and $D - K_X$ is a nef and big Cartier divisor. These results are immediate consequences of the Hirzebruch-Riemann-Roch theorem and some classical results on surfaces and 3-folds. However in higher dimensional case, it is

Received August 6, 2013; revised October 6, 2014
Mathematics Subject Classification: 14C20 (Primary), 14J40 (Secondary)
Key words and phrases: Polarized manifolds, adjoint bundles, effective nonvanishing

rather difficult to calculate $\dim H^0(X, \mathcal{O}_X(D))$. In fact, Conjecture 1.1 is still widely open for the case of $\dim X \geq 4$.

Concerning the effective nonvanishing of global sections of pluri-adjoint line bundles, Y. Fukuma ([7]) proposed the following problem:

PROBLEM 1.2. *Let (X, L) be a polarized manifold of dimension n with $\kappa(K_X + L) \geq 0$. Find a positive integer m_n depending only on n such that*

$$H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$$

holds for every $m \geq m_n$.

For the case of $\dim X \leq 4$, several results have been obtained (cf. [7], [8], [10]). In higher dimensional case, the problem is yet to be investigated. In fact the existence of the uniform bound m_n as above still remains open.

The purpose of this article is to give the following partial answer to Problem 1.2 in higher dimensional case.

THEOREM 1.3. *Let (X, L) be a polarized manifold of dimension n such that $K_X + L$ is nef. Then*

$$H^0(X, \mathcal{O}_X(m(K_X + L))) \neq 0$$

holds for every positive integer $m \geq n(n+1)/2 + 2$.

We prove the theorem above in Section 4. The outline of the proof is as follows: By the nefness of $K_X + L$ and the base point free theorem, we obtain a surjective morphism $f : X \longrightarrow Y$ to a smooth projective variety Y. Moreover, we obtain a big divisor B on Y such that $K_Y + B$ is nef and big (Lemmas 4.2 and 4.3). Then we can reduce the problem to the study of the effective nonvanishing of global sections of multiples of $K_Y + B$. Namely, it suffices to show the following:

THEOREM 1.4. *Let Y be a smooth projective variety of dimension d, N a nef and big divisor on Y and B a big divisor on Y. Then*

$$H^0(Y, \mathcal{O}_Y(K_Y + mN + B)) \neq 0$$

holds for every $m \geq d(d+1)/2 + 1$.

Letting $N := K_Y + B$, by the theorem above, we deduce Theorem 1.3. In Section 3, we prove Theorem 1.4, using the Kawamata-Shokurov concentration method ([13, Chapter 2]) and the technique adopted by Angehrn and Siu ([2]), and Tsuji ([17]) in their study of Fujita's freeness conjecture.

Throughout this paper, we work over the field of complex numbers $\mathbf{C}$. Furthermore we use the words *line bundle* and *invertible sheaf* interchangeably. If D is a Cartier divisor on a variety X, then $\mathcal{O}_X(D)$ denotes the corresponding line bundle over X.

2. Preliminaries

In this section, we review some algebraic and analytic notions.

2.1. Nef and big line bundles. In this subsection, we shall recall some properties of nef and big line bundles.

DEFINITION 2.1. Let X be a normal variety and D a Cartier divisor on X. Then

1. D is said to be *nef*, if $D \cdot C \geq 0$ holds for every irreducible curve C on X.

2. D is said to be *big*, if $\kappa(X, D) = \dim X$ holds,

where $\kappa(X, D)$ denotes the Iitaka-Kodaira dimension of D defined by

$$\kappa(X, D) = \limsup_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(mD))}{\log m} .$$

The following proposition called Kodaira's lemma is fundamental.

PROPOSITION 2.2. *Let X be a normal complete variety and D a big divisor on X. Then for an arbitrary divisor M, we have $|mD - M| \neq \emptyset$ for $m \gg 0$.*

PROOF. See [13, Lemma 0-0-3]. □

By using Kodaira's lemma and the Riemann-Roch theorem, we have the following:

PROPOSITION 2.3. *Let X be a smooth projective variety of dimension n and D a nef and big divisor on X. Then*

$$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1})$$

holds, where $h^0(X, \mathcal{O}_X(mD)) := \dim H^0(X, \mathcal{O}_X(mD))$.

PROOF. See [5, Corollary 4.3]. □

2.2. Singularities of divisors. In this subsection, we review the notion of singularities of pairs.

DEFINITION 2.4. Let X be a normal variety and $U := X_{\mathrm{reg}}$ the nonsingular locus of X. Since $\mathrm{codim}(X \setminus U) \geq 2$ holds, every divisor on X is uniquely determined by its restriction to U (cf. [9, Chapter II]). Then we can define the *canonical sheaf* $\omega_X = \mathcal{O}_X(K_X)$ of X by

$$\omega_X := i_* \mathcal{O}_U(K_U) ,$$

where $i : U \hookrightarrow X$ is the inclusion.

DEFINITION 2.5. Let (X, D) be a pair of a normal variety X and an effective $\mathbf{Q}$-divisor D on X. A *log resolution* of (X, D) is a proper birational morphism $\mu : Y \longrightarrow X$ such that Y is smooth and $\mathrm{Exc}(\mu) \cup \mu_*^{-1} D$ has a simple normal crossing support, where $\mathrm{Exc}(\mu) \subset X$ denotes the exceptional locus of μ, and $\mu_*^{-1} D$ denotes the strict transform of D on Y.

In order to consider singularities of pairs, we shall introduce the following notion.

DEFINITION 2.6. Let (X, D) be a pair of a normal variety and an effective $\mathbf{Q}$-divisor on X. Suppose that $K_X + D$ is $\mathbf{Q}$-Cartier. Let $\mu : Y \to X$ be a log resolution of (X, D). Then we have the formula:

$$K_Y = \mu^*(K_X + D) + \sum_i a_i E_i ,$$

where E_i is a prime divisor and $a_i \in \mathbf{Q}$. We call a_i the *discrepancy coefficient* for E_i. The pair (X, D) is said to have only *Kawamata log terminal singularities* (**KLT**) (resp. *log canonical singularities* (**LC**)), if $a_i > -1$ (resp. $a_i \geq -1$) holds for every i. The pair (X, D) is said to be KLT (resp. LC) at a point $x \in X$, if $(U, D|_U)$ is KLT (resp. LC) for some neighborhood U of x.

DEFINITION 2.7. Let (X, D) be a pair of a normal variety and an effective $\mathbf{Q}$-divisor on X. A subvariety W of X is said to be a *center of log canonical singularities* for (X, D), if there are a log resolution $\mu : Y \longrightarrow X$ and a prime divisor E on Y with the discrepancy coefficient $e \leq -1$ such that $\mu(E) = W$. The set of all centers of log canonical singularities for (X, D) is denoted by $CLC(X, D)$. For a point $x \in X$, we set $CLC(X, x, D) := \{W \in CLC(X, D) \mid x \in W\}$.

PROPOSITION 2.8. *Let (X, D) be a pair of a normal variety and an effective $\mathbf{Q}$-Cartier divisor such that $K_X + D$ is $\mathbf{Q}$-Cartier. Assume that X is KLT and (X, D) is LC. If W_1 and W_2 are the elements of $CLC(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. This implies that if (X, D) is LC but not KLT at a point $x \in X$, there exists the unique minimal element of $CLC(X, x, D)$.* (This minimal element is called the *minimal center of log canonical singularities* of (X, D) at x.)

PROOF. See [11, Proposition 1.5]. □

2.3. Singular hermitian metrics and multiplier ideal sheaves. In the proof of Theorem 1.3, we use singular hermitian metrics as in [5]. We here review the notions of singular hermitian metrics and multiplier ideal sheaves.

DEFINITION 2.9. Let L be a holomorphic line bundle over a complex manifold X. A *singular hermitian metric h* on L is given by

$$h = h_0 \cdot e^{-\varphi} ,$$

where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(X)$ is an L^1-local function on X. Then we call φ the *weight function* of h with respect to h_0, and (L, h) a *singular hermitian line bundle* over X. The *curvature current Θ_h* of h is defined by

$$\Theta_h = \Theta_{h_0} + \sqrt{-1}\partial\overline{\partial}\varphi ,$$

where Θ_{h_0} denotes the curvature form of h_0 and $\partial\overline{\partial}$ is taken in the sense of currents.

EXAMPLE 2.10. Let L be a holomorphic line bundle over a complex manifold X. Suppose that there exists a positive integer m with $\Gamma(X, L^{\otimes m}) \neq 0$. Let $\sigma \in \Gamma(X, L^{\otimes m})$ be a nontrivial section. Then

$$h := \frac{1}{|\sigma|^{2/m}} = \frac{h_0}{h_0^{\otimes m}(\sigma, \sigma)^{1/m}}$$

is a singular hermitian metric on L, where h_0 is an arbitrary C^∞-hermitian metric on L (the right hand side is independent of h_0). Then by Poincaré-Lelong's formula, the curvature current Θ_h is given by

$$\Theta_h = \frac{2\pi}{m}(\sigma)\,,$$

where (σ) denotes the current of integration over the divisor of σ.

DEFINITION 2.11. Let (L, h) be a singular hermitian line bundle over a complex manifold X. The L^2-sheaf $\mathcal{L}^2(L, h)$ of (L, h) is defined by

$$\mathcal{L}^2(L, h)(U) := \{\sigma \in \Gamma(U, L) \mid h(\sigma, \sigma) \in L^1_{\mathrm{loc}}(U)\}\,,$$

where U runs over the open subsets of X.

Now we shall write h as $h = h_0 \cdot e^{-\varphi}$, where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{\mathrm{loc}}(X)$ is the weight function of h with respect to h_0. Then we define the *multiplier ideal sheaf* $\mathcal{I}(h)$ of (L, h) by

$$\mathcal{I}(h) := \mathcal{L}^2(\mathcal{O}_X, e^{-\varphi})\,.$$

Then we see that $\mathcal{L}^2(L, h) = \mathcal{O}_X(L) \otimes \mathcal{I}(h)$ holds.

Multiplier ideal sheaves are very useful in investigating singularities of pairs. Using the notation above, we shall define the multiplier ideal sheaves of divisors as follows:

DEFINITION 2.12. Let X be a smooth projective variety and let $D = \sum_i a_i D_i$ be an effective $\mathbf{Q}$-divisor on X. For every i, let $\sigma_i \in \Gamma(X, \mathcal{O}_X(D_i))$ be a global section with divisor D_i, and let h_i be a C^∞-hermitian metric on $\mathcal{O}_X(D_i)$. Then we define the multiplier ideal sheaf $\mathcal{I}(D)$ of the divisor D by

$$\mathcal{I}(D) := \mathcal{I}\left(\frac{1}{\prod_i h_i(\sigma_i, \sigma_i)^{a_i}}\right)\,.$$

The following proposition reveals a relation between multiplier ideal sheaves and singularities of pairs (cf. Definition 2.6).

PROPOSITION 2.13. *Let X be smooth projective variety of dimension n and let D be an effective $\mathbf{Q}$-divisor on X. Then (X, D) is KLT at a point x of X, if and only if $\mathcal{I}(D)_x = \mathcal{O}_{X,x}$ holds. In particular, if the multiplicity of D at x is greater than or equal to n, then $\mathcal{I}(D)_x \neq \mathcal{O}_{X,x}$.*

PROOF. See [14, Proposition 3.20]. □

In proving Theorem 1.4, the following vanishing theorem by Nadel ([15]) plays an important role.

THEOREM 2.14. *Let (L, h) be a singular hermitian line bundle over a compact Kähler manifold X and ω be a Kähler form on X. Suppose that the curvature current Θ_h of h is strictly positive, i.e., there exists a positive constant $\varepsilon > 0$ such that $\Theta_h - \varepsilon\omega$ is a positive $(1,1)$-current. Then $\mathcal{I}(h)$ is a coherent sheaf on X, and*

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h)) = 0$$

holds for every $q \geq 1$.

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4. We set $\mu_0 := N^d$ and fix a point y_0 on Y. First we shall construct a singular hemitian metric on some multiple of N with sufficiently large singularity at y_0.

LEMMA 3.1. *Let ε be a positive number with $\varepsilon < 1$. Then*

$$H^0(Y, \mathcal{O}_Y(mN) \otimes \mathfrak{m}_{y_0}^{\lceil \sqrt[d]{\mu_0}(1-\varepsilon)m \rceil}) \neq 0$$

holds for any sufficiently large integer m.

PROOF. We consider the exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_Y(mN) \otimes \mathfrak{m}_{y_0}^{\lceil \sqrt[d]{\mu_0}(1-\varepsilon)m \rceil} &\longrightarrow \mathcal{O}_Y(mN) \\ &\longrightarrow \mathcal{O}_Y(mN) \otimes \mathcal{O}_Y/\mathfrak{m}_{y_0}^{\lceil \sqrt[d]{\mu_0}(1-\varepsilon)m \rceil} \longrightarrow 0\,. \end{aligned}$$

Since N is nef and big, by Proposition 2.3, we have

$$h^0(Y, \mathcal{O}_Y(mN)) = \frac{\mu_0}{d!} m^d + O(m^{d-1}) \tag{3.1}$$

for $m \gg 1$.

On the other hand, for a positive integer k, the sheaf $\mathcal{O}_Y/\mathfrak{m}_{y_0}^k$ is a skyscraper sheaf at y_0 and its rank is equal to

$$\binom{d+k-1}{d} = \frac{k^d}{d!} + O(k^{d-1})\,.$$

Then letting $k := \lceil \sqrt[d]{\mu_0}(1-\varepsilon)m \rceil$, by the exact sequence above and (3.1), we obtain the conclusion. □

Let us fix a positive number $\varepsilon < 1$ and a positive integer m_0 with

$$H^0(Y, \mathcal{O}_Y(m_0 N) \otimes \mathfrak{m}_{y_0}^{\lceil \sqrt[d]{\mu_0}(1-\varepsilon)m_0 \rceil}) \neq 0$$

as in Lemma 3.1. Then we take a nontrivial global section

$$\sigma_0 \in H^0(Y, \mathcal{O}_Y(m_0 N) \otimes \mathfrak{m}_{y_0}^{\lceil \sqrt[d]{\mu_0}(1-\varepsilon)m_0 \rceil}) .$$

We define the effective $\mathbf{Q}$-divisor Δ_0 on Y by

$$\Delta_0 = \frac{1}{m_0}(\sigma_0)$$

and the singular hermitian metric h_0 on N by

$$h_0 = \frac{1}{|\sigma_0|^{2/m_0}} .$$

Let α_0 be the positive number defined by $\alpha_0 := \inf\{\alpha > 0 \mid \mathcal{I}(h_0^{\alpha})_{y_0} \neq \mathcal{O}_{Y,y_0}\}$. Then by Lemma 2.13, we get the inequality:

$$\alpha_0 \leq \frac{d}{\sqrt[d]{\mu_0}(1-\varepsilon)} .$$

Fix an arbitrary positive number $\lambda \ll 1/d$. Since $\mu_0 \geq 1$ holds, taking ε sufficiently small, we may assume that $\alpha_0 \leq d + \lambda$ holds.

Let Y_1 be the minimal center of log canonical singularities of $(Y, \alpha_0 \Delta_0)$ at y_0, i.e., Y_1 is the unique minimal element of $CLC(Y, y_0, \alpha_0 \Delta_0)$ (cf. Proposition 2.8). Then one of the following two cases occurs.

1. $\dim Y_1 = 0$, i.e., $Y_1 = \{y_0\}$ holds.
2. $\dim Y_1 > 0$ holds.

First we consider the case of $\dim Y_1 = 0$. In this case, we can conclude the following:

LEMMA 3.2. *If* $\dim Y_1 = 0$ *holds, then*

$$H^0(Y, \mathcal{O}_Y(K_Y + mN + B)) \neq 0$$

holds for every $m \geq d + 1$.

PROOF. We fix an integer $m > \alpha_0$. Then by $\alpha_0 \leq d + \lambda$, we have $m \geq d + 1$.

Since N is big, by Proposition 2.2, there exists an effective $\mathbf{Q}$-divisor G on Y such that $N - G$ is ample. We may assume that the support of G does not contain y_0. Let $\delta > 0$ be a sufficiently small rational number, and we set $A := (m - \alpha_0)N - \delta G$. Note that A is ample, since N is nef. We fix a C^∞-hermitian metric h_A on A with strictly positive curvature. Then we define the singular hermitian metric h on mN by

$$h := \frac{h_0^{\alpha_0} \cdot h_A}{|\sigma_G|^{2\delta}} ,$$

where σ_G is a multivalued holomorphic section of the $\mathbf{Q}$-line bundle G with divisor G. Since $h \cdot h_B$ has strictly positive curvature current, it follows from Theorem 2.14 that

$$H^1(Y, \mathcal{O}_Y(K_Y + mN + B) \otimes \mathcal{I}(h \cdot h_B)) = 0$$

holds. This implies that the restriction map:

$$H^0(Y, \mathcal{O}_Y(K_Y + mN + B)) \longrightarrow H^0(Y, \mathcal{O}_Y(K_Y + mN + B) \otimes \mathcal{O}_Y/\mathcal{I}(h \cdot h_B)) \tag{3.2}$$

is surjective. Now we may assume that y_0 is not on the singular locus of h_B, and hence $\mathcal{O}_Y/\mathcal{I}(h \cdot h_B)$ has isolated support at y_0. Therefore by the surjectivity of (3.2), there exists a global section $\tau \in H^0(Y, \mathcal{O}_Y(K_Y + mN + B)$ with $\tau(y_0) \neq 0$. In particular we obtain $H^0(Y, \mathcal{O}_Y(K_Y + mN + B)) \neq 0$. We have thus proved the lemma. □

Next we consider the second case: $\dim Y_1 > 0$. In this case, we need to cut down the support of $\mathcal{O}_Y/\mathcal{I}(h_0^{\alpha_0})$. Let Y° be the set of points y on Y such that $|mN|$ is free at y and $\Phi_{|mN|}$ is an isomorphism on a neighborhood of y for some $m \geq 1$. Let y_1 be a regular point of Y_1 with $y_1 \in Y^\circ$, and we set $\mu_1 := N^{d_1} \cdot Y_1$. Then we see that $\mu_1 \geq 1$ holds. In the same way as the proof of Lemma 3.1, we have the following:

LEMMA 3.3.

$$H^0(Y_1, \mathcal{O}_{Y_1}(mN) \otimes \mathfrak{m}_{y_1}^{\lceil \sqrt[d_1]{\mu_1}(1-\varepsilon)m \rceil}) \neq 0$$

holds for every positive number $\varepsilon < 1$ and every integer $m \gg 1$.

Let l_1 be an arbitrary positive integer, which will be specified later. Let $\varepsilon' < 1$ be a sufficiently small positive number and m_1 a sufficiently large integer with

$$H^0(Y_1, \mathcal{O}_{Y_1}(m_1 l_1 N) \otimes \mathfrak{m}_{y_1}^{\lceil \sqrt[d_1]{\mu_1}(1-\varepsilon')m_1 l_1 \rceil}) \neq 0$$

as in Lemma 3.3. Then we take a nontrivial section

$$\sigma_1' \in H^0(Y_1, \mathcal{O}_{Y_1}(m_1 l_1 N) \otimes \mathfrak{m}_{y_1}^{\lceil \sqrt[d_1]{\mu_1}(1-\varepsilon')m_1 l_1 \rceil}).$$

As in the proof of Lemma 3.2, G is the effective $\mathbf{Q}$-divisor on Y such that $N - G$ is ample. Let k_1 be a positive integer such that $A_1 := k_1(N - G)$ is Cartier. Then we have the following:

LEMMA 3.4. *There exists a global section $\sigma_1 \in H^0(Y, \mathcal{O}_Y(m_1(l_1 + k_1)N))$ such that its vanishing order at y_1 is at least $\lceil \sqrt[d_1]{\mu_1}(1 - \varepsilon')m_1 l_1 \rceil$, and the divisor (σ_1) is smooth on $Y \setminus (Y_1 \cup \operatorname{Supp} G)$.*

PROOF. By Serre's vanishing theorem, taking m_1 sufficiently large, we may assume that $H^1(Y, \mathcal{O}_Y(m_1(l_1 N + A_1)) \otimes \mathcal{I}_{Y_1}) = 0$ holds, where $\mathcal{I}_{Y_1}$ denotes the ideal sheaf of Y_1. We may also assume that $H^0(Y_1, \mathcal{O}_{Y_1}(m_1 A_1)) \neq 0$ holds. Then the restriction map:

$$H^0(Y, \mathcal{O}_Y(m_1(l_1 N + A_1))) \longrightarrow H^0(Y_1, \mathcal{O}_{Y_1}(m_1(l_1 N + A_1))) \tag{3.3}$$

is surjective.

Let τ be a general section in $H^0(Y_1, \mathcal{O}_{Y_1}(m_1 A_1))$. Then by the surjectivity of (3.3), the section:

$$\sigma_1' \otimes \tau \in H^0(Y_1, \mathcal{O}_{Y_1}(m_1(l_1 N + A_1)) \otimes \mathfrak{m}_{y_1}^{\lceil \sqrt[d_1]{\mu_1}(1-\varepsilon')m_1 l_1 \rceil})$$

extends to a global section $\sigma_1 \in H^0(Y, \mathcal{O}_Y(m_1(l_1 N + A_1)))$. Since G is effective, we may consider σ_1 as an element of $H^0(Y, \mathcal{O}_Y(m_1(l_1 + k_1)N))$. By virtue of Bertini's theorem and the surjectivity of (3.3), we can take σ_1 such that (σ_1) is smooth on $Y \setminus (Y_1 \cup \operatorname{Supp} G)$. So we are done. □

We define the **Q**-divisor Δ_1 on Y by

$$\Delta_1 := \frac{1}{m_1(l_1 + k_1)}(\sigma_1)\,,$$

and the singular hermitian metric h_1 on N by

$$h_1 := \frac{1}{|\sigma_1|^{2/m_1(l_1+k_1)}}\,.$$

Now let us fix a sufficiently small rational number $0 < \varepsilon_0 < 1$. We set

$$\alpha_1 := \inf\{\alpha > 0 \mid \mathcal{I}(h_0^{\alpha_0-\varepsilon_0} \cdot h_1^{\alpha})_{y_1} \neq \mathcal{O}_{Y,y_1}\}\,.$$

Then we get the following:

LEMMA 3.5. *$\alpha_1 \leq d_1 + \lambda$ holds, where $\lambda > 0$ is the fixed number as mentioned earlier.*

For the proof, we use the following elementary lemma (cf. [17, Lemma 6]).

LEMMA 3.6. *Let a, b be positive numbers. Then*

$$\int_0^1 \frac{r_2^{2d_1-1}}{(r_1^2 + r_2^{2a})^b} dr_2 = r_1^{2\left(\frac{d_1}{a} - b\right)} \int_0^{r_1^{-a}} \frac{r_3^{2d_1-1}}{(1 + r_3^{2a})^b} dr_3$$

holds, where $r_3 := r_2 \cdot r_1^{-1/a}$.

PROOF OF LEMMA 3.5. Let $(z_1, \ldots, z_d)$ be a local coordinate on a neighbourhood U of y_1 in Y with $Y_1 \cap U = \{q \in U \mid z_{d_1+1}(q) = \cdots = z_d(q) = 0\}$. Then we set $r_1 := (\sum_{i=d_1+1}^{d} |z_i|^2)^{1/2}$ and $r_2 := (\sum_{i=1}^{d_1} |z_i|^2)^{1/2}$. Now we fix a C^∞-hermitian metric h_N on N and set $a := \lceil \sqrt[d_1]{\mu_1}(1 - \varepsilon) m_1 l_1 \rceil$. By the construction of σ_1, there exists a positive constant C such that $|\sigma_1|^2 \leq C(r_1^2 + r_2^{2a})$ holds on a neighbourhood of y_1. Here $|\cdot|$ denotes the norm with respect to $h_N^{m_1(l_1+k_1)}$.

On the other hand, there exists a positive integer M such that $|\sigma_0|^{-2} = O(r_1^{-M})$ holds on a neighbourhood of the generic point of $Y_1 \cap U$, where $|\cdot|$ denotes the norm with respect to $h_N^{m_0}$.

Then by the definition of α_1 and Lemma 3.6, we have the inequality:

$$\alpha_1 \leq \frac{l_1 + k_1}{l_1} \cdot \frac{d_1}{\sqrt[d_1]{\mu_1}(1 - \varepsilon')} + \frac{m_1(l_1 + k_1)}{m_0} M \varepsilon_0\,.$$

Taking l_1 sufficiently large and taking ε' and ε_0 sufficiently small, we obtain the desired inequality. □

Let Y_2 denote the minimal center of LC singularities of $(Y, (\alpha_0 - \varepsilon_0)\Delta_0 + \alpha_1 \Delta_1)$ at y_1. Since $(Y, (\alpha_0 - \varepsilon_0)\Delta_0)$ is KLT and (σ_1) is smooth on $Y \setminus (Y_1 \cup \operatorname{Supp} G)$, we see that Y_2 is a proper subvariety of Y_1.

Repeating the same process, we finally obtain the strictly decreasing sequence of subvarieties of Y:

$$Y \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_r \supsetneq Y_{r+1} = \{y_r\},$$

points on Y:

$$y_0, y_1, \ldots, y_r,$$

where y_i is a regular point on Y_i with $y_i \in Y^\circ$ for each i, and positive numbers:

$$\alpha_0, \alpha_1, \ldots, \alpha_r,$$

with the estimates $\alpha_i \le d_i + \lambda$, where $d_i := \dim Y_i$. Furthermore, we obtain global sections:

$$\sigma_0 \in H^0(Y, \mathcal{O}_Y(m_0 N)), \quad \sigma_i \in H^0(Y, \mathcal{O}_Y(m_i(l_i + k_1)N))) \quad (i = 1, \ldots, r)$$

for some positive integers $m_0, \ldots, m_r$ and $l_1, \ldots, l_r$.

Then for $2 \le i \le r$, let Δ_i be the effective $\mathbf{Q}$-divisor defined by

$$\Delta_i := \frac{1}{m_i(l_i + k_1)}(\sigma_i),$$

and h_i the singular hermitian metric on N defined by

$$h_i := \frac{1}{|\sigma_i|^{2/m_i(l_i + k_1)}}.$$

We note that Y_j is the minimal center of log canonical singularities of the pair $(Y, (\sum_{i=0}^{j-2}(\alpha_i - \varepsilon_i)\Delta_i + \alpha_{j-1}\Delta_{j-1})$ at y_{j-1} for every $j = 2, \ldots, r+1$. Then we can complete the proof by an argument similar to that in the proof of Lemma 3.2 as follows.

Let m be a positive integer with $m > \sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i) + \alpha_r$. Since $\alpha_i \le d_i + \lambda$ holds for every i, we have $m \ge d(d+1)/2 + 1$. Then we define the singular hermitian metric h_+ on $(\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i) + \alpha_r)N$ by

$$h_+ := \left(\prod_{i=0}^{r-1} h_i^{\alpha_i - \varepsilon_i}\right) \cdot h_r^{\alpha_r}.$$

By the construction of h_i, we see that $\mathcal{O}_Y/\mathcal{I}(h_+)$ has isolated support at y_r.

Let us fix a sufficiently small rational number $\delta > 0$. Let A' be the $\mathbf{Q}$-divisor on Y defined by $A' := (m - \sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i) - \alpha_r)N - \delta G$. Since A' is ample, there exists a C^∞-hermitian metric $h_{A'}$ on A' with strictly positive curvature. Let h' be the singular hermitian metric on mN defined by

$$h' := \frac{h_+ \cdot h_{A'}}{|\sigma_G|^{2\delta}},$$

where σ_G is a multivalued holomorphic section of G with divisor G. Taking y_0 generic, we may assume that y_r is not contained in the support of G. Hence we see that h' has strictly positive curvature current and the support of $\mathcal{O}_Y/\mathcal{I}(h')$ is isolated at y_r. Then by Theorem 2.14, we have $H^1(Y, \mathcal{O}_Y(K_Y + mN + B) \otimes \mathcal{I}(h' \cdot h_B)) = 0$, and so the restriction map:

$$H^0(Y, \mathcal{O}_Y(K_Y + mN + B)) \longrightarrow H^0(Y, \mathcal{O}_Y(K_Y + mN + B) \otimes \mathcal{O}_Y/\mathcal{I}(h' \cdot h_B))$$

is surjective. Now we may assume that y_r is not on the singular locus of h_B, and hence $\mathcal{O}_Y/\mathcal{I}(h' \cdot h_B)$ has isolated support at y_r. Thus there exists a global section $\tau' \in H^0(Y, \mathcal{O}_Y(K_Y + mN + B))$ with $\tau'(y_r) \neq 0$. This completes the proof of Theorem 1.4.

4. Proof of Theorem 1.3

We shall prove Theorem 1.3. Let X be a smooth projective variety of dimension n and L an ample line bundle over X such that $K_X + L$ is nef. By [16, Theorem 0.2], we have $\kappa(X, K_X + L) \geq 0$. If $\kappa(X, K_X + L) = 0$ holds, using [3, Lemma 3.3.2], we have $\mathcal{O}_X(K_X + L) = \mathcal{O}_X$, and so $H^0(X, \mathcal{O}_X(K_X + L)) \cong \mathbf{C}$. Hence it suffices to consider the case of $\kappa(X, K_X + L) > 0$.

Since $K_X + L$ is nef, we see that $(m-1)(K_X + L) + L$ is ample for every $m \geq 1$. Then by the base point free theorem ([13, Theorem 3-1-1]), we have the following:

LEMMA 4.1. *The complete linear system $|m(K_X + L)|$ is base point free for every positive integer $m \gg 1$. Moreover, there exists a C^∞-hermitian metric h_0 on $K_X + L$ with semipositive curvature.*

Let $\Phi_m : X \dashrightarrow \mathbf{P}H^0(X, \mathcal{O}_X(m(K_X + L)))^*$ denote the rational map associated with $|m(K_X + L)|$, where m is a positive integer with $|m(K_X + L)| \neq \emptyset$. Taking a sufficiently large integer a, by Lemma 4.1, we obtain a surjective morphism $f := \Phi_a : X \longrightarrow Y$, where Y denotes the image of X. Further we may assume that $\kappa(X, K_X + L) = \dim Y$ holds, and that $\kappa(F, K_F + L|_F) = 0$ holds for a general fiber F of f. Taking a suitable modification, we may also suppose that Y is smooth. Then again by [3, Lemma 3.3.2], we see that $K_F + L|_F$ is linearly equivalent to $\mathcal{O}_F$.

Let l be a non-negative integer. We define the reflexive sheaf B_l on Y by

$$B_l := f_*\mathcal{O}_X(K_{X/Y} + L + l(K_X + L))^{**},$$

where $K_{X/Y} := K_X - f^*K_Y$ is the relative canonical sheaf and $**$ denotes the double dual. Since $K_F + L|_F$ is trivial, B_l is an invertible sheaf on Y for every $l \geq 0$. Moreover we have the following:

LEMMA 4.2. *B_l is big for every $l \geq 0$.*

PROOF. Let h_L be a C^∞-hermitian metric on L with strictly positive curvature and h_0 a C^∞-hermitian metric on $K_X + L$ as in Corollary 4.1. Then we define the singular hermitian metric h_{B_l} on B_l by

$$h_{B_l}(\sigma, \sigma) := \int_{X/Y} h_0^l \cdot h_L \cdot \sigma \wedge \overline{\sigma}\,, \tag{4.1}$$

where $\sigma \in \Gamma(Y, B_l)$ is a global section of B_l. Here the fiber integral in the right hand side is defined by the following property:

$$\int_U \left(\int_{X/Y} h_0^l \cdot h_L \cdot \sigma \wedge \overline{\sigma} \right) = \int_{f^{-1}(U)} h_0^l \cdot h_L \cdot \sigma \wedge \overline{\sigma}$$

holds for any open set U in Y. We note that h_{B_l} is smooth on the smooth locus of f, and may have singularities on the discriminant locus of f. Then by [18, Theorem 1.5] (see also [4, Theorem 0.1]), we see that h_{B_l} has strictly positive curvature current. This completes the proof of Lemma 4.2. □

We set $B := B_0 = f_*\mathcal{O}_X(K_{X/Y} + L)^{**}$. Then we obtain the $\mathbf{Q}$-linear equivalence:

$$K_Y + B \sim_{\mathbf{Q}} \frac{1}{l}(K_Y + B_{l-1})\,, \tag{4.2}$$

and $f_*\mathcal{O}_X(l(K_X + L))^{**} \cong \mathcal{O}_Y(K_Y + B_{l-1})$ for every $l \geq 1$. Consequently, we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(m(K_X + L))) &\cong H^0(Y, f_*\mathcal{O}_X(m(K_X + L))^{**}) \\ &\cong H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \end{aligned} \tag{4.3}$$

for every $m \geq 1$. In order to apply Theorem 1.4, we need to show the following:

LEMMA 4.3. *$K_Y + B$ is nef and big.*

PROOF. It follows immediately from (4.3) that $K_Y + B$ is big.

Let dV_Y be a C^∞-volume form on Y. Then we may consider dV_Y^{-1} as a C^∞-hermitian metric on K_Y. For every $l \geq 1$, let h_l be the singular hermitian metric on $(1/l)(K_Y + B_{l-1})$ defined by $h_l := (dV_Y^{-1} \cdot h_{B_{l-1}})^{1/l}$. By (4.2), we may regard h_l as a singular hermitian metric on $K_Y + B$.

Since $h_{B_{l-1}}$ has semipositive curvature current, we see that

$$\Theta_{h_l} = \frac{1}{l}\Theta_{h_{B_{l-1}}} + \frac{1}{l}\Theta_{dV_Y^{-1}} \geq \frac{1}{l}\Theta_{dV_Y^{-1}} \tag{4.4}$$

holds for every $l \geq 1$. Let

$$\Theta_{h_{B_{l-1}}} = (\Theta_{h_{B_{l-1}}})_{\mathrm{ac}} + (\Theta_{h_{B_{l-1}}})_{\mathrm{sing}}$$

be the Lebesgue decomposition, where $(\Theta_{h_{B_{l-1}}})_{\mathrm{ac}}$ denotes the absolutely continuous part and $(\Theta_{h_{B_{l-1}}})_{\mathrm{sing}}$ denotes the singular part. Then by (4.1), we see that $(\Theta_{h_{B_{l-1}}})_{\mathrm{sing}}$ is independent of l. Hence by taking a suitable modification of Y, we may assume that $(\Theta_{h_{B_{l-1}}})_{\mathrm{sing}}$ is an effective divisor independent of l. Then by the inequality (4.4), we have

$$\begin{aligned}(K_Y + B) \cdot C &= \frac{1}{l} \int_C \{(\Theta_{h_{B_{l-1}}})_{\mathrm{ac}} + \Theta_{dV_Y^{-1}}\} + \frac{1}{l}(\Theta_{h_{B_{l-1}}})_{\mathrm{sing}} \cdot C \\ &\geq \frac{1}{l} K_Y \cdot C + \frac{1}{l}(\Theta_{h_{B_{l-1}}})_{\mathrm{sing}} \cdot C\end{aligned}$$

for an irreducible curve C. Letting l tend to infinity, we can conclude that $K_Y + B$ is nef after taking a suitable modification of Y. By the functoriality, we see that $K_Y + B$ is nef, without taking such a modification. We have thus proved Lemma 4.3. □

Then applying Theorem 1.4 for $N = K_Y + B$, we conclude the following:

LEMMA 4.4.

$$H^0(Y, \mathcal{O}_Y(m(K_Y + B))) \neq 0$$

holds for every $m \geq d(d+1)/2 + 2$.

By (4.3) and the lemma above, we have completed the proof of Theorem 1.3.

ACKNOWLEDGMENT. I would like to express my thanks to Professor Hajime Tsuji for his suggestions and constant support.

References

[1] F. AMBRO, *Ladders on Fano varieties*, Algebraic geometry, 9. J. Math. Sci. (New York), **94** (1999), no. 1, 1126–1135.

[2] U. ANGEHRN and Y.-T. SIU, Effective freeness and point separation for adjoint bundles, Invent. Math. **122** (1995), 291–308.

[3] M. C. BELTRAMETTI and A. J. SOMMESE, *The adjunction theory of complex projective varieties*, de Gruyter Expositions in Math. **16** Walter de Gruyter, Berlin, New York, (1995).

[4] B. BERNDTSSON and M. PAUN, Bergman kernels and the pseudoeffectivity of relative canonical bundles, Duke Math. J. **145** (2008), no. 2, 341–378.

[5] J.-P. DEMAILLY, *Singular hermitian metrics on positive line bundles*, Proc. Conf. Complex algebraic varieties (Bayreuth, April 26, 1990), edited by K. Hulek, T. Peternell, M. Schneider, F. Schreyer, Lecture Notes in Math., Vol. **1507**, Springer-Verlag, Berlin, 1992.

[6] T. FUJITA, *On polarized manifolds whose adjoint bundles are not semipositive*, Algebraic Geometry, Sendai, 1985, Advanced Studies Pure Math., Vol. **10**, North-Holland, Amsterdam (1987), 167–178.

[7] Y. FUKUMA, Effective non-vanishing of global sections of multiple adjoint bundles for polarized 3-folds, J. Pure Appl. Algebra **215** (2011), 168–184.

[8] Y. FUKUMA, Effective non-vanishing of global sections of multiple adjoint bundles for polarized 4-folds, J. Pure Appl. Algebra **217** (2013), 1535–1547.

[9] R. HARTSHORNE, *Algebraic Geometry*, Springer-Verlag (1977).

[10] A. HÖRING, On a conjecture of Beltrametti and Sommese, Journal of Algebraic Geometry **21** (2012), 721–751.

[11] Y. KAWAMATA, Fujita's freeness conjecture for 3-folds and 4-folds, Math. Ann. **308** (1997), 491–505.

[12] Y. KAWAMATA, On effective nonvanishing and base point freeness, Kodaira's issue, Asian J. Math. **4** (2000), 173–181.

[13] Y. KAWAMATA, K. MATSUDA and K. MATSUKI, Introduction to the minimal model problem, Adv. St. Pure Math. **10** (1987), 283–360.

[14] J. KOLLÁR, Singularities of pairs, *Algebraic geometry-Santa Cruz 1995*, Proc. Sympos. Pure Math., vol. **62**, Amer. Math. Soc., Providence, RI, 1997, 221–287.

[15] A. M. NADEL, Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. **132** (1990), 549–596.

[16] V. V. SHOKUROV, Theorems on non-vanishing, Math. USSR Izv. **26** (1986), 591–604.

[17] H. TSUJI, Global generation of adjoint line bundles, Nagoya Math. J. **142** (1996), 5–16.

[18] H. TSUJI, Variation of Bergman kernels of adjoint line bundles, math.CV/0511342 (2005).

Present Address:
DEPARTMENT OF MATHEMATICS,
SOPHIA UNIVERSITY,
KIOICHO, CHIYODA-KU, TOKYO 102–8554, JAPAN.
e-mail: tomoki-a@sophia.ac.jp