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On Tamely Ramified Iwasawa Modules for $\mathbb{Z}_p$-extensions of Imaginary Quadratic Fields

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Abstract. We study the Iwasawa modules related to certain tamely ramified extensions (tamely ramified Iwasawa modules). Let p be an odd prime number, and k an imaginary quadratic field. In the present paper, we shall give some results concerning the μ-invariant of tamely ramified Iwasawa modules for $\mathbb{Z}_p$-extensions of k.

1. Introduction

Let p be an odd prime number, and k an imaginary quadratic field. We denote by $\mathbb{Z}_p$ the ring of p-adic integers. Moreover, let K be a $\mathbb{Z}_p$-extension of k. That is, K/k is an infinite Galois extension and $\mathrm{Gal}(K/k)$ is (topologically) isomorphic to the additive group of $\mathbb{Z}_p$.

In the present paper, we shall treat "tamely ramified" Iwasawa modules for $\mathbb{Z}_p$-extensions. However, we firstly state some basic facts about "unramified" (usual) Iwasawa modules. Let $L(K)$ be the maximal unramified abelian pro-p extension of K. It is known that the unramified Iwasawa module $X(K) := \mathrm{Gal}(L(K)/K)$ is a finitely generated torsion module over the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$. Then the λ-invariant $\lambda = \lambda(K/k)$ and the μ-invariant $\mu = \mu(K/k)$ are defined from the structure of $X(K)$ (see Section 2.1). We note that $\mu = 0$ if and only if $X(K)$ is finitely generated as a $\mathbb{Z}_p$-module. Hence, to study the structure of $X(K)$, it is important to know whether $\mu = 0$ or not. (We assumed that k is an imaginary quadratic field, but these facts hold when the base field is an arbitrary algebraic number field.)

We shall state some known results about this "unramified" μ-invariant (for the case when k is an imaginary quadratic field). Let K^c/k be the cyclotomic $\mathbb{Z}_p$-extension. We see that $\mu(K^c/k) = 0$ by Ferrero-Washington's theorem [6]. Gillard [10], [11], Schneps [26] (and recently Oukhaba-Viguié [20]) showed $\mu = 0$ for certain non-cyclotomic $\mathbb{Z}_p$-extensions. Bloom-Gerth [1] gave an upper bound of the number of $\mathbb{Z}_p$-extensions satisfying $\mu > 0$ for a fixed k (see Section 3.2). Note that Iwasawa [16] gave a method to construct a $\mathbb{Z}_p$-extension (over a certain algebraic number field) which satisfies $\mu > 0$ (see also Ozaki [21]). However,

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it seems hard to apply this method to construct a $\mathbb{Z}_p$-extension satisfying $\mu > 0$ over an imaginary quadratic field.

Next, we shall introduce the Iwasawa module relating to certain tamely ramified extensions. (This object was already studied by several authors. See, e.g., Salle [24], Mizusawa-Ozaki [18], Itoh-Mizusawa-Ozaki [14].) Take a non-empty finite set S of (finite) primes of k *not lying above* p. For a $\mathbb{Z}_p$-extension K/k, we denote by $M_S(K)$ the maximal abelian pro-p extension of K unramified outside S (i.e., unramified outside the primes of K lying above the primes of S). We put $X_S(K) = \mathrm{Gal}(M_S(K)/K)$. This is an analog of the unramified Iwasawa module $X(K)$, and called the "S-ramified (or tamely ramified) Iwasawa module". It can be shown that $X_S(K)$ is also a finitely generated torsion module over $\mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$. Similar to $X(K)$, the λ-invariant λ_S and the μ-invariant μ_S for $X_S(K)$ can be defined.

We shall consider about the invariant μ_S in the present paper. In Section 2, we will state basic facts about the theory of $\mathbb{Z}_p$-extensions and the tamely ramified Iwasawa modules. In Section 3, we consider the $\mathbb{Z}_p$-extensions whose μ_S-invariant is positive. In particular, there exists a $\mathbb{Z}_p$-extension K/k and a set S which satisfy $\mu_S > 0$ (this seems essentially shown by Iwasawa). We also give an upper bound of the number of $\mathbb{Z}_p$-extensions satisfying $\mu_S > 0$ for given k and S (this follows as a corollary of Bloom-Gerth's result [1]). In Section 4, we introduce a question (Question 4.1) about the vanishing of μ_S. We will give some sufficient conditions such that this question has an affirmative answer in Sections 4 and 5. Especially, Proposition 4.8 seems a non-trivial result on this question. We also give calculation examples in Section 5.

2. Notation and basic facts

2.1. Notation. In the present paper, we *always* assume that p is an odd prime number and k is an imaginary quadratic field. (Moreover, we suppose that $p > 3$ when $k = \mathbb{Q}(\sqrt{-3})$ in Section 5.)

For a finite set $\mathcal{S}$, we denote by $|\mathcal{S}|$ the number of elements of $\mathcal{S}$. For an algebraic number field F (a finite extension of $\mathbb{Q}$), let $\mathcal{O}_F$ be the ring of integers in F, $E(F)$ the group of units in F, and $h(F)$ the class number of F (i.e., the order of the ideal class group of F). In the present paper, a prime of an algebraic number field always denotes a finite prime (and we will identify it with the corresponding prime ideal of the ring of integers). For an integral ideal $\mathfrak{a}$ of an algebraic number field, we denote by $N(\mathfrak{a})$ the absolute norm of $\mathfrak{a}$. For a finitely generated $\mathbb{Z}_p$-module N, we call $\dim_{\mathbb{F}_p} N/p$ the p-rank of N (we abbreviate N/pN to N/p), and $\dim_{\mathbb{Q}_p} N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ the $\mathbb{Z}_p$-rank of N.

Let $\mathfrak{F}$ be a $\mathbb{Z}_p$-extension of an algebraic number field F, and γ a fixed topological generator of $\mathrm{Gal}(\mathfrak{F}/F)$. We put $\Lambda = \mathbb{Z}_p[[T]]$ (the ring of formal power series of T). Then there exists an isomorphism $\mathbb{Z}_p[[\mathrm{Gal}(\mathfrak{F}/F)]] \simeq \Lambda$ with $\gamma \mapsto 1+T$. We shall regard a $\mathbb{Z}_p[[\mathrm{Gal}(\mathfrak{F}/F)]]$-module also as a Λ-module. For non-negative integers $m > n$, we put $\omega_n = (1+T)^{p^n} - 1$ and $\nu_{m,n} = \omega_m/\omega_n$. We denote by $\mathfrak{F}_n$ the nth layer of $\mathfrak{F}/F$ (note that $\mathfrak{F}_0 = F$).

We briefly recall the definition of the λ-, μ-invariants, and the characteristic polynomial (for the details, see, e.g., [15], [19], [28]). Let X be a finitely generated torsion Λ-module. Then there exists a pseudo-isomorphism from X to an elementary torsion Λ-module

$$E = \Lambda/(f_1^{m_1}) \oplus \cdots \oplus \Lambda/(f_r^{m_r}) \oplus \Lambda/(p^{n_1}) \oplus \cdots \oplus \Lambda/(p^{n_s}),$$

where $f_1, \dots, f_r$ are irreducible distinguished polynomials of Λ. (It can be occurred that E does not contain a factor of the form $\Lambda/(f^m)$ or $\Lambda/(p^n)$. In particular, X is pseudo-isomorphic to $E = 0$ when the order of X is finite.) By using this pseudo-isomorphism, we define the λ-invariant of X as $\sum_{i=1}^{r} m_i \deg(f_i)$, and the μ-invariant of X as $\sum_{j=1}^{s} n_j$. When E does not contain a factor of the form $\Lambda/(f^m)$ (resp. $\Lambda/(p^n)$), the λ-invariant (resp. μ-invariant) of X is defined to be 0. We note that the μ-invariant of X is 0 if and only if X is finitely generated as a $\mathbb{Z}_p$-module. We also define the characteristic polynomial of X as $p^{n_1+\cdots+n_s} f_1^{m_1} \dots f_r^{m_r}$. (These invariants and the characteristic polynomial are determined uniquely.)

2.2. *S*-ramified Iwasawa modules. Recall that k is an imaginary quadratic field. Let S be a non-empty finite set of primes of k not lying above p, and $\mathbb{K}$ a (finite or infinite) abelian extension of k. We denote by $M_S(\mathbb{K})$ the maximal abelian pro-p extension of $\mathbb{K}$ unramified outside S. We also denote by $L(\mathbb{K})$ the maximal unramified abelian pro-p extension of $\mathbb{K}$. Put $X_S(\mathbb{K}) = \mathrm{Gal}(M_S(\mathbb{K})/\mathbb{K})$ and $X(\mathbb{K}) = \mathrm{Gal}(L(\mathbb{K})/\mathbb{K})$. Let K/k be a $\mathbb{Z}_p$-extension and N/k a finite abelian extension. Then $\mathfrak{N} := NK$ is a $\mathbb{Z}_p$-extension of N. It is well known that $X(\mathfrak{N})$ is a finitely generated torsion $\mathbb{Z}_p[[\mathrm{Gal}(\mathfrak{N}/N)]](\simeq \Lambda)$-module. Since S is a set of primes of k, we can see that Λ also acts on $X_S(\mathfrak{N})$. We denote by $M'_S(\mathfrak{N})$ the maximal abelian pro-p extension of $\mathfrak{N}$ unramified outside S in which all primes ramifying in $\mathfrak{N}/N$ split completely. (In the present paper, we mainly treat the case when all primes lying above p ramify in $\mathfrak{N}/N$.) We put $X'_S(\mathfrak{N}) = \mathrm{Gal}(M'_S(\mathfrak{N})/\mathfrak{N})$. For $n \geq 0$, we define $M'_S(\mathfrak{N}_n)$, and $X'_S(\mathfrak{N}_n)$ similarly (see also [24]).

PROPOSITION 2.1. *Let the notation be as above, and choose $e \geq 0$ such that all primes which ramify in $\mathfrak{N}/N$ are totally ramified in $\mathfrak{N}/\mathfrak{N}_e$. Let S be a finite set of primes of k.*

(1) *There exists a finite index submodule Z_S of $X_S(\mathfrak{N})$ such that*

$$X_S(\mathfrak{N})/\nu_{n,e} Z_S \simeq X_S(\mathfrak{N}_n) \quad for\ n \geq e.$$

(2) *There exists a finite index submodule Z'_S of $X'_S(\mathfrak{N})$ such that*

$$X'_S(\mathfrak{N})/\nu_{n,e} Z'_S \simeq X'_S(\mathfrak{N}_n) \quad for\ n \geq e.$$

PROOF. The proof is essentially the same as that of a similar result for the unramified Iwasawa module $X(\mathfrak{N})$. See, e.g., [28, Chapter 13], [19, Chapter XI]. □

In particular, if $\mathfrak{N}/N$ is a $\mathbb{Z}_p$-extension in which exactly one prime of N is ramified and it is totally ramified, we can obtain the following:

$$X_S(\mathfrak{N})/\omega_n X_S(\mathfrak{N}) \simeq X_S(\mathfrak{N}_n) \quad \text{and} \quad X'_S(\mathfrak{N})/\omega_n X'_S(\mathfrak{N}) \simeq X'_S(\mathfrak{N}_n) \quad \text{for } n \geq 0\,.$$

Note that both of $X_S(\mathfrak{N}_n)$ and $X'_S(\mathfrak{N}_n)$ are finite because all primes of S do not divide p. Hence we can see that $X_S(\mathfrak{N})$ and $X'_S(\mathfrak{N})$ are finitely generated torsion Λ-modules (by using Proposition 2.1 and the same method given in, e.g., [28, Chapter 13]). We denote by $\lambda_S = \lambda_S(\mathfrak{N}/N)$ (resp. $\mu_S = \mu_S(\mathfrak{N}/N)$) the λ-invariant (resp. μ-invariant) of $X_S(\mathfrak{N})$ as a finitely generated torsion Λ-module. We also denote by $\lambda = \lambda(\mathfrak{N}/N)$ (resp. $\mu = \mu(\mathfrak{N}/N)$) the λ-invariant (resp. μ-invariant) of $X(\mathfrak{N})$.

2.3. Multiplicative groups of residue classes. For this subsection, see also [21], [24], [18], [14], [13], etc. Let K be a $\mathbb{Z}_p$-extension of an imaginary quadratic field k, and K_n the nth layer of K/k for $n \geq 0$ (recall that $K_0 = k$). For a prime $\mathfrak{q}$ of k which does not divide p, we put

$$R_{\mathfrak{q},n} = (\mathcal{O}_{K_n}/\mathfrak{q})^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p\,.$$

We remark that $R_{\mathfrak{q},n}$ is non-trivial for all n if and only if $R_{\mathfrak{q},0}$ is non-trivial because K_n/k is a cyclic extension of degree p^n. Moreover $R_{\mathfrak{q},0}$ is non-trivial if and only if p divides $N(\mathfrak{q}) - 1$. We also put $R_{\mathfrak{q}} = \varprojlim R_{\mathfrak{q},n}$, where the projective limit is taken with respect to the mappings induced from the norm mapping. Since the mapping $R_{\mathfrak{q},m} \to R_{\mathfrak{q},n}$ induced from the norm mapping is surjective for all $m > n \geq 0$, we note that $R_{\mathfrak{q}}$ is non-trivial if and only if $p \mid N(\mathfrak{q}) - 1$. When $\mathfrak{q}$ does not split completely in K, we see that $R_{\mathfrak{q}}$ is a finitely generated $\mathbb{Z}_p$-module. However, when $\mathfrak{q}$ splits completely in K, we see that $R_{\mathfrak{q}}$ is not finitely generated over $\mathbb{Z}_p$ if it is not trivial. (For example, we consider the case that $|R_{\mathfrak{q},0}| = p$ and $\mathfrak{q}$ splits completely in K/k. In this case, we can show that $R_{\mathfrak{q},n}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}[\mathrm{Gal}(K_n/k)]$, and then $R_{\mathfrak{q}}$ is isomorphic to $\Lambda/(p)$. See also p.790 and p.797 of [21].)

Let S be a finite set of primes of k not lying above p. We put $Y_S(K_n) = \mathrm{Gal}(M_S(K_n)/L(K_n))$ for $n \geq 0$, and $Y_S(K) = \mathrm{Gal}(M_S(K)/L(K))$. We can obtain the following exact sequences:

$$0 \to Y_S(K_n) \to X_S(K_n) \to X(K_n) \to 0$$

$$E(K_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \bigoplus_{\mathfrak{q} \in S} R_{\mathfrak{q},n} \to Y_S(K_n) \to 0\,.$$

(The second exact sequence follows from class field theory.) We put $E_\infty = \varprojlim E(K_n) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where the projective limit is taken with respect to the mappings induced from the norm mapping. Then we also obtain the following exact sequences:

$$0 \to Y_S(K) \to X_S(K) \to X(K) \to 0$$

$$E_\infty \to \bigoplus_{\mathfrak{q}\in S} R_{\mathfrak{q}} \to Y_S(K) \to 0\,.$$

In the rest of the present paper, we mainly treat a finite set S of primes of k satisfying the following condition.

(N) S is not empty, every prime $\mathfrak{q}$ of S does not divide p and satisfies $p \mid N(\mathfrak{q}) - 1$.

For a finite set S of primes of k not lying above p, let S_0 be the maximal subset of S which satisfies (N). Then we obtain that $X_S(K) \cong X_{S_0}(K)$. (Recall that $R_{\mathfrak{q}}$ is trivial when p does not divide $N(\mathfrak{q}) - 1$. If S_0 is empty, then $X_S(K) \cong X(K)$.) Hence, it is sufficient to consider only for the case that S satisfies (N).

2.4. Decomposition of primes in a $\mathbb{Z}_p$-extension. Let K^c/k be the cyclotomic $\mathbb{Z}_p$-extension, and K^a/k the anti-cyclotomic $\mathbb{Z}_p$-extension. K^c is the unique $\mathbb{Z}_p$-extension which is abelian over $\mathbb{Q}$. K^a is a Galois extension over $\mathbb{Q}$, and ι acts on $\mathrm{Gal}(K^a/k)$ by inversion, where ι is the generator of $\mathrm{Gal}(k/\mathbb{Q})$. We note that K^a is uniquely determined because k is an imaginary quadratic field. We shall state some basic (known) results.

LEMMA 2.2. *Let $\mathfrak{q}$ be a prime of k not lying above p. Then there is a unique $\mathbb{Z}_p$-extension of k in which $\mathfrak{q}$ splits completely.*

PROOF. The authors could not find a literature which states the assertion explicitly. However, this assertion is contained in Theorem (11) of [4] when the prime number q lying below $\mathfrak{q}$ does not split in k, and the rest case (when q splits in k) also can be shown by using the facts given in the proof of that theorem. We will state here briefly. Let $\widetilde{k}$ be the composite of all $\mathbb{Z}_p$-extensions of k, then $\mathrm{Gal}(\widetilde{k}/k)$ is isomorphic to $\mathbb{Z}_p^{\oplus 2}$ because k is an imaginary quadratic field (see, e.g., [4], [15], [19], [28]). We recall the fact that every finite prime does not split completely in K^c/k. Hence the $\mathbb{Z}_p$-rank of the decomposition subgroup of $\mathrm{Gal}(\widetilde{k}/k)$ for $\mathfrak{q}$ is just 1 (note that the $\mathbb{Z}_p$-rank of this decomposition subgroup is at most 1 because $\mathfrak{q}$ does not divide p). From this, the assertion follows. □

LEMMA 2.3. *Let q be a prime number which is not equal to p.*

(1) *Suppose that q does not split in k, and let $\mathfrak{q}$ be the unique prime of k lying above q. Then $\mathfrak{q}$ splits completely in K^a.*

(2) *Suppose that q splits in k, and let $\mathfrak{q}$ be a prime of k lying above q. Then $\mathfrak{q}$ does not split completely in K^a.*

PROOF. (1) This is well known ([4, Theorem (11)], [3, p.2132], etc.). (2) For example, see [3]. □

3. $\mathbb{Z}_p$-extension having a positive μ_S-invariant

3.1. Sufficient condition. The following proposition gives a sufficient condition for being $\mu_S > 0$. It seems that this is essentially shown by Iwasawa in his work [16] on giving examples of $\mathbb{Z}_p$-extensions having a positive unramified μ-invariant (see also Ozaki [21]).

PROPOSITION 3.1. *Let S be a finite set of primes of k satisfying* (N), *and K a $\mathbb{Z}_p$-extension of k. If S contains at least two primes which split completely in K, then $\mu_S(K/k) > 0$.*

PROOF. When $S \subseteq S'$, there is a surjection $X_{S'}(K) \to X_S(K)$, and then we obtain an inequality $\mu_{S'}(K/k) \geq \mu_S(K/k)$. Hence it suffices to prove for the case that $S = \{\mathfrak{q}_1, \mathfrak{q}_2\}$ and both of $\mathfrak{q}_1, \mathfrak{q}_2$ split completely in K.

We note that $\mu_S(K/k) > 0$ if and only if the p-rank of $X_S(K_n)$ is unbounded as $n \to \infty$. (This follows from the argument given in the proof of [28, Proposition 13.23].) We shall consider the following exact sequence:

$$E(K_n)/p \to R_{\mathfrak{q}_1,n}/p \oplus R_{\mathfrak{q}_2,n}/p \to Y_S(K_n)/p \to 0\,.$$

Since k is an imaginary quadratic field, we have

$$\dim_{\mathbb{F}_p} E(K_n)/p \leq p^n$$

by Dirichlet's unit theorem. On the other hand, since both of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ split completely in K_n,

$$\dim_{\mathbb{F}_p}(R_{\mathfrak{q}_1,n}/p \oplus R_{\mathfrak{q}_2,n}/p) = 2p^n\,.$$

Therefore, the p-rank of $Y_S(K_n)$ is unbounded as $n \to \infty$, and that of $X_S(K_n)$ is also. □

3.2. Analog of Bloom-Gerth's result. Bloom-Gerth [1] gave an upper bound for the number of $\mathbb{Z}_p$-extensions having a positive unramified μ-invariant of a fixed imaginary quadratic field k. We can give a similar result for the μ_S-invariant.

Put

$$\delta = \begin{cases} 1 & \text{if } p \text{ splits in } k/\mathbb{Q}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\lambda(K^c/k)$ is the unramified λ-invariant of the cyclotomic $\mathbb{Z}_p$-extension K^c/k. The following result is known.

THEOREM A (Corollary 1 of [1]). *The number of $\mathbb{Z}_p$-extensions of k having positive unramified μ-invariant is at most $\lambda(K^c/k) - \delta$.*

Note that the number of $\mathbb{Z}_p$-extensions having positive unramified μ-invariant can be smaller than $\lambda(K^c/k) - \delta$ (see, e.g., Sands [25], Fujii [8]). By using Theorem A, we can obtain the following:

PROPOSITION 3.2. *Let S be a finite set of primes of k satisfying* (N). *Denote by ι the generator of* $\mathrm{Gal}(k/\mathbb{Q})$, *and put*

$$S_1 = \{\mathfrak{q} \in S \mid \mathfrak{q} \neq \mathfrak{q}^\iota\}, \quad S_2 = \{\mathfrak{q} \in S \mid \mathfrak{q} = \mathfrak{q}^\iota\}.$$

Let d (resp. d_S) be the number of $\mathbb{Z}_p$-extensions satisfying $\mu > 0$ (resp. $\mu_S > 0$). Then we have the following inequalities.

$$d_S \leq |S_1| + \min\{1, |S_2|\} + d \leq |S| + \lambda(K^c/k) - \delta.$$

PROOF. For a $\mathbb{Z}_p$-extension K/k, we recall the following exact sequence:

$$E_\infty \to \bigoplus_{\mathfrak{q} \in S} R_\mathfrak{q} \to X_S(K) \to X(K) \to 0.$$

From this, we can conclude that $\mu_S(K/k) > 0$ only if

(a) the unramified μ-invariant is positive, or

(b) $R_\mathfrak{q}$ is not finitely generated as a $\mathbb{Z}_p$-module (i.e., $\mathfrak{q}$ splits completely in K/k).

For each $\mathfrak{q} \in S$, there is a unique $\mathbb{Z}_p$-extension such that $\mathfrak{q}$ splits completely by Lemma 2.2. We also note that every prime of S_2 splits completely in K^a/k by Lemma 2.3 (1). From these facts, we can obtain the left inequality. The right inequality follows from Theorem A. □

EXAMPLE 3.3. Assume that $\mu = 0$ for all $\mathbb{Z}_p$-extensions of k. Let q_1, q_2 ($\neq p$) be prime numbers which are inert in k. We denote by $\mathfrak{q}_1, \mathfrak{q}_2$ the prime ideals of k lying above q_1, q_2, respectively. We put $S = \{\mathfrak{q}_1, \mathfrak{q}_2\}$. Assume also that S satisfies (N). Then we see that $d_S \leq 1$ by Proposition 3.2. On the other hand, both of $\mathfrak{q}_1$ and $\mathfrak{q}_2$ split completely in K^a/k by Lemma 2.3 (1), and hence $\mu_S(K^a/k) > 0$ by Proposition 3.1. In this case, there is *exactly* one $\mathbb{Z}_p$-extension of k satisfying $\mu_S > 0$.

4. Sufficient conditions for satisfying $\mu_S = 0$

4.1. Our question. Let S be a finite set of primes of an imaginary quadratic field k satisfying (N). We showed in Proposition 3.1 that if at least two primes of S split completely in K/k then $\mu_S(K/k) > 0$. On the other hand, if no prime of S splits completely in K/k, we can see that $\mu_S(K/k) = \mu(K/k)$. (In particular, $\mu_S(K^c/k) = 0$. This is known. See, e.g., [14].) Relating these facts, the following question arises.

QUESTION 4.1. Let K/k be a $\mathbb{Z}_p$-extension such that only one prime of S splits completely. Assume that $\mu(K/k) = 0$. Then, is $\mu_S(K/k)$ also zero?

Considering this question, it is sufficient to treat the case that S consists of one prime (which splits completely in K/k) by the following proposition.

PROPOSITION 4.2. *Let K/k be a $\mathbb{Z}_p$-extension, and $S = \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ a finite set of primes of k satisfying* (N). *Assume that $\mathfrak{q}_1$ is the only prime of S which splits completely in K/k, and put $S_1 = \{\mathfrak{q}_1\}$. Then, $\mu_{S_1}(K/k) = 0$ if and only if $\mu_S(K/k) = 0$.*

PROOF. Note that $\mu_S(K/k) = 0$ implies $\mu_{S_1}(K/k) = 0$ because $\mu_{S_1}(K/k) \leq \mu_S(K/k)$. We shall show the converse.

Our proof uses the idea given in, e.g., [21, p. 799], [18], [14]. We note that the unramified μ-invariant of K/k is zero since $\mu_{S_1}(K/k)$ is zero. This implies that the p-rank of $X(K_n)$ is bounded as $n \to \infty$. Then, to see the assertion, it suffices to prove that the p-rank of $Y_S(K_n)$ is bounded. We consider the following exact sequence:

$$E(K_n)/p \xrightarrow{\phi_n} \bigoplus_{i=1}^{r} R_{\mathfrak{q}_i,n}/p \longrightarrow Y_S(K_n)/p \longrightarrow 0\,.$$

At first, we shall prove that the p-rank of $\operatorname{Ker}\phi_n$ (the kernel of ϕ_n) is bounded as $n \to \infty$. To show this, we consider the following exact sequence:

$$E(K_n)/p \xrightarrow{\phi'_n} R_{\mathfrak{q}_1,n}/p \longrightarrow Y_{S_1}(K_n)/p \longrightarrow 0\,.$$

By the assumption, the p-rank of $Y_{S_1}(K_n)$ is bounded as $n \to \infty$. From this, there exists a constant a such that $\dim_{\mathbb{F}_p} Y_{S_1}(K_n)/p \leq a$ for $n \geq 0$. Moreover, since $\mathfrak{q}_1$ splits completely in K/k, we see that $\dim_{\mathbb{F}_p} R_{\mathfrak{q}_1,n}/p = p^n$. By Dirichlet's unit theorem, we obtain

$$p^n - 1 \leq \dim_{\mathbb{F}_p} E(K_n)/p \leq p^n$$

for $n \geq 0$ (recall that k is an imaginary quadratic field). From these facts,

$$\dim_{\mathbb{F}_p} \operatorname{Ker}\phi'_n + p^n = \dim_{\mathbb{F}_p} E(K_n)/p + \dim_{\mathbb{F}_p} Y_{S_1}(K_n)/p \leq p^n + a\,,$$

and hence $\dim_{\mathbb{F}_p} \operatorname{Ker}\phi'_n$ is bounded as $n \to \infty$. We consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ker}\phi_n & \longrightarrow & E(K_n)/p & \xrightarrow{\phi_n} & \bigoplus_{i=1}^{r} R_{\mathfrak{q}_i,n}/p \\
 & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & \operatorname{Ker}\phi'_n & \longrightarrow & E(K_n)/p & \xrightarrow{\phi'_n} & R_{\mathfrak{q}_1,n}/p\,,
\end{array}$$

where the right vertical mapping is the natural projection. By the above diagram, we can obtain that $\operatorname{Ker}\phi_n \subseteq \operatorname{Ker}\phi'_n$, then $\dim_{\mathbb{F}_p} \operatorname{Ker}\phi_n$ is bounded as $n \to \infty$. Hence, there exists a constant b such that $\dim_{\mathbb{F}_p} \operatorname{Ker}\phi_n \leq b$ for $n \geq 0$. Moreover, since $\mathfrak{q}_i$ does not split completely in K/k for $i \neq 1$, there exists a constant c such that $\dim_{\mathbb{F}_p} \bigoplus_{i=1}^{r} R_{\mathfrak{q}_i,n}/p \leq p^n + c$ for $n \geq 0$. Therefore,

$$\begin{aligned}
(p^n - 1) + \dim_{\mathbb{F}_p} Y_S(K_n)/p &\leq \dim_{\mathbb{F}_p} E(K_n)/p + \dim_{\mathbb{F}_p} Y_S(K_n)/p \\
&= \dim_{\mathbb{F}_p} \operatorname{Ker}\phi_n + \dim_{\mathbb{F}_p} \bigoplus_{i=1}^{r} R_{\mathfrak{q}_i,n}/p
\end{aligned}$$

$$\leq b + (p^n + c)\,,$$

and we can prove the p-rank of $Y_S(K_n)$ is bounded as $n \to \infty$. □

We also remark the relation between the μ_S-invariant and the unramified μ-invariant of a certain p-extension.

PROPOSITION 4.3. *Let* $\mathfrak{q}$ *be a prime of* k *not lying above* p, *and* K *a* $\mathbb{Z}_p$*-extension of* k *such that* $\mathfrak{q}$ *splits completely in* K. *Assume that there exists a cyclic extension* M/k *of degree* p *which is unramified outside* $\mathfrak{q}$ *and totally ramified at* $\mathfrak{q}$. *Put* $S = \{\mathfrak{q}\}$. *Then,* $\mu_S(K/k) = 0$ *if and only if* $\mu(MK/M) = 0$.

PROOF. (see also [16], [21].) We note that $M \cap K = k$ since $\mathfrak{q}$ is ramified in M/k. Put $M_n = MK_n$ for all $n \geq 0$, then $MK = \bigcup M_n$. Let $L^e(M_n)$ be the maximal unramified elementary abelian p-extension of M_n, and L'_n the maximal abelian extension of K_n contained in $L^e(M_n)$. Let σ be a generator of $\mathrm{Gal}(M_n/K_n)$. Then we can see that

$$\mathrm{Gal}(L'_n/M_n) \simeq \mathrm{Gal}(L^e(M_n)/M_n)/(\sigma - 1)\mathrm{Gal}(L^e(M_n)/M_n)\,.$$

We note that $L'_n \subseteq M_S(K_n)$ since L'_n/K_n is unramified outside primes of K_n lying above $\mathfrak{q}$.

Suppose that $\mu_S(K/k) = 0$, then the p-rank of $X_S(K_n)$ is bounded as $n \to \infty$, and that of $\mathrm{Gal}(L'_n/M_n)$ is also. We can obtain the following (see, e.g., [16, p. 6]):

$$\begin{aligned}\dim_{\mathbb{F}_p} \mathrm{Gal}(L^e(M_n)/M_n) &\leq p \times \dim_{\mathbb{F}_p} \mathrm{Gal}(L^e(M_n)/M_n)/(\sigma - 1)\mathrm{Gal}(L^e(M_n)/M_n)\\ &= p \times \dim_{\mathbb{F}_p} \mathrm{Gal}(L'_n/M_n)\,.\end{aligned}$$

Hence the p-rank of $\mathrm{Gal}(L^e(M_n)/M_n)$ is bounded as $n \to \infty$. We note that $X(M_n)/p \simeq \mathrm{Gal}(L^e(M_n)/M_n)$. Therefore, the p-rank of $X(M_n)$ is bounded, that is, $\mu(MK/M) = 0$.

Conversely, we assume that $\mu_S(K/k) > 0$. Let $M^e_S(K_n)$ be the maximal elementary abelian p-extension of K_n contained in $M_S(K_n)$. Then the p-rank of $X_S(K_n)$ is equal to that of $\mathrm{Gal}(M^e_S(K_n)/K_n)$. Since M_n/K_n is a cyclic extension of degree p unramified outside S, we see that $M_n \subseteq M^e_S(K_n)$. Let $\mathfrak{Q}$ be a prime of K_n lying above $\mathfrak{q}$. Since $\mathfrak{Q}$ is tamely ramified in $M^e_S(K_n)/K_n$, the inertia subgroup of $\mathrm{Gal}(M^e_S(K_n)/K_n)$ for $\mathfrak{Q}$ is cyclic. Moreover, all primes of K_n lying above $\mathfrak{q}$ are totally ramified in M_n. From these facts, we can conclude that $M^e_S(K_n)/M_n$ is an unramified extension. By the assumption that $\mu_S(K/k) > 0$, the p-rank of $\mathrm{Gal}(M^e_S(K_n)/K_n)$ is unbounded as $n \to \infty$, and then the p-rank of $\mathrm{Gal}(M^e_S(K_n)/M_n)$ is also unbounded. Consequently, the p-rank of $X(M_n)$ is unbounded because $M^e_S(K_n)$ is an intermediate field of $L(M_n)/M_n$. Therefore, $\mu(MK/M) > 0$. □

4.2. Sufficient conditions. We shall give some sufficient conditions for the vanishing of μ_S. At first, we treat the "exceptional case".

PROPOSITION 4.4. *We put* $p = 3$ *and* $k = \mathbb{Q}(\sqrt{-3})$. *Let* $\mathfrak{q}$ *be a prime of* k *which satisfies the following conditions*:

$$3 \mid N(\mathfrak{q}) - 1 \quad \textit{and} \quad 9 \nmid N(\mathfrak{q}) - 1\,.$$

(*Under the conditions*, $\mathfrak{q}$ *does not divide* 3.) *Put* $S = \{\mathfrak{q}\}$. *Then* $X_S(K)$ *is trivial for every* $\mathbb{Z}_3$*-extension* K *of* k.

PROOF. By the assumptions, $(\mathcal{O}_k/\mathfrak{q})^\times \otimes_\mathbb{Z} \mathbb{Z}_3$ is a cyclic group of order 3, and $E(k)$ contains a primitive third root of unity. These facts imply that the natural mapping $E(k) \otimes_\mathbb{Z} \mathbb{Z}_3 \to (\mathcal{O}_k/\mathfrak{q})^\times \otimes_\mathbb{Z} \mathbb{Z}_3$ is surjective (cf. [14]). Hence $Y_S(k)$ is trivial, and then $X_S(k)$ is also trivial because $h(k) = 1$. Let K/k be an arbitrary $\mathbb{Z}_3$-extension. Since K/k is totally ramified at the unique prime lying above 3, we see $X_S(K)/\omega_0 X_S(K) \simeq X_S(k)$. (See the paragraph after Proposition 2.1.) Hence by using a well known argument (see, e.g., [28, Proposition 13.22]), we can obtain the assertion. □

Next, we state a sufficient condition which can be obtained easily. (Similar arguments and results can be found in other papers.)

PROPOSITION 4.5 (cf. p. 799 of [21], Theorem 3.1 of [13], for example). *Assume that* p *does not split in* $k/\mathbb{Q}$. *Let* $\mathfrak{q}$ *be a prime of* k *not dividing* p *and satisfying* $p \mid N(\mathfrak{q}) - 1$. *We put* $S = \{\mathfrak{q}\}$. *Let* K/k *be a* $\mathbb{Z}_p$*-extension. If the* (*unique*) *prime of* k *lying above* p *does not split in* $M_S(k)/k$, *then* $X_S(K) \simeq X_S(k)$. *In particular*, $X_S(K)$ *is a finite cyclic* p*-group.*

PROOF. We denote by $\mathfrak{p}$ the unique prime of k lying above p. Then the order of the ideal class containing $\mathfrak{p}$ is 1 or 2 because p does not split in $k/\mathbb{Q}$. If p divides $h(k)$, then $\mathfrak{p}$ splits in $L(k)/k$, and hence it also splits in $M_S(k)/k$. Thus, under the assumptions of this proposition, we see that $p \nmid h(k)$. Put $M = M_S(k)$. Since $X(k)$ is trivial, we see that $X_S(k)$ is cyclic. From this, we can show that $X_S(M)$ is trivial. We also see that $X(M)$ is trivial, and hence the $\mathbb{Z}_p$-extension MK/M is totally ramified at the unique prime lying above p. In this case, as noted in the paragraph after Proposition 2.1, the isomorphism $X_S(MK)/\omega_0 X_S(MK) \simeq X_S(M)$ holds. Then we can obtain that $X_S(MK)$ is trivial because $X_S(M)$ is trivial. Consequently, we see $MK = M_S(K)$, and $X_S(K) \simeq X_S(k)$ which is a finite cyclic p-group. □

EXAMPLE 4.6. Assume that p is inert in $k/\mathbb{Q}$ and p does not divide $h(k)$. Let q be a prime number satisfying the following conditions:

$$p \mid q - 1\,, \quad \text{and } q \text{ is inert in } k/\mathbb{Q}\,.$$

Put $\mathfrak{p} = p\mathcal{O}_k$, $\mathfrak{q} = q\mathcal{O}_k$, and $S = \{\mathfrak{q}\}$. In this case, $|(\mathcal{O}_k/\mathfrak{q})^\times| = q^2 - 1$ and p does not divide $q + 1$. Let d be the largest integer such that $p^d \mid q - 1$. We can see that

$$X_S(k) \simeq (\mathcal{O}_k/\mathfrak{q})^\times \otimes_\mathbb{Z} \mathbb{Z}_p \simeq \mathbb{Z}/p^d\mathbb{Z}\,.$$

If $p^{\frac{q^2-1}{p}} \not\equiv 1 \pmod{\mathfrak{q}}$, then the class of (a certain power of) p generates the p-Sylow subgroup of $(\mathcal{O}_k/\mathfrak{q})^\times$, and this implies that $\mathfrak{p}$ does not split in $M_S(k)/k$. Moreover, we can obtain the following:

$$p^{\frac{q^2-1}{p}} \equiv 1 \pmod{\mathfrak{q}} \Leftrightarrow p^{\frac{q^2-1}{p}} \equiv 1 \pmod{q}$$

$$\Leftrightarrow p^{\frac{q-1}{p}} \equiv 1 \pmod{q}.$$

Hence by Proposition 4.5, if $p^{\frac{q-1}{p}} \not\equiv 1 \pmod{q}$ then $X_S(K) \simeq \mathbb{Z}/p^d\mathbb{Z}$ for every $\mathbb{Z}_p$-extension K/k. (See also, e.g., [13] for the case of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.)

REMARK 4.7. Assume that p is inert in $k/\mathbb{Q}$ and p does not divide $h(k)$. Let q be a prime number satisfying the following condition (slightly different from Example 4.6):

$$p \mid q+1\,, \text{ and } q \text{ is inert in } k/\mathbb{Q}\,.$$

Put $\mathfrak{p} = p\mathcal{O}_k$, $\mathfrak{q} = q\mathcal{O}_k$, and $S = \{\mathfrak{q}\}$. We note that S satisfies (N). In this case, we can see that $p^{\frac{q^2-1}{p}} \equiv 1 \pmod{q}$. This implies that $\mathfrak{p}$ always splits in $M_S(k)/k$. Hence q does not satisfy the assumption of Proposition 4.5.

We can also give a sufficient condition when p splits in k.

PROPOSITION 4.8. *Let q be a prime number which is inert in $k/\mathbb{Q}$. Put $\mathfrak{q} = q\mathcal{O}_k$ and $S = \{\mathfrak{q}\}$. Moreover, we assume that p and q satisfy all of the following conditions*:

(i) *p splits in $k/\mathbb{Q}$, p does not divide $h(k)$, and $\lambda(K^c/k) = 1$,*
(ii) *p divides $q+1$,*
(iii) *$\mathfrak{q}$ does not split in K^c/k,*
(iv) *both primes of k lying above p do not split in $M_S(k)/k$.*

Then $X_S(K^a)$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}_p$-module. In particular, $\mu_S(K^a/k) = 0$.

We note that S satisfies (N). We also remark that $\mathfrak{q}$ splits completely in K^a/k by Lemma 2.3 (1). We denote by $\mathfrak{p}, \mathfrak{p}^\iota$ the primes of k lying above p. Put $M = M_S(k)$. We note that p^2 does not divide q^2-1 by the assumption (iii). Hence M/k is a cyclic extension of degree p, and totally ramified at $\mathfrak{q}$ because $p \nmid h(k)$. Recall that K_1^a (resp. K_1^c) is the initial layer of K^a/k (resp. K^c/k). From the assumption that $p \nmid h(k)$, both of $\mathfrak{p}$ and $\mathfrak{p}^\iota$ are totally ramified in K^a/k (see, e.g., [25, p. 680]). We also note that $K_1^aK_1^c/K_1^a$ is an unramified extension (see, e.g., [25, pp. 680–681]). Put $\mathcal{K} = MK_1^aK_1^c$, then $\mathrm{Gal}(\mathcal{K}/k) \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus 3}$ and $\mathcal{K}/k$ is unramified outside $\{\mathfrak{q}, \mathfrak{p}, \mathfrak{p}^\iota\}$. The following is the "key lemma" of our proof of Proposition 4.8.

LEMMA 4.9. *Assume that k, p, and q satisfy the conditions of Proposition* 4.8, *and keep the notation as above. Then p does not divide $h(\mathcal{K})$.*

PROOF OF LEMMA 4.9. Our proof uses the central class field (see, e.g., [5], [23], [27], [29]). Let $\mathcal{K}_g$ be the genus field of $\mathcal{K}/k$, that is, the maximal unramified abelian extension of $\mathcal{K}$ which is also an abelian extension over k. Let $\mathcal{K}_z$ be the central class field of $\mathcal{K}/k$, that is, the maximal unramified abelian extension of $\mathcal{K}$ which is a Galois extension over k and $\mathrm{Gal}(\mathcal{K}_z/\mathcal{K})$ is contained in the center of $\mathrm{Gal}(\mathcal{K}_z/k)$. We note that $\mathcal{K}_g \subseteq \mathcal{K}_z$. It is well known that $p \nmid h(\mathcal{K})$ if and only if p does not divide $[\mathcal{K}_z : \mathcal{K}]$. Hence we shall show that p does not divide $[\mathcal{K}_z : \mathcal{K}]$.

We shall consider $[\mathcal{K}_g : \mathcal{K}]$ at first. We see that $\mathfrak{q}$ is unramified in $K_1^c K_1^a/k$. On the other hand, $\mathfrak{q}$ is totally ramified in M/k. Hence the ramification index of $\mathfrak{q}$ in $\mathcal{K}/k$ is p. For the prime $\mathfrak{p}$, it is unramified in M/k. Since the ramification index of $\mathfrak{p}$ in $K_1^a K_1^c/k$ is p (see, e.g., [25, pp. 680–681]), that of in $\mathcal{K}/k$ is also p. Similarly, we can see that the ramification index of $\mathfrak{p}^\iota$ in $\mathcal{K}/k$ is p. If p divides $[\mathcal{K}_g : \mathcal{K}]$, then there must be a non-trivial unramified abelian p-extension over k because the ramification indices of $\mathfrak{q}$, $\mathfrak{p}$, and $\mathfrak{p}^\iota$ are equal to p. This contradicts to the fact that $p \nmid h(k)$. Hence we see that p does not divide $[\mathcal{K}_g : \mathcal{K}]$.

Therefore, to see the assertion of this lemma, it suffices to prove that p does not divide $[\mathcal{K}_z : \mathcal{K}_g]$. Note that $\mathrm{Gal}(\mathcal{K}_z/\mathcal{K}_g)$ is an abelian p-group in our situation (see, e.g., [23], [27]), thus we will show that $\mathcal{K}_z = \mathcal{K}_g$. Put $G = \mathrm{Gal}(\mathcal{K}/k)$ and $V = \{\mathfrak{q}, \mathfrak{p}, \mathfrak{p}^\iota\}$. For $\mathfrak{r} \in V$, we denote by $G_\mathfrak{r}$ the decomposition subgroup of G for $\mathfrak{r}$, and $D_\mathfrak{r}$ the decomposition field for $\mathfrak{r}$ in $\mathcal{K}/k$.

Similar to [5], we can show that $\mathcal{K}_z = \mathcal{K}_g$ if both of the following conditions are satisfied.

(a) $|G_\mathfrak{r}| = p^2$ for each $\mathfrak{r} \in V$, and $G_\mathfrak{r} \neq G_{\mathfrak{r}'}$ for $\mathfrak{r} \neq \mathfrak{r}'$,

(b) $G_\mathfrak{q} \cap G_\mathfrak{p}$, $G_\mathfrak{q} \cap G_{\mathfrak{p}^\iota}$, and $G_\mathfrak{p} \cap G_{\mathfrak{p}^\iota}$ generate G.

(This follows from, e.g., Theorem 3, Example 2, and the facts stated in pp. 290–291 of [23]. See also [27, p. 423] and [5, p. 458, Lemma].) Moreover, they are also equivalent to the following conditions:

(a′) $[D_\mathfrak{r} : k] = p$ for each $\mathfrak{r} \in V$, and $D_\mathfrak{r} \neq D_{\mathfrak{r}'}$ for $\mathfrak{r} \neq \mathfrak{r}'$,

(b′) $D_\mathfrak{q} D_\mathfrak{p} D_{\mathfrak{p}^\iota} = \mathcal{K}$.

(This follows from the argument given in the proof of [5, Theorem 2].) Hence it suffices to prove (a′) and (b′). (See also [29], etc., for the case when the base field is $\mathbb{Q}$.)

Firstly, we shall prove (a′). Since q is inert in $k/\mathbb{Q}$, $\mathfrak{q}$ splits completely in K_1^a/k by Lemma 2.3 (1). By the assumption (iii), all primes of K_1^a lying above q are inert in $K_1^a K_1^c/K_1^a$. Moreover, since $\mathfrak{q}$ is totally ramified in M/k, all primes of $K_1^a K_1^c$ lying above q are totally ramified in $\mathcal{K}/K_1^a K_1^c$. Hence it follows that $D_\mathfrak{q} = K_1^a$ and $[D_\mathfrak{q} : k] = p$. We already noted that both of $\mathfrak{p}$ and $\mathfrak{p}^\iota$ are ramified in $D_\mathfrak{q}/k$. Let $k(\mathfrak{p}^\iota)$ be the inertia field of $\mathfrak{p}$ in $K_1^a K_1^c/k$. We note the fact that $K_1^a K_1^c$ coincides with L_1 which is defined in [12, p. 371, Lemma] (see [25, pp. 680–681]). Then we can see that $[k(\mathfrak{p}^\iota) : k] = p$ ([12, p. 371]), and $\mathfrak{p}$ is inert in $k(\mathfrak{p}^\iota)/k$ ([12, Theorem 3]). By the assumption (iv), $\mathfrak{p}$ is also inert in M/k. We see that the decomposition field $D'_\mathfrak{p}$ for $\mathfrak{p}$ in $Mk(\mathfrak{p}^\iota)/k$ is a cyclic extension over k of degree p, and $D'_\mathfrak{p} \neq k(\mathfrak{p}^\iota), M$. All primes of $D'_\mathfrak{p}$ lying above $\mathfrak{p}$ are inert in $Mk(\mathfrak{p}^\iota)/D'_\mathfrak{p}$. Since the ramification index of $\mathfrak{p}$ in $\mathcal{K}/k$ is p, the primes of $Mk(\mathfrak{p}^\iota)$ lying above $\mathfrak{p}$ are ramified in $\mathcal{K}/Mk(\mathfrak{p}^\iota)$. Hence it follows that $D_\mathfrak{p} = D'_\mathfrak{p}$ and $[D_\mathfrak{p} : k] = p$. We note that both of $\mathfrak{p}^\iota$ and $\mathfrak{q}$ are ramified in $D_\mathfrak{p}/k$ because $D_\mathfrak{p} \neq k(\mathfrak{p}^\iota), M$. Similarly, we can obtain that $[D_{\mathfrak{p}^\iota} : k] = p$, and both of $\mathfrak{p}$ and $\mathfrak{q}$ are ramified in $D_{\mathfrak{p}^\iota}/k$. From these facts, we can also see that $D_v \neq D_{v'}$ for $v \neq v'$.

Secondly, we shall prove (b′). Put $D' = D_\mathfrak{q} D_\mathfrak{p} D_{\mathfrak{p}^\iota}$. Suppose that $D' \subsetneq \mathcal{K}$. We note that $D_v \neq D_{v'}$ for $v \neq v'$, hence it follows that $[D' : k] = p^2$. We note that $M = M_S(k)$ is a Galois extension over $\mathbb{Q}$ because q is inert in k. From this, we see that $\mathcal{K}/\mathbb{Q}$ is also a Galois extension. Put $G' = \mathrm{Gal}(\mathcal{K}/D')$. Since $D_{\mathfrak{q}} = K_1^a$ and $D_{\mathfrak{p}}D_{\mathfrak{p}^\iota}$ are Galois extensions over $\mathbb{Q}$, and $D'/\mathbb{Q}$ is also. Hence $\mathrm{Gal}(k/\mathbb{Q}) = \langle \iota \rangle$ acts on G, and G' is closed under this action. Put $G^{\pm} = \{\tau \in G | \iota(\tau) = \tau^{\pm 1}\}$, then $G \simeq G^+ \oplus G^-$. Let $\mathcal{K}^{G^+}$ and $\mathcal{K}^{G^-}$ be the fixed fields of G^+ and G^- in $\mathcal{K}/k$, respectively. We can see that $K_1^c \subseteq \mathcal{K}^{G^-}$ since ι acts on $\mathrm{Gal}(K_1^c/k)$ trivially. Moreover, we see that $K_1^a \subseteq \mathcal{K}^{G^+}$ since ι acts on $\mathrm{Gal}(K_1^a/k)$ by inversion. We shall prove $M \subseteq \mathcal{K}^{G^+}$. Recall that $M/\mathbb{Q}$ is a Galois extension, and hence ι acts on $\mathrm{Gal}(M/k)$. We put $\mathrm{Gal}(M/k) = \langle \sigma \rangle (\simeq \mathbb{Z}/p\mathbb{Z})$. Then $\iota(\sigma)$ is equal to either σ or σ^{-1}. If $\iota(\sigma) = \sigma$, then M is an abelian extension over $\mathbb{Q}$. In this case, we can see that

$$\mathrm{Gal}(M/k) \simeq \mathrm{Gal}(M_{\{q\}}(\mathbb{Q})/\mathbb{Q}) \simeq (\mathbb{Z}/q\mathbb{Z})^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p\,,$$

where $M_{\{q\}}(\mathbb{Q})$ is the maximal abelian p-extension of $\mathbb{Q}$ unramified outside q. Since p divides $q+1$, we conclude that $\mathrm{Gal}(M/\mathbb{Q})$ is trivial. This is a contradiction. Consequently, ι must acts on $\mathrm{Gal}(M/k)$ by inversion, and hence $M \subseteq \mathcal{K}^{G^+}$. Therefore $\mathcal{K}^{G^-} = K_1^c$ and $\mathcal{K}^{G^+} = MK_1^a$, i.e., $|G^+| = p$ and $|G^-| = p^2$. Put $G' = \langle \tau \rangle (\simeq \mathbb{Z}/p\mathbb{Z})$. Since G' is closed under the action of ι, we see that $\iota(\tau)$ equals either τ or τ^{-1}. If $\iota(\tau) = \tau^{-1}$, then $G' \subseteq G^-$, and hence $K_1^c \subseteq D'$. Moreover, by the facts that $D_{\mathfrak{q}} = K_1^a \subseteq D'$ and $[D' : k] = p^2$, we can obtain $D' = K_1^a K_1^c$. However, since $\mathfrak{p}$ does not split in $K_1^a K_1^c$, it is a contradiction. Next, we assume $\iota(\tau) = \tau$. Then we see that $G' = G^+$ (i.e., $D' = MK_1^a$). However, since $\mathfrak{p}$ does not split in MK_1^a, it is a contradiction. Hence $D' = \mathcal{K}$. Therefore, we have shown that $\mathcal{K}_z = \mathcal{K}_g$, and then we can obtain our assertion. □

REMARK 4.10. It seems that one can also obtain this lemma by using [27, (2,4) Theorem].

PROOF OF PROPOSITION 4.8. By Lemma 4.9, it follows that $L(K_1^a) = K_1^a K_1^c$. We also note that $X(K^a) \simeq \mathbb{Z}_p$ (as a $\mathbb{Z}_p$-module) in this case (see, e.g., [8, p. 297]).

Since $\mathcal{K}/K_1^a$ is an abelian p-extension unramified outside the primes of K_1^a lying above $\mathfrak{q}$, it follows that $\mathcal{K} \subseteq M_S(K_1^a)$. Suppose that $\mathcal{K} \subsetneq M_S(K_1^a)$. Since $\mathfrak{q}$ splits completely in K_1^a/k, we denote by $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_p$ the primes of K_1^a lying above $\mathfrak{q}$. We consider the following exact sequence:

$$\bigoplus_{i=1}^{p} (\mathcal{O}_{K_1^a}/\mathfrak{q}_i)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \to X_S(K_1^a) \to X(K_1^a) \to 0\,.$$

We note that $\mathrm{Ker}(X_S(K_1^a) \to X(K_1^a)) = \mathrm{Gal}(M_S(K_1^a)/L(K_1^a))$. Thus $M_S(K_1^a)/L(K_1^a)$ is an elementary abelian p-extension because $|(\mathcal{O}_{K_1^a}/\mathfrak{q}_i)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p| = p$ for all i. Since the primes of $L(K_1^a)$ lying above $\mathfrak{q}$ are tamely ramified in $M_S(K_1^a)/L(K_1^a)$, the inertia subgroup of $\mathrm{Gal}(M_S(K_1^a)/L(K_1^a))$ for every prime lying above $\mathfrak{q}$ is cyclic. Furthermore, since $\mathfrak{q}$ is totally ramified in $M_S(k)/k$ and unramified in $L(K_1^a)/k$, all primes lying above $\mathfrak{q}$ are actually ramified in $\mathcal{K}/L(K_1^a)$. We can see that $M_S(K_1^a)/\mathcal{K}$ is non-trivial unramified abelian

p-extension because of the cyclicity of inertia subgroups. This contradicts to Lemma 4.9. Therefore, we can obtain $M_S(K_1^a) = \mathcal{K}$.

We denote by $\mathfrak{P}$ (resp. $\mathfrak{P}^\iota$) the unique prime of K_1^a lying above $\mathfrak{p}$ (resp. $\mathfrak{p}^\iota$). For $v \in \{\mathfrak{P}, \mathfrak{P}^\iota\}$, let G_v be the decomposition subgroup of $\mathrm{Gal}(\mathcal{K}/K_1^a)$ for v, and D_v the decomposition field for v in $\mathcal{K}/K_1^a$. We shall use the notations and results given in the proof of Lemma 4.9. Since $D_\mathfrak{q} = K_1^a$, we see that $D_\mathfrak{P} = D_\mathfrak{q} D_\mathfrak{p}$, and $D_{\mathfrak{P}^\iota} = D_\mathfrak{q} D_{\mathfrak{p}^\iota}$. We already showed that $\mathcal{K} = D_\mathfrak{q} D_\mathfrak{p} D_{\mathfrak{p}^\iota}$. Hence, we can obtain

$$D_\mathfrak{P} D_{\mathfrak{P}^\iota} = \mathcal{K} \text{ and } D_\mathfrak{P} \cap D_{\mathfrak{P}^\iota} = K_1^a .$$

Thus, $X_S(K_1^a)(= \mathrm{Gal}(\mathcal{K}/K_1^a) \simeq (\mathbb{Z}/p\mathbb{Z})^2)$ is generated by $G_\mathfrak{P}$ and $G_{\mathfrak{P}^\iota}$. This implies that $X'_S(K_1^a)$ is trivial. (We note that $X'_S(k)$ is also trivial.) Since both primes lying above p are totally ramified in K^a/k, we can obtain that

$$X'_S(K^a)/Z'_S \simeq X'_S(k) , \quad X'_S(K^a)/\nu_{1,0} Z'_S \simeq X'_S(K_1^a)$$

with a finite index submodule Z'_S of $X'_S(K^a)$ by Proposition 2.1 (2). From the fact that both of $X'_S(K^a)/Z'_S$ and $X'_S(K^a)/\nu_{1,0}Z'_S$ are trivial, we can see that $X'_S(K^a)$ is also trivial by using the same argument given in the proof of [9, Theorem 1 (1)] (cf. also [24]).

We also see that $X'_S(K_n^a)$ is trivial for $n \geq 0$. Hence $X_S(K_n^a)$ is generated by the decomposition subgroups for the primes lying above p. (Note that these decomposition subgroups are cyclic.) We see that the p-rank of $X_S(K_n^a)$ is at most 2 because the number of primes of K_n^a lying above p is 2. When $n \geq 2$, there is a natural surjection $X_S(K_n^a) \to X_S(K_1^a)$. Since the p-rank of $X_S(K_1^a)$ is 2, we see that the p-rank of $X_S(K_n^a)$ must be 2 for $n \geq 2$. This implies that the p-rank of $X_S(K^a)$ is also 2 (see, e.g., the proof of [9, Theorem 1 (2)]).

We see that $R_\mathfrak{q} \simeq \Lambda/(p)$ because $|(\mathcal{O}_k/\mathfrak{q})^\times \otimes_\mathbb{Z} \mathbb{Z}_p| = p$ and $\mathfrak{q}$ splits completely in K^a/k. Hence we have the following exact sequence:

$$\Lambda/(p) \to X_S(K^a) \to X(K^a) \to 0 .$$

Considering this sequence, we can conclude that $X_S(K^a) \simeq \mathbb{Z}_p \oplus \mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}_p$-module. □

REMARK 4.11. By Proposition 3.2, we see that the number of $\mathbb{Z}_p$-extensions satisfying $\mu_S > 0$ is at most 1 under the assumptions of Proposition 4.8. In this case, K^a/k is the only $\mathbb{Z}_p$-extension which has a possibility of being $\mu_S > 0$. Hence the assertion of Proposition 4.8 also implies that $\mu_S = 0$ for all $\mathbb{Z}_p$-extensions of k.

From Propositions 4.3 and 4.8 we can obtain the following:

COROLLARY 4.12. *Under the assumptions of Proposition* 4.8, *the unramified μ-invariant of a $\mathbb{Z}_p$-extension $K^a M_S(k)/M_S(k)$ is zero.*

5. Calculation examples

5.1. Criteria. Let K be a $\mathbb{Z}_p$-extension of an imaginary quadratic field k. In this section, for simplicity, we assume that p, k, and K satisfy either of the following (I) or (II).

(I) p does not split in $k/\mathbb{Q}$, and p does not divide $h(k)$. Moreover, $p > 3$ if $k = \mathbb{Q}(\sqrt{-3})$.

(II) p splits in $k/\mathbb{Q}$, p does not divide $h(k)$, and $\lambda(K^c/k) = 1$. Moreover, both primes of k lying above p are totally ramified in K/k.

REMARK 5.1. For every $\mathbb{Z}_p$-extension K/k satisfying (I), the unramified Iwasawa module $X(K)$ is trivial. For every $\mathbb{Z}_p$-extension K/k satisfying (II), the unramified Iwasawa invariants satisfy $\lambda(K/k) = 1$ and $\mu(K/k) = 0$ (see, e.g., [25, Theorem]).

Let $\mathfrak{q}$ be a prime of k not lying above p, and put $S = \{\mathfrak{q}\}$. Under the assumptions, if $\mathfrak{q}$ does not split completely in K/k then we see that $\mu_S(K/k) = 0$. In the rest of this section, we assume that $\mathfrak{q}$ splits completely in K, and satisfies the following:

(H) $p \mid N(\mathfrak{q}) - 1$, $p^2 \nmid N(\mathfrak{q}) - 1$.

Under the assumptions, we can obtain that $|X_S(k)| = p$. (Recall that $p > 3$ when $k = \mathbb{Q}(\sqrt{-3})$. See also Proposition 4.4.) Since $\mathfrak{q}$ splits completely in K/k, we see that $R_{\mathfrak{q}} \simeq \Lambda/(p)$. In the following, we shall give some criteria for the vanishing of $\mu_S(K/k)$. These criteria need an information on $X_S(K_n)$ or $X'_S(K_n)$.

We can obtain a lower bound for $|X_S(K_n)|$ and $|X'_S(K_n)|$ by using properties of finitely generated Λ-modules. (Similar results for the case of unramified Iwasawa modules are well-known. See, e.g., [7], [25].) Recall that $X_S(K)$ and $X'_S(K)$ are finitely generated torsion Λ-modules. Hence there exists an elementary torsion Λ-module E (resp. E') and a pseudo-isomorphism $X_S(K) \to E$ (resp. $X'_S(K) \to E'$). In our situation, all primes which are ramified in K/k are totally ramified. Hence, by using the method given in the proof of [25, (2.1) Proposition] (and Proposition 2.1), we can obtain the following estimations for all $n \geq 0$:

$$|X_S(K_n)| \geq |X_S(k)| \cdot |E/\nu_{n,0}E|, \quad |X'_S(K_n)| \geq |X'_S(k)| \cdot |E'/\nu_{n,0}E|.$$

We mention the fact that if $E = \Lambda/(p^m)$ then $|E/\nu_{n,0}E| = p^{m(p^n-1)}$ for all $n \geq 0$ (see, e.g., [25, (2.2) Proposition], [28, pp. 281–282]). In particular, if the μ-invariant of $X'_S(K)$ is positive, then the elementary torsion Λ-module E' which is pseudo isomorphic to $X'_S(K)$ contains a factor of the type $\Lambda/(p^m)$ with some $m \geq 1$, and hence we obtain that

$$|X'_S(K_n)| \geq |X'_S(k)| \cdot p^{m(p^n-1)} \geq |X'_S(k)| \cdot p^{p^n-1}$$

all $n \geq 0$. The same type result also holds for $X_S(K)$.

We also remark that if the μ-invariant of $X'_S(K)$ is 0, then the μ-invariant of $X_S(K)$ is also 0. (Recall that all primes lying above p are totally ramified in K/k. See also [15, pp. 262–263].)

PROPOSITION 5.2. *Assume that p, k, and K satisfy* (I). *Let $\mathfrak{q}$ be a prime of k which splits completely in K and satisfies* (H). *We put $S = \{\mathfrak{q}\}$. Moreover, we assume that the unique prime of k lying above p splits in $M_S(k)/k$. Then $|X'_S(K_n)| < p^{p^n}$ for some n implies $\mu_S(K/k) = 0$.*

REMARK 5.3. For the case that the unique prime of k lying above p does not split in $M_S(k)/k$, we see that $X_S(K)$ is finite by Proposition 4.5.

PROOF OF PROPOSITION 5.2. We note that $|X'_S(k)| = p$ by the assumptions. Hence the assertion follows from the facts stated in the last two paragraphs before the statement of this proposition. □

PROPOSITION 5.4. *Assume that p, k, and K satisfy* (II). *Let $\mathfrak{q}$ be a prime of k which splits completely in K and satisfies* (H). *We put $S = \{\mathfrak{q}\}$. Then $|X_S(K_n)| < p^{p^n+n}$ for some n implies $\mu_S(K/k) = 0$.*

PROOF. Under the assumption (II), we can see that the characteristic polynomial of $X(K)$ is T by using the fact that $L(K) = \widetilde{k}$, where $\widetilde{k}$ is the composite of all $\mathbb{Z}_p$-extensions of k. (See, e.g., [8, p. 297]. See also [25].) Since $R_{\mathfrak{q}} \simeq \Lambda/(p)$, we can obtain the following exact sequence:

$$\Lambda/(p) \to X_S(K) \to X(K) \to 0\,.$$

Assume that $\mu_S(K/k) > 0$. Then the characteristic polynomial of $X_S(K)$ must be pT. Put $E_1 = \Lambda/(p)$ and $E_2 = \Lambda/(T)$. We can see that there is a pseudo-isomorphism $X_S(K) \to E_1 \oplus E_2$. We note that $|X_S(k)| = p$ from the assumptions (II) and (H). By using [25, (2.2) Proposition], we can obtain the following for all $n \geq 0$:

$$\begin{aligned} |X_S(K_n)| &= |X_S(K)/\nu_{n,0}Z_S| \\ &\geq |X_S(k)| \cdot |E_1/\nu_{n,0}E_1| \cdot |E_2/\nu_{n,0}E_2| \\ &= p \cdot p^{p^n-1} \cdot p^n = p^{p^n+n}\,. \end{aligned}$$

Hence, the assertion follows. □

PROPOSITION 5.5. *Assume that p, k, and K satisfy* (II). *Let $\mathfrak{q}$ be a prime of k which splits completely in K and satisfies* (H). *We put $S = \{\mathfrak{q}\}$. Then $|X'_S(K_n)| < |X'_S(k)|p^{p^n-1}$ for some n implies $\mu_S(K/k) = 0$.*

PROOF. This also follows from the arguments given in the paragraphs before Proposition 5.2. □

We shall apply the above criteria for some imaginary quadratic fields. Recall that K/k satisfies (I) or (II), and $\mathfrak{q}$ satisfies (H). We also assumed that $\mathfrak{q}$ splits completely in K/k. Let $\mathcal{C}_{\mathfrak{q},1}$ be the ray class group of K_1 modulo $\mathfrak{q}\mathcal{O}_{K_1}$, and $\mathcal{A}_{\mathfrak{q},1}$ the Sylow p-subgroup of $\mathcal{C}_{\mathfrak{q},1}$. Since the primes lying above $\mathfrak{q}$ do not divide p, we see that $X_S(K_1) \simeq \mathcal{A}_{\mathfrak{q},1}$ by class field theory. We also see that $X'_S(K_1) \simeq \mathcal{A}_{\mathfrak{q},1}/(\mathcal{D}_1 \cap \mathcal{A}_{\mathfrak{q},1})$, where $\mathcal{D}_1$ is the subgroup of $\mathcal{C}_{\mathfrak{q},1}$ generated by the ray classes containing a prime of K_1 lying above p. The second author calculated $|\mathcal{A}_{\mathfrak{q},1}|$ and $|\mathcal{A}_{\mathfrak{q},1}/(\mathcal{D}_1 \cap \mathcal{A}_{\mathfrak{q},1})|$ by using Magma [2]. (PARI/GP [22] was also used to check a part of calculation results.) Moreover, the defining polynomials of K_1^a which are written in Kim-Oh [17, Table I] and Brink [3, p. 2136] were used in these calculations.

5.2. Calculation for the case (I) with $p = 3$. We assume that $p = 3$ and $k = \mathbb{Q}(\sqrt{-1})$. In this case, we can apply Proposition 5.2. Let q be a prime number satisfying the following condition:

$$q \text{ is inert in } k/\mathbb{Q} \quad \text{and} \quad \mathfrak{q} = q\mathcal{O}_k \text{ satisfies (H)}.$$

Then $\mathfrak{q}$ splits completely in K^a/k by Lemma 2.3 (1). Put $S = \{\mathfrak{q}\}$. We note that $|X_S(k)| = 3$ because q satisfies (H). We classify q into the following four types:

(1-a) $q \equiv 1 \pmod 3$ and $|X'_S(k)| = 1$,
(1-b) $q \equiv 1 \pmod 3$ and $|X'_S(k)| = 3$,
(2-a) $q \equiv 2 \pmod 3$ and $|X'_S(k)| = 1$,
(2-b) $q \equiv 2 \pmod 3$ and $|X'_S(k)| = 3$.

By Proposition 4.5 and Proposition 5.2, either

$$|X'_S(k)| = 1 \text{ or } |X'_S(K_1^a)| < 3^3$$

implies $\mu_S(K^a/k) = 0$. Thus, we see that $\mu_S(K^a/k) = 0$ for the types (1-a) and (2-a). We note that there is no prime number q satisfying (2-a) by Remark 4.7. For the primes $q < 500000$ satisfying the above assumptions, we obtained the following.

$p = 3, k = \mathbb{Q}(\sqrt{-1})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-b)	2320	3^3	3^2	0	1495	64.4
		3^3	3^3	?	825	35.6
(2-b)	6928	3^2	3^1	0	4621	66.7
		3^3	3^3	?	2307	33.3

The number of primes q satisfying (1-b) and $q < 500000$ is 2320, and 1495 of these primes satisfy $|X_S(K_1^a)| = 3^3$, $|X'_S(K_1^a)| = 3^2$ (and then $\mu_S(K^a/k) = 0$ for such primes). Similarly, we see that $\mu_S(K^a/k) = 0$ for about 66.7% of 6928 primes satisfying (2-b) and $q < 500000$. (Note that the percentage is rounded off at the first decimal place.) For both of (1-b) and (2-b), only two kinds of the pair $(|X_S(K_1^a)|, |X'_S(K_1^a)|)$ were found in our calculation results. It is a question whether this also holds for $q > 500000$ or not. (See also the below data and the other examples.)

By Proposition 3.2 and its proof, $\mu_S(K^a/k) = 0$ implies that $\mu_S = 0$ for all $\mathbb{Z}_p$-extensions of k. Moreover, $\mu_S(K^a/k) = 0$ also implies that $\mu(M_S(k)K^a/M_S(k)) = 0$ by Proposition 4.3.

For other fields satisfying (I) with $p = 3$, we obtained the following ($q < 500000$).

$p = 3, k = \mathbb{Q}(\sqrt{-7})$

type	total	$\|X_S(K_1^a)\|$	$\|X'_S(K_1^a)\|$	$\mu_S(K^a/k)$	number of q	%
(1-b)	2341	3^3	3^2	0	1577	67.4
		3^3	3^3	?	764	32.6
(2-b)	6944	3^2	3^1	0	4629	66.7
		3^3	3^3	?	2315	33.3

$p = 3, k = \mathbb{Q}(\sqrt{-19})$

type	total	$\|X_S(K_1^a)\|$	$\|X'_S(K_1^a)\|$	$\mu_S(K^a/k)$	number of q	%
(1-b)	2315	3^3	3^2	0	1558	67.3
		3^3	3^3	?	757	32.7
(2-b)	6959	3^2	3^1	0	4636	66.6
		3^3	3^3	?	2323	33.4

$p = 3, k = \mathbb{Q}(\sqrt{-43})$

type	total	$\|X_S(K_1^a)\|$	$\|X'_S(K_1^a)\|$	$\mu_S(K^a/k)$	number of q	%
(1-b)	2323	3^3	3^2	0	1582	68.1
		3^3	3^3	?	741	31.9
(2-b)	6934	3^2	3^1	0	4600	66.3
		3^3	3^3	?	2334	33.7

$p = 3, k = \mathbb{Q}(\sqrt{-67})$

type	total	$\|X_S(K_1^a)\|$	$\|X'_S(K_1^a)\|$	$\mu_S(K^a/k)$	number of q	%
(1-b)	2326	3^3	3^2	0	1580	67.9
		3^3	3^3	?	746	32.1
(2-b)	6972	3^2	3^1	0	4642	66.6
		3^3	3^3	?	2330	33.4

$p = 3, k = \mathbb{Q}(\sqrt{-163})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-b)	2374	3^3	3^2	0	1595	67.2
		3^3	3^3	?	779	32.8
(2-b)	6893	3^2	3^1	0	4619	67.0
		3^3	3^3	?	2274	33.0

5.3. Calculation for the case (I) with $p = 5$. We assume that $p = 5$ and $k = \mathbb{Q}(\sqrt{-2})$. Assume also that a prime number q is inert in $k/\mathbb{Q}$ and $\mathfrak{q} = q\mathcal{O}_k$ satisfies (H). Put $S = \{\mathfrak{q}\}$. Then $\mathfrak{q}$ splits completely in K^a/k by Lemma 2.3 (1). We classify q into the following four types:

(1-a) $q \equiv 1 \pmod 5$ and $\lvert X'_S(k)\rvert = 1$,
(1-b) $q \equiv 1 \pmod 5$ and $\lvert X'_S(k)\rvert = 5$,
(4-a) $q \equiv 4 \pmod 5$ and $\lvert X'_S(k)\rvert = 1$,
(4-b) $q \equiv 4 \pmod 5$ and $\lvert X'_S(k)\rvert = 5$.

Either

$$\lvert X'_S(k)\rvert = 1 \ \text{ or } \ \lvert X'_S(K_1^a)\rvert < 5^5$$

implies $\mu_S(K^a/k) = 0$. For the primes $q < 500000$ satisfying the above assumptions, we obtained the following.

$p = 5, k = \mathbb{Q}(\sqrt{-2})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-b)	809	5^3	5^2	0	657	81.2
		5^5	5^4	0	125	15.5
		5^5	5^5	?	27	3.3
(4-b)	4147	5^2	5^1	0	3320	80.1
		5^4	5^3	0	670	16.2
		5^5	5^5	?	157	3.8

For other fields satisfying (I) with $p = 5$, we obtained the following ($q < 500000$).

$p = 5, k = \mathbb{Q}(\sqrt{-3})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-b)	842	5^3	5^2	0	670	79.6
		5^5	5^4	0	137	16.3
		5^5	5^5	?	35	4.2
(4-b)	4171	5^2	5^1	0	3326	79.7
		5^4	5^3	0	676	16.2
		5^5	5^5	?	169	4.1

$p = 5, k = \mathbb{Q}(\sqrt{-5})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-b)	833	5^3	5^2	0	672	80.7
		5^5	5^4	0	136	16.3
		5^5	5^5	?	25	3.0
(4-b)	4165	5^2	5^1	0	3317	79.6
		5^4	5^3	0	670	16.1
		5^5	5^5	?	178	4.3

$p = 5, k = \mathbb{Q}(\sqrt{-7})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-b)	801	5^3	5^2	0	670	83.6
		5^5	5^4	0	100	12.5
		5^5	5^5	?	31	3.9
(4-b)	4161	5^2	5^1	0	3318	79.7
		5^4	5^3	0	675	16.2
		5^5	5^5	?	168	4.0

Since the percentages are rounded off, their sum is not necessarily to be 100%.

5.4. Other $\mathbb{Z}_p$-extensions (case (I)). We put $p = 3$ and $k = \mathbb{Q}(\sqrt{-1})$. Here, we consider the case that q splits in $k/\mathbb{Q}$. We denote by $\mathfrak{q}$, $\mathfrak{q}^\iota$ the primes of k lying above q. Assume that $\mathfrak{q}$ satisfies (H). Although $\mathfrak{q}$ does not split completely in K^a/k by Lemma 2.3 (2), there exists a unique $\mathbb{Z}_3$-extension of k such that $\mathfrak{q}$ splits completely by Lemma 2.2. (It also holds for $\mathfrak{q}^\iota$.) There are only four fields which can be the initial layer of a $\mathbb{Z}_3$-extension of k. Two of them are K_1^a and K_1^c. We denote by F_1, F_1^ι the other initial layers of $\mathbb{Z}_3$-extensions of

k (they are conjugate over $\mathbb{Q}$). Since defining polynomials of K_1^a and K_1^c are known, we can obtain a defining polynomial of an intermediate field of $K_1^a K_1^c/k$. In this case, we can take

$$f = x^6 - 6x^5 - 99x^4 + 1354x^3 + 5526x^2 - 13668x + 237977$$

as a defining polynomial of F_1. (Note that $x^3 - 3x - 1$ was used as a defining polynomial of the first layer of the cyclotomic $\mathbb{Z}_3$-extension.) Let K/k be the unique $\mathbb{Z}_3$-extension such that $\mathfrak{q}$ splits completely. We note that $\mathfrak{q}$ does not split in K^c/k by our assumption. Hence we see that K_1 is the unique cubic subextension of $K_1^a K_1^c/k$ such that $\mathfrak{q}$ splits completely. (Note that it can be occurred that $K_1 = K_1^a$.) Moreover, we may assume that $\mathfrak{q}$ splits completely in K_1^a or F_1. (If the primes lying above q do not split in K_1^a/k, just one prime lying above q splits in F_1/k.) Put $S = \{\mathfrak{q}\}$. Note that $\mathfrak{q}$ does not satisfy (H) when $q \equiv 2 \pmod 3$. Hence we shall classify q into the following two types:

(a) $q \equiv 1 \pmod 3$ and $|X'_S(k)| = 1$,

(b) $q \equiv 1 \pmod 3$ and $|X'_S(k)| = 3$.

In this case, either

$$|X'_S(k)| = 1 \text{ or } |X'_S(K_1)| < 3^3$$

implies $\mu_S(K/k) = 0$. Thus, $\mu_S(K/k) = 0$ for the type (a). For the type (b), we obtained the following result for $q < 500000$.

$p = 3, k = \mathbb{Q}(\sqrt{-1})$

K_1	total	$\lvert X_S(K_1)\rvert$	$\lvert X'_S(K_1)\rvert$	$\mu_S(K/k)$	number of q	%
F_1	1529	3^2	3^1	0	1008	65.9
		3^3	3^2	0	343	22.4
		3^3	3^3	?	178	11.6
K_1^a	773	3^2	3^1	0	524	67.8
		3^3	3^2	0	170	22.0
		3^3	3^3	?	79	10.2

For other fields satisfying (I) with $p = 3$, we obtained the following ($q < 500000$).

$p = 3, k = \mathbb{Q}(\sqrt{-7})$

$$f = x^6 - 6x^5 + 96x^4 - 4637x^3 + 516390x^2 - 5900613x + 68794273$$

K_1	total	$\lvert X_S(K_1)\rvert$	$\lvert X'_S(K_1)\rvert$	$\mu_S(K/k)$	number of q	%
F_1	1541	3^2	3^1	0	1042	67.6
		3^3	3^2	0	320	20.8
		3^3	3^3	?	179	11.6
K_1^a	740	3^2	3^1	0	508	68.6
		3^3	3^2	0	149	20.1
		3^3	3^3	?	83	11.2

$p = 3, k = \mathbb{Q}(\sqrt{-19})$

$$f = x^6 - 183x^5 + 59058x^4 - 5638684x^3 + 846963261x^2 - 31483317837x + 2880007852283$$

K_1	total	$\lvert X_S(K_1)\rvert$	$\lvert X'_S(K_1)\rvert$	$\mu_S(K/k)$	number of q	%
F_1	1548	3^2	3^1	0	1050	67.8
		3^3	3^2	0	332	21.4
		3^3	3^3	?	166	10.7
K_1^a	759	3^2	3^1	0	515	67.9
		3^3	3^2	0	171	22.5
		3^3	3^3	?	73	9.6

$p = 3, k = \mathbb{Q}(\sqrt{-43})$

$$f = x^6 - 6x^5 + 337947x^4 - 927794x^3 + 37453878699x^2 - 58156440513x + 1371920398285159$$

K_1	total	$\lvert X_S(K_1)\rvert$	$\lvert X'_S(K_1)\rvert$	$\mu_S(K/k)$	number of q	%
F_1	1535	3^2	3^1	0	1029	67.0
		3^3	3^2	0	344	22.4
		3^3	3^3	?	162	10.6
K_1^a	764	3^2	3^1	0	511	66.9
		3^3	3^2	0	159	20.8
		3^3	3^3	?	94	12.3

$p = 3, k = \mathbb{Q}(\sqrt{-67})$

$$f = x^6 - 6x^5 + 1395234x^4 - 2718680x^3 + 637961231943x^2 - 801945922254x + 96282167114135501$$

K_1	total	$\lvert X_S(K_1)\rvert$	$\lvert X'_S(K_1)\rvert$	$\mu_S(K/k)$	number of q	%
F_1	1555	3^2	3^1	0	1034	66.5
		3^3	3^2	0	340	21.9
		3^3	3^3	?	181	11.6
K_1^a	740	3^2	3^1	0	491	66.4
		3^3	3^2	0	167	22.6
		3^3	3^3	?	82	11.1

$p = 3, k = \mathbb{Q}(\sqrt{-163})$

$$f = x^6 + 1683x^5 + 14095938x^4 + 14591467188x^3 + 61493922898743x^2 + 30803779397034963x + 83715673074662296513$$

K_1	total	$\lvert X_S(K_1)\rvert$	$\lvert X'_S(K_1)\rvert$	$\mu_S(K/k)$	number of q	%
F_1	1508	3^2	3^1	0	1001	66.4
		3^3	3^2	0	367	24.3
		3^3	3^3	?	140	9.3
K_1^a	740	3^2	3^1	0	502	67.8
		3^3	3^2	0	170	23.0
		3^3	3^3	?	68	9.2

5.5. Calculation for the case (II) with $p = 3$. We assume that $p = 3$ and $k = \mathbb{Q}(\sqrt{-2})$. In this case, we can apply Propositions 5.4 and 5.5. Let q be a prime number satisfying the following condition:

$$q \text{ is inert in } k/\mathbb{Q} \text{ and } \mathfrak{q} = q\mathcal{O}_k \text{ satisfies (H).}$$

Then $\mathfrak{q}$ splits completely in K^a/k by Lemma 2.3 (1). Put $S = \{\mathfrak{q}\}$. We classify q into the following four types:

(1-a) $q \equiv 1 \pmod 3$ and $|X'_S(k)| = 1$,
(1-b) $q \equiv 1 \pmod 3$ and $|X'_S(k)| = 3$,
(2-a) $q \equiv 2 \pmod 3$ and $|X'_S(k)| = 1$,
(2-b) $q \equiv 2 \pmod 3$ and $|X'_S(k)| = 3$.

By Propositions 5.4 and 5.5, either

$$|X_S(K_1^a)| < 3^4 \text{ or } |X'_S(K_1^a)| < |X'_S(k)|3^2$$

implies $\mu_S(K^a/k) = 0$. We can see $\mu_S(K^a/k) = 0$ for the type (2-a) by Proposition 4.8. For the primes $q < 500000$ satisfying the above assumptions, we obtained the following.

$p = 3, k = \mathbb{Q}(\sqrt{-2})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-a)	4606	3^3	3^1	0	3018	65.5
		3^4	3^2	?	1588	34.5
(1-b)	2324	3^3	3^1	0	1552	66.8
		3^4	3^3	?	772	33.2
(2-b)	2277	3^4	3^2	0	1537	67.5
		3^4	3^3	?	740	32.5

For other fields satisfying (II) with $p = 3$, we obtained the following ($q < 500000$).

$p = 3, k = \mathbb{Q}(\sqrt{-5})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-a)	4642	3^3	3^1	0	3077	66.3
		3^4	3^2	?	1565	33.7
(1-b)	2315	3^3	3^1	0	1541	66.6
		3^4	3^3	?	774	33.4
(2-b)	2345	3^4	3^2	0	1539	65.6
		3^4	3^3	?	806	34.4

$p = 3, k = \mathbb{Q}(\sqrt{-11})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-a)	4622	3^3	3^1	0	3123	67.6
		3^4	3^2	?	1499	32.4
(1-b)	2333	3^3	3^1	0	1586	68.0
		3^4	3^3	?	747	32.0
(2-b)	2317	3^4	3^2	0	1566	67.6
		3^4	3^3	?	751	32.4

5.6. Calculation for the case (II) with $p = 5$. We assume that $p = 5$ and $k = \mathbb{Q}(\sqrt{-1})$. Assume also that a prime number q is inert in $k/\mathbb{Q}$ and $\mathfrak{q} = q\mathcal{O}_k$ satisfies (H). Put $S = \{\mathfrak{q}\}$. Then $\mathfrak{q}$ splits completely in K^a/k by Lemma 2.3 (1). We classify q into the following four types:

(1-a) $q \equiv 1 \pmod 5$ and $\lvert X'_S(k)\rvert = 1$,

(1-b) $q \equiv 1 \pmod 5$ and $|X'_S(k)| = 5$,
(4-a) $q \equiv 4 \pmod 5$ and $|X'_S(k)| = 1$,
(4-b) $q \equiv 4 \pmod 5$ and $|X'_S(k)| = 5$.

Either

$$|X_S(K_1^a)| < 5^6 \text{ or } |X'_S(K_1^a)| < |X'_S(k)|5^4$$

implies $\mu_S(K^a/k) = 0$. We see that $\mu_S(K^a/k) = 0$ for the type (4-a) by Proposition 4.8. For the primes $q < 500000$ satisfying the above assumptions, we obtained the following.

$p = 5, k = \mathbb{Q}(\sqrt{-1})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-a)	3349	5^3	5^1	0	2671	79.8
		5^5	5^3	0	554	16.5
		5^6	5^4	?	124	3.7
(1-b)	833	5^3	5^1	0	675	81.0
		5^5	5^3	0	121	14.5
		5^6	5^5	?	37	4.4
(4-b)	817	5^4	5^2	0	671	82.1
		5^6	5^4	0	120	14.7
		5^6	5^5	?	26	3.2

When $k = \mathbb{Q}(\sqrt{-19})$ and $p = 5$, we obtained the following ($q < 500000$).

$p = 5, k = \mathbb{Q}(\sqrt{-19})$

type	total	$\lvert X_S(K_1^a)\rvert$	$\lvert X'_S(K_1^a)\rvert$	$\mu_S(K^a/k)$	number of q	%
(1-a)	3346	5^3	5^1	0	2692	80.5
		5^5	5^3	0	522	15.6
		5^6	5^4	?	132	3.9
(1-b)	831	5^3	5^1	0	672	80.9
		5^5	5^3	0	130	15.6
		5^6	5^5	?	29	3.5
(4-b)	818	5^4	5^2	0	649	79.3
		5^6	5^4	0	139	17.0
		5^6	5^5	?	30	3.7

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