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Fourier Multipliers from L^p-spaces to Morrey Spaces on the Unit Circle

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Abstract. Let p, λ be real numbers such that $1 \leq p \leq \infty$, and $0 \leq \lambda \leq 1$. Also we let $L^p(\mathbf{T})$ be the L^p-spaces on the unit circle $\mathbf{T}$, $L^{p,\lambda}(\mathbf{T})$ Morrey spaces on $\mathbf{T}$ (cf. [14]), and $M(L^p, L^{p,\lambda})$ the set of all translation invariant bounded linear operators from $L^p(\mathbf{T})$ to $L^{p,\lambda}(\mathbf{T})$. Figa-Talamanca and Gaudry [2] showed $M(L^p, L^p) \neq M(L^q, L^q)$ $(1 < p < q \leq 2)$. In this paper, we generalize Gaudry's result. Our main results are $M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu})$ for $\lambda/p \neq \nu/q$ $(1 < p, q < \infty,\ 0 < \lambda, \nu < 1)$, and $M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu})$ for $2 < p < q$ and $\lambda/p = \nu/q$ $(0 < \lambda, \nu < 1)$. Moreover, we show a relation between $M(L^p, L^{p,\lambda})$ and the measure whose distribution function satisfies a Lipschitz condition (cf. [4]).

1. Introduction

Let $1 \leq p \leq \infty$ and $0 \leq \lambda \leq 1$. Then $L^p(\mathbf{T})$ denotes the L^p-spaces on the unit circle $\mathbf{T}$ and $L^{p,\lambda}(\mathbf{T})$ denotes Morrey spaces defined by

$$L^{p,\lambda}(\mathbf{T}) = \left\{ f \;\middle|\; ||f||_{p,\lambda} := \sup_{\substack{I \subset \mathbf{T} = [-\pi,\pi) \\ I \neq \phi:\text{interval}}} \left(\frac{1}{|I|^\lambda} \int_I |f|^p \frac{dx}{2\pi} \right)^{\frac{1}{p}} < \infty \right\}.$$

We note $L^{p,0}(\mathbf{T}) = L^p(\mathbf{T})$, $L^{p,1}(\mathbf{T}) = L^\infty(\mathbf{T})$ and $L^{p,\lambda}(\mathbf{T})$ is a Banach space (cf. [10], [14; p.215]). We remark $L^{p,\lambda}(\mathbf{T}) \neq L^p(\mathbf{T})$ for $0 < \lambda < 1$ ([15]).

For Banach spaces X and Y which are translation invariant function spaces in $L^1(\mathbf{T})$, we denote by $M(X, Y)$ the set of all operators which are translation invariant bounded linear operators from X to Y. We note $M(X, Y)$ is a Banach space with respect to the operator norm $\|\cdot\|_{M(X,Y)}$. An element of $M(X, Y)$ is called a Fourier multiplier (operator). When $X = L^p$ and $Y = L^q$, an element of $M(L^p, L^q) \cap M(\mathbf{T})$ for $1 \leq p < q$ is called an L^p-improving measure ([6] cf. [5], [7]), where $M(\mathbf{T})$ is the set of all bounded regular Borel measures on $\mathbf{T}$. Let μ be a non-negative measure on $\mathbf{T}$. For $0 < \alpha < 1$, we denote $\mu \in Lip_\alpha(M(\mathbf{T}))$,

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if there exists a positive constant C such that $\mu(I) \leq C|I|^\alpha$ for any non-empty interval $I \subset \mathbf{T}$. μ_f is called that the distribution function of μ_f satisfies the Lipschitz condition, if $\mu_f \in Lip_\alpha(M(\mathbf{T}))$ for some $0 < \alpha < 1$, where $\mu_f(E) = \int_E f(x)\frac{dx}{2\pi}$ for a measurable set E on $\mathbf{T}$ and a nonnegative function $f \in L^1(\mathbf{T})$. For $M(L^p, L^q)$ and $Lip_\alpha(M(\mathbf{T}))$, the following results are known.

THEOREM A ([2] cf. [3], [11]). *Let $1 < p < q \leq 2$. Then we have*

$$M(L^p, L^p) \neq M(L^q, L^q) .$$

THEOREM B ([4]). *There exists $f \in L^1(\mathbf{T})$ with $f \geq 0$ such that*

$$T_f \notin \bigcup_{1 \leq p < q < \infty} M(L^p, L^q), \ \ \mu_f \in \bigcap_{0<\alpha<1} Lip_\alpha(M(\mathbf{T})) .$$

Then we study those results in Morrey spaces.

Our main results are as follows:

THEOREM 1.1. *Let $1 \leq p, q < \infty$ and $0 < \lambda, \nu < 1$. Suppose $\frac{\lambda}{p} \neq \frac{\nu}{q}$. Then we have*

$$M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu}) .$$

THEOREM 1.2. *Let $0 < \lambda, \nu < 1$. Also let p, q be positive numbers with $1+\lambda < p < q$ and $\frac{1}{p} + \frac{1}{q} < 1$. Suppose $\frac{\lambda}{p} = \frac{\nu}{q}$. Then we have*

$$M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu}) .$$

THEOREM 1.3. *Let $f \in L^1(\mathbf{T})$ be a non-negative function. Then we have that μ_f is in $Lip_\alpha(M(\mathbf{T}))$ for some $0 < \alpha < 1$, if and only if $T_f \in M(L^p, L^{p,\lambda})$ for some $1 < p < \infty$ and $0 < \lambda < 1$, where $T_f g = f * g$.*

The paper is organized as follows: In §2, we investigate the inclusion relation between $L^p(\mathbf{T})$ and $L^{p,\lambda}(\mathbf{T})$. In §3, we prove Theorem 1.1 by the norm estimate of the Dirichlet kernel in $M(L^p, L^{p,\lambda})$. In §4, we prove Theorem 1.2 by using the norm estimate of the Rudin-Shapiro polynomials in $M(L^p, L^{p,\lambda})$. In §5, we prove Theorem 1.3. Throughout this paper, we denote by $|E|$ the normalized Haar measure of $E \subset \mathbf{T}$.

The letter C stands for a constant not necessarily the same at each occurrence. $A \sim B$ stands for $C^{-1}A \leq B \leq CA$ for some $C > 0$.

2. $L^p(\mathbf{T})$ and $L^{p,\lambda}(\mathbf{T})$

In this section, we will consider the inclusion relation between the L^p-spaces and Morrey spaces on $\mathbf{T}$.

PROPOSITION 2.1 (cf. [8; Proposition 5.1], [13; Lemma 1.3]). *Let $1 \leq r, p < \infty$ and $0 < \lambda < 1$. Then, we have the following*:

(1) $L^{p,\lambda}(\mathbf{T}) \subsetneq L^r(\mathbf{T})$ *if* $1 \le r \le p < \infty$;
(2) $L^{p,\lambda}(\mathbf{T}) \not\subset L^r(\mathbf{T})$ *and* $L^r(\mathbf{T}) \not\subset L^{p,\lambda}(\mathbf{T})$ *if* $p < r < \frac{p}{1-\lambda}$;
(3) $L^r(\mathbf{T}) \subsetneq L^{p,\lambda}(\mathbf{T})$ *if* $r \ge \frac{p}{1-\lambda}$.

PROOF. (1) Since $L^{p,\lambda}(\mathbf{T}) \subsetneq L^p(\mathbf{T})$ (see [15; p.587]), we get the desired result.
(2) By the assumption on r, we can choose $0 < \lambda_0 < \lambda$ as $r = \frac{p}{1-\lambda_0}$, and $\mu > 0$ such that $\frac{1-\lambda}{p} < \mu < \frac{1}{r}$. Set $f(x) = \chi_{(0,1)}(x)x^{-\mu} \in L^r(\mathbf{T})$. Then we have $f \notin L^{p,\lambda}(\mathbf{T})$. Let $I = (a, b)$ for $0 < a < b < 1$. By the mean value theorem, we have

$$\begin{aligned}\frac{1}{|I|^\lambda}\int_I |f|^p \frac{dx}{2\pi} &= (b-a)^{-\lambda}\int_a^b x^{-p\mu}\frac{dx}{2\pi}\\ &= C(b-a)^{1-\lambda}(a+\theta(b-a))^{-p\mu}\\ &\ge C(b-a)^{1-\lambda}b^{-p\mu}\end{aligned}$$

for some $0 < \theta < 1$. So, putting $a = \frac{b}{2}$, we have

$$\frac{1}{|I|^\lambda}\int_I |f|^p \frac{dx}{2\pi} \ge Cb^{1-\lambda-p\mu}$$

for all $0 < b < 1$. Since $\mu > \frac{1-\lambda}{p}$, we have $f \notin L^{p,\lambda}(\mathbf{T})$. Therefore, we get $f \in L^r(\mathbf{T})$ and $f \notin L^{p,\lambda}(\mathbf{T})$.

Next we show $L^{p,\lambda}(\mathbf{T}) \not\subset L^r(\mathbf{T})$ for all $\lambda_0 < \lambda < 1$. Suppose $L^{p,\lambda}(\mathbf{T}) \subset L^r(\mathbf{T})$. By the closed graph theorem, there exists a constant C such that

$$\|f\|_r \le C\|f\|_{p,\lambda}$$

for all $f \in L^{p,\lambda}(\mathbf{T})$. Now let δ be in $0 < \delta < \frac{1}{10}$, and $N \in \mathbf{N}$. Also we denote $I(k,\delta) = \{x \in (0,1) | \frac{k}{N} - \frac{\delta}{2} < x < \frac{k}{N} + \frac{\delta}{2}\}$ for $k = 1, \dots, N-1$, $I(N,\delta) = \{x \in (0,1) | 1 - \frac{\delta}{2} < x < 1\}$, and $E = \cup_{k=1}^N I(k,\delta)$. Then we choose a natural number N such that $\delta N \sim \delta^{1-\lambda}$. Hence, we have $|E| \sim \delta N \sim \delta^{1-\lambda}$. When we define $g_\delta = \delta^{-\frac{1}{r}}\chi_E$. For any non-empty interval $I \subset \mathbf{T}$, we have

$$\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \le |I|^{-\lambda}\delta^{-\frac{p}{r}}|E \cap I|\,.$$

Here, we investigate the left-hand sides of the inequality for $k = Card\{\ell | I(\ell,\delta) \cap (E \cap I) \ne \phi\} \ge 4$. Since $\frac{k}{2N} \le |I| \le \frac{k+1}{N}$ and $(k-2)\delta \le |E \cap I| \le k\delta$, we have

$$|I|^{-\lambda}\delta^{-\frac{p}{r}}|E \cap I| \le |I|^{-\lambda}\delta^{-\frac{p}{r}}k\delta \le |I|^{-\lambda}\delta^{-\frac{p}{r}}(2N|I|)\delta \le C\delta^{\lambda_0-\lambda}\,,$$

and

$$\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \le C\delta^{\lambda_0-\lambda}\,.$$

Next we estimate $\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi}$ for $k = Card\{\ell | I(\ell,\delta)\cap(E\cap I)\neq\phi\} \leq 3$. Since $|E\cap I| \leq C\min\{3\delta, |I|\}$, we have

$$\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \leq C\min\{|I|^{1-\lambda}\delta^{-\frac{p}{r}}, |I|^{-\lambda}\delta^{1-\frac{p}{r}}\}.$$

Hence, we have $\frac{1}{|I|^\lambda}\int_I |g_\delta|^p \frac{dx}{2\pi} \leq C\delta^{1-\lambda-\frac{p}{r}}$ by using the case $|I|\leq\delta$ or $|I|>\delta$. Thus, we obtain $\|g_\delta\|_{p,\lambda} \leq C\delta^{\frac{\lambda_0-\lambda}{p}}$ for sufficiently small $\delta>0$. By the assumption $L^{p,\lambda}(\mathbf{T})\subset L^r(\mathbf{T})$, we have

$$\delta^{-\frac{\lambda}{r}} \sim \|g_\delta\|_r \leq C\|g_\delta\|_{p,\lambda} \leq C\delta^{\frac{\lambda_0-\lambda}{p}}.$$

This contradicts $\delta^{\frac{\lambda-\lambda_0}{p}-\frac{\lambda}{r}} \leq C$ with $\frac{\lambda-\lambda_0}{p}-\frac{\lambda}{r} = \frac{\lambda_0}{p}(\lambda-1) < 0$ for $0<\lambda<1$. Hence we have $L^{p,\lambda}(\mathbf{T}) \not\subset L^r(\mathbf{T})$.

(3) By the Hölder inequality, we have $\|f\|_{p,\lambda} \leq C\|f\|_r$ for all $f\in L^r(\mathbf{T})$, and thus $L^r(\mathbf{T})\subset L^{p,\lambda}(\mathbf{T})$. Suppose $r_0 = \frac{p}{1-\lambda}$. When we define $f(x) = \chi_{(0,1)}(x)x^{-\frac{1}{r_0}}$, it is easy to show $f\notin L^{r_0}(\mathbf{T})$ and $f\in L^{p,\lambda}(\mathbf{T})$ similar to (1). Thus, we have $L^r(\mathbf{T}) \subsetneq L^{p,\lambda}(\mathbf{T})$ for $r\geq\frac{p}{1-\lambda}$. □

COROLLARY 2.2. *Let D_N be the Dirichlet kernel $D_N(x) = \sum_{k=-N}^{N} e^{ikx}$ of degree N. Then, we have*

$$\|D_N\|_{p,\lambda} \sim N^{\frac{\lambda}{p}+\frac{1}{p'}}$$

for any $1\leq p<\infty$ and $0<\lambda<1$.

PROOF. Since we have $L^r(\mathbf{T})\subset L^{p,\lambda}(\mathbf{T})$ for $r=\frac{p}{1-\lambda}$ by Proposition 2.1 (3), there exists a constant $C>0$ such that $\|D_N\|_{p,\lambda} \leq C\|D_N\|_r$. By Edwards [1; Exercise 7.5], we have

$$\|D_N\|_{p,\lambda} \leq C\|D_N\|_r \sim N^{\frac{1}{r'}} = N^{\frac{\lambda}{p}+\frac{1}{p'}}.$$

For the interval $I_N = [-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}]$, we have

$$|I_N|^{-\lambda}\int_{I_N} |D_N|^p \frac{dx}{2\pi} \geq |I_N|^{-\lambda}\int_0^{\frac{\pi}{2N+1}} \left(\frac{(N+\frac{1}{2})x\frac{2}{\pi}}{\frac{x}{2}}\right)^p \frac{dx}{2\pi} \sim N^{p+\lambda-1},$$

and $\|D_N\|_{p,\lambda} \geq CN^{\frac{\lambda}{p}+\frac{1}{p'}}$. Therefore, we get the desired result. □

REMARK 2.3. Similarly, for the Poisson kernel $P_r(x) = \frac{1-r^2}{1-2r\cos x+r^2}$ $(0<r<1)$, we have

$$\|P_r\|_{p,\lambda} \sim ((1-r)^{-1})^{\frac{\lambda}{p}+\frac{1}{p'}}.$$

3. $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$ $(\frac{\lambda}{p} \neq \frac{\nu}{q})$

In this section, we consider between $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$.

First we obtain the following:

LEMMA 3.1. *Let $0 < \lambda < 1$ and $1 \leq p, q < \infty$. Suppose $q > p(1-\lambda)$. We define the operator $T \in M(L^p, L^{q,\lambda})$ such that $Tf = D_N * f$. Then, we have*

$$\|D_N\|_{M(L^p,L^{q,\lambda})} = \|T\|_{M(L^p,L^{q,\lambda})} \sim N^{\frac{1}{p}-\frac{1-\lambda}{q}} .$$

In particular, $\|D_N\|_{M(L^p,L^{p,\lambda})} \sim N^{\frac{\lambda}{p}}$.

PROOF. Since we have $L^r(\mathbf{T}) \subset L^{q,\lambda}(\mathbf{T})$ for $r = \frac{q}{1-\lambda}$ and $L^r(\mathbf{T}) \subset L^p(\mathbf{T})$ by the assumption, we obtain $\|T\|_{M(L^p,L^{q,\lambda})} \leq \|T\|_{M(L^p,L^r)}$. By the norm estimate of D_N in $M(L^p, L^r)$ (cf. [1]), we get

$$\|T\|_{M(L^p,L^r)} \leq CN^{\frac{1}{p}-\frac{1}{r}} .$$

Conversely, we have $\|T\|_{M(L^p,L^{q,\lambda})} \geq CN^{\frac{1}{p}-\frac{1-\lambda}{q}}$, by $\|D_N\|_{q,\lambda} \leq \|T\|_{M(L^p,L^{q,\lambda})}\|D_N\|_p$ and Corollary 2.2. Hence, we obtain

$$\|D_N\|_{M(L^p,L^{q,\lambda})} = \|T\|_{M(L^p,L^{q,\lambda})} \sim N^{\frac{1}{p}-\frac{1-\lambda}{q}} ,$$

and we get the desired result. □

Now we can prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let $0 < \lambda, \nu < 1$, $1 \leq p, q < \infty$, and $\frac{\lambda}{p} \neq \frac{\nu}{q}$. By Lemma 3.1, we have $\|D_N\|_{M(L^p,L^{p,\lambda})} \sim N^{\frac{\lambda}{p}}$. Thus, we obtain $M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu})$. □

COROLLARY 3.2. *Let $0 < \lambda, \nu < 1$ and $1 \leq p, q < \infty$. Suppose $\frac{\lambda}{p} > \frac{\nu}{q}$. Then there exists $f \in L^1(\mathbf{T})$ such that $T_f \in M(L^q, L^{q,\nu})$ and $T_f \notin M(L^p, L^{p,\lambda})$, where $T_f g = f * g$.*

PROOF. Let a be a positive number with $\frac{\nu}{q} < a < \frac{\lambda}{p}$. Also we define $k_n = 2^{n+4}$. Then, we have $k_n + 2^n < k_{n+1} - 2^{n+1}$ $(n \geq 1)$. When we define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^{an}} D_{2^n}(x) e^{ik_n x} ,$$

we show that T_f satisfies the desired conditions. When we choose r such that $\frac{1}{r'} < \frac{\nu}{q}$ with $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$\|f\|_r \leq C \sum_{n=1}^{\infty} \frac{1}{2^{an}} \|D_{2^n}(x) e^{ik_n x}\|_r$$

$$\leq C \sum_{n=1}^{\infty} 2^{n(-a+\frac{1}{r'})} < \infty ,$$

and $f \in L^r(\mathbf{T}) \subset L^1(\mathbf{T})$. Also we obtain $T_f \in M(L^q, L^{q,\nu})$, since

$$\begin{aligned} \|f * g\|_{q,\nu} &\leq C \sum_{n=1}^{\infty} \frac{1}{2^{an}} \|D_{2^n}(x)e^{ik_n x} * g\|_{q,\nu} \\ &\leq C\|g\|_q \end{aligned}$$

by Lemma 3.1 and $a > \frac{\nu}{q}$. Similarly, since $T_f(D_{2^n}(x)e^{ik_n x}) = 2^{-an}D_{2^n}(x)e^{ik_n x}$, we have $T_f \notin M(L^p, L^{p,\lambda})$. Thus, we get the desired result. □

REMARK 3.3. We have $M(L^p, L^{p,\lambda}) = M(L^p, L_0^{p,\lambda})$ $(1 \leq p < \infty,\ 0 < \lambda < 1)$, where $L_0^{p,\lambda}(\mathbf{T})$ is the closure of $C(\mathbf{T})$ in $L^{p,\lambda}(\mathbf{T})$.

REMARK 3.4. We remark $M(L^1, L^{p,\lambda}) = L^{p,\lambda}(\mathbf{T})$ $(1 < p < \infty,\ 0 < \lambda < 1)$. In fact, let f_0 be in $L^{p,\lambda}(\mathbf{T})$, and g in $L^1(\mathbf{T})$. Then we have $\|f_0 * g\|_{p,\lambda} \leq \|f_0\|_{p,\lambda}\|g\|_1$ by the Hölder inequality, and $L^{p,\lambda}(\mathbf{T}) \subset M(L^1, L^{p,\lambda})$. Conversely, let T be in $M(L^1, L^{p,\lambda})$, and $K_N(x) = \sum_{j=-N}^{N}(1 - \frac{|j|}{N+1})e^{ijx}$ the Fejér kernel of degree N. Then we obtain $TK_N \in L^{p,\lambda}(\mathbf{T})$ and $\|TK_N\|_{p,\lambda} \leq \|T\|_{M(L^1,L^{p,\lambda})}$ $(N \geq 1)$. Hence, there exists $\{TK_{N_j}\}_j$, a subsequence of $\{TK_N\}_N$, such that TK_{N_j} converges in the weak*-topology of $L^{p,\lambda}(\mathbf{T})$ for some $f \in L^{p,\lambda}(\mathbf{T})$. By the Banach-Alaoglu theorem, since we have the predual of $L^{p,\lambda}(\mathbf{T})$ ([15]), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Tg(x)h(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f * g(x)h(x)dx$$

for all $h \in C(\mathbf{T})$ and any trigonometric polynomial g. Therefore, we obtain $Tg = f * g$ $(g \in L^1(\mathbf{T}))$. Then we get $M(L^1, L^{p,\lambda}) = L^{p,\lambda}(\mathbf{T})$.

PROPOSITION 3.5. *Let* $0 < \lambda, \nu < 1$ *and* $1 < p, q < \infty$. *Suppose* $2 < p < q$ *or* $q < p \leq 2$. *For* $\lambda = \frac{p-2}{q-2}\nu$, *we have*

$$M(L^q, L^{q,\nu}) \subsetneq M(L^p, L^{p,\lambda}) .$$

PROOF. Since $L^{q,\nu}(\mathbf{T}) \subset L^q(\mathbf{T})$, we have $M(L^q, L^{q,\nu}) \subset M(L^2, L^2)$. First let $2 < p < q$, and $T \in M(L^q, L^{q,\nu})$. Since T is bounded from $L^q(\mathbf{T})$ to $L^{q,\nu}(\mathbf{T})$ and from $L^2(\mathbf{T})$ to $L^2(\mathbf{T})$, we obtain that T is bounded from $L^p(\mathbf{T})$ to $L^{p,\kappa}(\mathbf{T})$ by the Peetre interpolation theorem [12; Theorem 4.1], where p and κ are defined by $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}$ and $\frac{\kappa}{p} = \frac{\theta}{q}\nu + \frac{1-\theta}{2}0$. Then an arithmetic shows $\kappa = \frac{p-2}{q-2}\nu$. Since $\frac{\lambda}{p} \neq \frac{\nu}{q}$, we have $M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu})$. □

4. $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$ ($\frac{\lambda}{p} = \frac{\nu}{q}$)

In this section, we consider the inclusion relation between $M(L^p, L^{p,\lambda})$ and $M(L^q, L^{q,\nu})$ for $\frac{\lambda}{p} = \frac{\nu}{q}$, and $0 < \lambda, \nu < 1$, $1 < p < q < \infty$. For this, we recall the Rudin-Shapiro polynomials (cf. [9], [14]).

DEFINITION 4.1. Let m be a non-negative integer. We define trigonometric polynomials $P_m(x)$, $Q_m(x)$ such that

(1) $P_0(x) = Q_0(x) = 1$;

(2) $P_{m+1}(x) = P_m(x) + e^{i2^m x} Q_m(x)$, $Q_{m+1}(x) = P_m(x) - e^{i2^m x} Q_m(x)$.

We prepare the following lemmas which will be used in the proof of Theorem 1.2.

LEMMA 4.2 (cf. [9], [14]). *The Rudin-Shapiro polynomials P_m, Q_m have the following properties*:

(1) $P_m(x) = \sum_{k=0}^{2^m-1} \varepsilon_k e^{ikx}$, $Q_m(x) = \sum_{k=0}^{2^m-1} \eta_k e^{ikx}$ *for some* $\varepsilon_k, \eta_k \in \{-1, 1\}$;

(2) $|P_m(x)| \le C(2^m)^{\frac{1}{2}}$ $(x \in \mathbf{T})$;

(3) $\|T_m\|_{M(L^q,L^q)} \sim (2^m)^{|\frac{1}{2}-\frac{1}{q}|}$ $(1 < q < \infty)$, *where* $T_m f = P_m * f$.

By Lemma 4.2 and the Peetre interpolation theorem [12], we obtain the following:

LEMMA 4.3. *Let* $0 < \lambda < 1$, *and* $p > 1 + \lambda$. *Then we have the estimates*:

$$\|T_m\|_{M(L^p,L^{p,\lambda})} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}} \quad (p \ge 2);$$

$$\|T_m\|_{M(L^p,L^{p,\lambda})} \le C(2^m)^{\frac{\lambda}{p}+\frac{1}{p}-\frac{1}{2}} \quad (1 + \lambda < p < 2);$$

$$\|T_m\|_{M(L^p,L^{p,\lambda})} \ge C(2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}} \quad (1 + \lambda < p < 2),$$

where $T_m f = P_m * f$.

PROOF. Step 1. We show $\|T_m\|_{M(L^2,L^{2,\lambda})} \sim (2^m)^{\frac{\lambda}{2}}$. Let P be a trigonometric polynomial such that $P(x) = \sum_{k=-n}^{n} a_k e^{ikx}$ for any positive integer n. Since $P_m * P(x) = \sum_{k=0}^{\min(2^m-1,n)} \varepsilon_k a_k e^{ikx}$, we have $|P_m * P(x)|^2 \le C 2^m \|P\|_2^2$ by the Schwarz inequality. Then for any interval I with $|I| < 2^{-m}$, we have

$$\frac{1}{|I|^\lambda} \int_I |P_m * P|^2 \frac{dx}{2\pi} \le C 2^{m\lambda} \|P\|_2^2$$

by the Parseval inequality. When $|I| \ge 2^{-m}$, we obtain

$$\frac{1}{|I|^\lambda} \int_I |P_m * P|^2 \frac{dx}{2\pi} \le \frac{1}{|I|^\lambda} \int_{-\pi}^{\pi} |P_m * P|^2 \frac{dx}{2\pi}$$

$$\leq \frac{1}{|I|^{\lambda}} \sum_{k=0}^{2^m-1} |a_k|^2$$
$$\leq C 2^{m\lambda} \|P\|_2^2$$

by the Parseval inequality. Hence, we get $\|T_m P\|_{2,\lambda} \leq C(2^m)^{\frac{\lambda}{2}} \|P\|_2$, and $\|T_m\|_{M(L^2, L^{2,\lambda})} \leq C(2^m)^{\frac{\lambda}{2}}$. On the other hand, since

$$\|P_m * P_m\|_{2,\lambda} \leq \|T_m\|_{M(L^2,L^{2,\lambda})} \|P_m\|_2$$
$$\leq C \|T_m\|_{M(L^2,L^{2,\lambda})} (2^m)^{\frac{1}{2}}$$

and $\|P_m * P_m\|_{2,\lambda} \sim (2^m)^{\frac{\lambda}{2}+\frac{1}{2}}$ by Lemma 4.2, we obtain $\|T_m\|_{M(L^2,L^{2,\lambda})} \sim (2^m)^{\frac{\lambda}{2}}$.

Step 2. When $p > 2$ and $0 < \lambda < 1$, we have

$$\|T_m\|_{M(L^p,L^{p,\lambda})} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}.$$

In fact, let $r > 2$ and $0 < \theta, \kappa < 1$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r}$ and $\frac{\lambda}{p} = \frac{\theta}{2}\kappa$. By Lemma 4.2, we have $\|T_m\|_{M(L^r,L^r)} \sim (2^m)^{\frac{1}{2}-\frac{1}{r}}$. Applying Step 1 and the Peetre interpolation theorem, we have

$$\|T_m\|_{M(L^p,L^{p,\lambda})} \leq C(2^m)^{\frac{\theta\kappa}{2}} (2^m)^{(\frac{1}{2}-\frac{1}{r})(1-\theta)}.$$

Hence, we obtain $\|T_m\|_{M(L^p,L^{p,\lambda})} \leq C(2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$. Conversely, we get

$$\|T_m\|_{M(L^p,L^{p,\lambda})} \geq C(2^m)^{\frac{\lambda}{p}+\frac{1}{p'}-\frac{1}{2}} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$$

by Corollary 2.2 and Lemma 4.2. Therefore we have $\|T_m\|_{M(L^p,L^{p,\lambda})} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$.

Step 3. We show $\|T_m\|_{M(L^p,L^{p,\lambda})} \leq C(2^m)^{\frac{\lambda}{p}+\frac{1}{p}-\frac{1}{2}}$ for $1+\lambda < p < 2$. First, we choose $1 < r < p$ and $0 < \theta, \kappa < 1$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{r}$ and $\frac{\lambda}{p} = \frac{\theta}{2}\kappa$. Then, we can show that

$$\|T_m\|_{M(L^p,L^{p,\lambda})} \leq C \|T_m\|_{M(L^2,L^{2,\lambda})}^{\theta} \|T_m\|_{M(L^r,L^r)}^{1-\theta}$$
$$\leq C(2^m)^{\frac{\lambda}{p}+\frac{1}{p}-\frac{1}{2}}$$

by applying the Peetre interpolation theorem. On the other hand, by $\|T_m(P_m)\|_{p,\lambda} \sim (2^m)^{\frac{\lambda}{p}+\frac{1}{p'}}$ we have $\|T_m\|_{M(L^p,L^{p,\lambda})} \geq C(2^m)^{\frac{\lambda}{p}+\frac{1}{2}-\frac{1}{p}}$, similarly in Step 2. After all, we get the desired result. □

PROOF OF THEOREM 1.2. By the assumption, we have $q > 2$, and $\|T_m\|_{M(L^q,L^{q,\nu})} \sim (2^m)^{\frac{\lambda}{q}+\frac{1}{2}-\frac{1}{q}}$ for m. If we have $M(L^p, L^{p,\lambda}) = M(L^q, L^{q,\nu})$, we obtain the contradiction to $p < q$ for $p > 2$. For $1+\lambda < p \leq 2$, we have $M(L^p, L^{p,\lambda}) \neq M(L^q, L^{q,\nu})$ by the estimate in Lemma 4.3. Then we get the desired result. □

COROLLARY 4.4. *Let* $0 < \lambda, \nu < 1$, $1+\lambda < p < q$, *and* $\frac{1}{p}+\frac{1}{q} < 1$. *Suppose* $\frac{\lambda}{p} = \frac{\nu}{q}$. *Then there exists* $T \in M(L^p, L^{p,\lambda})$ *such that* $T \notin M(L^q, L^{q,\nu})$.

PROOF. Let $2 < p < q$. Then there exists a in $\frac{\lambda}{p} + \frac{1}{2} - \frac{1}{p} < a < \frac{\nu}{q} + \frac{1}{2} - \frac{1}{q}$. Also we define $k_n = 2^{n+4}$. Then, we have $k_n + 2^{n+1} < k_{n+1} - 2^{n+2}$. We define

$$S_N(x) = \sum_{m=1}^{N} \frac{1}{2^{am}} P_m(x) e^{ik_m x}$$

for any $N \in \mathbf{N}$. Then, $\{S_N\}_N$ is a Cauchy sequence in $M(L^p, L^{p,\lambda})$ by the choice of a and Lemma 4.3, and there exists $S \in M(L^p, L^{p,\lambda})$ such that $\|S_N - S\|_{M(L^p,L^{p,\lambda})} \to 0$ as $N \to \infty$. Also let g be a function such that $g(x) = P_m(x)e^{ik_m x}$. We consider $\{S_N * g\}_{N>m}$. Then we can prove $S \notin M(L^q, L^{q,\nu})$ by the way similar to Corollary 3.2 in view of the choice of a. In case of $p \leq 2 \leq q$, we omit the details, since the proof is similar to it of the case $2 < p < q$. □

5. $M(L^p, L^{p,\lambda})$ and the Lipschitz conditions

DEFINITION 5.1. Let μ be in $M(\mathbf{T})$ and $0 < \alpha < 1$. We say that $\mu \in Lip_\alpha(M(\mathbf{T}))$ for $\mu \in M(\mathbf{T})$ with $\mu \geq 0$ if for any interval $I = [x, x+h]$,

$$\mu(I) \leq C|I|^\alpha = C|h|^\alpha$$

for some constant $C > 0$ independent of I. For $f \in L^1(\mathbf{T})$ with $f \geq 0$, we denote $\mu_f(E) = \frac{1}{2\pi}\int_E f(x)dx$ for any measurable set $E \subset \mathbf{T}$.

It is easy to prove the following:

PROPOSITION 5.2. *Let* f *be in* $L^1(\mathbf{T})$ *with* $f \geq 0$. *Then we have that* μ_f *is in* $Lip_\alpha(M(\mathbf{T}))$ *if and only if* $f \in L^{1,\alpha}(\mathbf{T})$.

By applying Proposition 5.2, we can show the following:

PROPOSITION 5.3. *Suppose* $f \in L^1(\mathbf{T})$ *and* $f \geq 0$. *Then we have the following*:

(1) *If* $\mu_f \in Lip_\alpha(M(\mathbf{T}))$ *for* $0 < \alpha < 1$, *then* $T_f \in M(L^p, L^{p,\alpha})$ *for all* $1 < p < \infty$.

(2) *If* $T_f \in M(L^p, L^{p,\lambda})$ *for some* $1 < p < \infty$ *and* $0 < \lambda < 1$, *then* $\mu_f \in Lip_{\frac{\lambda}{p}}(M(\mathbf{T}))$.

PROOF. (1) Since $\mu_f \in Lip_\alpha(M(\mathbf{T}))$ for all $0 < \alpha < 1$, we get $f \in L^{1,\alpha}(\mathbf{T})$ by Proposition 5.2. Let $I \subset \mathbf{T}$ be a nonempty interval. For $g \in L^p(\mathbf{T})$, we have

$$\frac{1}{|I|^\alpha}\int_I \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy\right|^p \frac{dx}{2\pi}$$

$$\le \frac{1}{|I|^\alpha}\int_I\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(y)|^p|f(x-y)|dy\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)|dy\right)^{\frac{p}{p'}}\frac{dx}{2\pi}$$
$$\le \|f\|_{1,\alpha}^p\|g\|_p^p$$

by the Hölder inequality. Hence, we obtain $\|f*g\|_{p,\alpha}\le\|f\|_{1,\alpha}\|g\|_p$ and $T_f\in M(L^p, L^{p,\alpha})$.

(2) Let f be in $L^1(\mathbf{T})$ with $f\ge 0$, and $T_f\in M(L^p, L^{p,\lambda})$. Now, let $I_\delta=[-\delta,\delta]$ $(0<\delta<1)$ and $g=\chi_{I_\delta}$. It is sufficient to show $\mu_f(I_\eta)\le C|I_\eta|^{\frac{\lambda}{p}}$ for sufficiently small $\eta>0$. First we remark

$$f*g(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(x-y)f(y)dy=\mu_f(I_\delta+x)\,,$$

and $I_{\frac{\delta}{2}}\subset I_\delta+x$ for $x\in I_{\frac{\delta}{2}}$. Hence, we obtain

$$\frac{1}{|I_\delta|^\lambda}\int_{I_\delta}|f*g|^p\frac{dx}{2\pi}\ge\frac{1}{|I_\delta|^\lambda}\int_{I_{\frac{\delta}{2}}}\mu_f(I_{\frac{\delta}{2}})^p\frac{dx}{2\pi}\,,$$

and

$$\begin{aligned}|I_\delta|^{-\lambda}\mu_f(I_{\frac{\delta}{2}})^p|I_{\frac{\delta}{2}}| &\le |I_\delta|^{-\lambda}\int_{I_\delta}|f*g|^p\frac{dx}{2\pi}\\ &\le\|f*g\|_{p,\lambda}^p\\ &\le\|T_f\|_{M(L^p,L^{p,\lambda})}^p\|g\|_p^p\\ &\le C|I_\delta|.\end{aligned}$$

Therefore, we get $\mu_f(I_{\frac{\delta}{2}})\le C|I_{\frac{\delta}{2}}|^{\frac{\lambda}{p}}$, and the desired result. □

As a corollary of Proposition 5.3, we have Theorem 1.3.

Moreover, by Theorem B and Proposition 5.3, we conclude that $M(L^r, L^{r,\lambda})$ are different from $M(L^p, L^q)$ $(1\le p<q<\infty)$. Precisely, we obtain the following corollary:

COROLLARY 5.4. *Let* $1\le p<q<\infty,\ 1\le r<\infty$, *and* $0<\lambda<1$. *Then we have*

$$M(L^p, L^q)\ne M(L^r, L^{r,\lambda})\,.$$

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