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Generalized Koszul Resolution

Satoshi MOCHIZUKI and Akiyoshi SANNAI

Chuo University and Nagoya University

Abstract. In this paper, we generalize a notion of Koszul resolutions and characterize modules which admits such resolutions. It turns out that for a noetherian ring A and a coherent A-module M, M has a two dimensional generalized Koszul resolution if and only if M is a pure weight two module in the sense of [HM10]. As an application, we attack the Gersten conjecture for weight two case.

1. Introduction

In [Ger73] p.28, Gersten proposed the following conjecture. Let A be a commutative noetherian ring with 1 and p a natural number such that $0 \leq p \leq \dim A$. We write $\mathcal{M}_A^p$ for the category of finitely generated A-modules M whose support has codimension $\geq p$ in $\operatorname{Spec} A$.

CONJECTURE 1.1 (Gersten's conjecture). *For any commutative regular local ring A and natural numbers n, p, the canonical inclusion $\mathcal{M}_A^{p+1} \hookrightarrow \mathcal{M}_A^p$ induces the zero map on K-groups*

$$\pi_p^n : K_n(\mathcal{M}_A^{p+1}) \to K_n(\mathcal{M}_A^p)\,.$$

In this paper, we consider Conjecture 1.1 for $n = 0$. For the related topics to Gersten's conjecture for the Grothendieck groups, please see the following references:

(1) For relationship Gersten's conjecture for the Grothendieck groups with Serre's intersection multiplicity conjecture [Ser65], please see the references [Dut93], [Dut95] and [Lev85]. (Please see also [CF68] and [GS87]).

(2) For Gersten's conjecture for regular local rings of equi-characteristic please see the references [Pan03] and [Qui73].

(3) For Gersten's conjecture for regular local rings smooth over some discrete valuation rings, please see the references [GL87] and [RS90]. (Please see also [Blo86], [Ger73], [She82] and [Lev85]).

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(4) For Gersten's conjecture for K-theory with finite coefficients, please see the references [Gil86] and [GL00].

(5) For non-regular case, please see the references [Bal09], [DHM85], [Lev88] and [Smo87].

It is well-known that Gersten's conjecture for Grothendieck groups is equivalent to the *generator conjecture* below (See [Lev85, 1.1]). Recall that a sequence of elements $f_1, \dots, f_q$ in A is said to be an *A-regular sequence* if all $(f_1, \dots, f_i)$ is not an unit ideal and if f_1 is not a zero divisor of A and if f_{i+1} is not a zero divisor of $A/(f_1, \dots, f_i)$ for any $1 \leq i \leq q-1$. Here is a statement of the generator conjecture:

If A is a commutative regular local ring, then the Grothendieck group $K_0(\mathcal{M}^p(A))$ is generated by cyclic modules $A/(f_1, \dots, f_p)$ where the sequence $f_1, \dots, f_p$ forms an A-regular sequence for $p = 1, \dots, \dim A$.

One of the main result in this paper is to find the generators of $K_0(\mathcal{M}_A^2)$ as follows.

THEOREM 1.2 (A part of Corollary 6.6). *For a regular A, $K_0(\mathcal{M}_A^2)$ is generated by modules of the form $\frac{A^{\oplus n}}{<\mathrm{Im}\, P, \mathrm{Im}\, Q>}$ where P and Q are endomorphisms on $A^{\oplus n}$ such that P and Q are similar to the matrices of the following type*:

$$P \sim \begin{pmatrix} fE_m & 0 \\ 0 & E_{n-m} \end{pmatrix}, \quad Q \sim \begin{pmatrix} gE_m & 0 \\ 0 & E_{n-m} \end{pmatrix}$$

where f, g forms a regular sequence and E_k is the k-th unit matrix. Moreover if P and Q are commutative, then the class of $\frac{A^{\oplus n}}{<\mathrm{Im}\, P, \mathrm{Im}\, Q>}$ is in the kernel of π_2^0.

For a non-regular ring A, we have similar statement by replacing with $K_0(\mathcal{M}_A^2)$ by the Grothendieck group of perfect complexes. (see 6.6).

The main point of this paper is to deal with such problems with a new theory of complexes and resolutions. More precisely, the theory of generalized Koszul resolutions and to relate this notion with pure weight modules defined in [HM10] and Koszul cubes defined in [Moc13]. To state the main theorem, we define the notion of generalized Koszul resolutions and Koszul cubes. An *n-cube* x in a category $\mathcal{C}$ is a contravariant functor from $[1]^{\times n}$ to $\mathcal{C}$ where $[1]$ is a totally ordered set $\{0, 1\}$ with the natural order $0 < 1$. For each $\mathfrak{i} \in [1]^{\times n}$, we call $x(\mathfrak{i})$ a *vertex of x*. For each $\mathfrak{i} = (i_1, \dots, i_n)$ such that $i_k = 1$, we write $x(\mathfrak{i} - \mathfrak{e}_k \to \mathfrak{i})$ by $d_{\mathfrak{i}}^k$ where $\mathfrak{e}_k$ is the k-th unit vector. Let us fix a sequence $f_1, \dots, f_n$ in A such that for any bijection σ on the set $\{1, 2, \dots, n\}$, a sequence $f_{\sigma(1)}, \dots, f_{\sigma(n)}$ forms an A-regular sequence. A *free Koszul cube* associated with $f_1, \dots, f_n$ is an n-cube x in $\mathcal{M}_A(= \mathcal{M}_A^0)$ satisfying the following conditions:

(1) Each vertex of x is a finitely generated free A-modules and their rank is constant. Therefore we can consider the determinant of its boundary maps.

(2) There are positive integers $m_1, \dots, m_n$ and $\det d_{\mathfrak{i}}^k = f_k^{m_s}$ for each $\mathfrak{i} = (i_1, \dots, i_n) \in [1]^{\times n}$ and $1 \leq k \leq n$ such that $i_k = 1$.

A *generalized Koszul resolution* associated with $f_1, \ldots, f_n$ is the totalized complex of a Koszul cubes associated with $f_1, \ldots, f_n$. Now we state the main theorem:

THEOREM 1.3. *For any M in $\mathcal{M}_A$ and an A-regular sequence f, g, the following conditions are equivalent*:

(1) *M is a pure weight two module supported on $V(f, g)$ in the sense of* [HM10].

(2) *M is resolved by a generalized Koszul resolution associated with f and g.*

(3) *There exist a free Koszul cube x associated with f, g such that* $\mathrm{H}_0(\mathrm{Tot}\, x)$ *is isomorphic to M.*

Finally the second author propose the conjecture 6.8 about purely linear algebra which implies Gersten's conjecture for K_0-groups for three dimensional regular local rings.

CONVENTIONS. Throughout this paper, we use the letter A to denote a commutative noetherian ring with a unit. We denote the category of finitely generated A-modules, the category of finitely generated projective A-modules by $\mathcal{M}_A$, $\mathcal{P}_A$ respectively.

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2. Definition of weight

In this section, we start from reviewing a notion of pure weight perfect modules over noetherian rings. For more information of pure weight perfect modules over any schemes, see [HM10]. Mainly we study fundamental properties of pure weight modules over a Cohen-Macaulay local ring.

DEFINITION 2.1. For an ideal I of A generated by a regular sequence of A and $0 \leq r \leq \infty$, let us denote the category of finitely generated A-modules of projective dimension $\leq r$ supported on $V(I)$ by $\mathcal{M}_A^I(r)$. One can easily check that $\mathcal{M}_A^I(r)$ is closed under the extensions and direct summands in $\mathcal{M}_A$. Therefore $\mathcal{M}_A^I(r)$ has a natural exact categorical structure. If I is generated by a regular sequence consisting of r elements, we denote $\mathcal{M}_A^I(r)$ by $\mathbf{Wt}_A^I$ and an A-module in $\mathbf{Wt}_A^I$ is said to be a *pure weight r A-module* (*supported on I*).

REMARK 2.2. The definition above is compatible with the definition in [HM10]. To confirm this, we see that the notion of torsion and projective dimension are equivalent for a finitely generated module over a noetherian ring (see [Wei94, Proposition 4.1.5]) and that in the notation above, $\mathrm{Spec}\, A/I \hookrightarrow \mathrm{Spec}\, A$ is a regular closed immersion.

EXAMPLE 2.3. (1) Let $f_1, \ldots, f_p$ be an A-regular sequence of A, then $A/(f_1, \ldots, f_p)$ is a typical example of a module of weight p. We call such a pure weight module a *simple pure weight module*.

(2) A module of weight 0 is a projective module whose support is the total space Spec A.

(3) If A is a Cohen-Macaulay local ring of Krull dimension d, a module of weight d is just a module of finite projective dimension and finite length.

(4) If A is not a Cohen-Macaulay ring, the class of A-modules of finite length and finite projective dimension does not work well as is in the following example in [Ger74]. Let $(A, \mathfrak{m})$ be a 2-dimensional local ring which is not normal and Spec $A \smallsetminus \{\mathfrak{m}\}$ is regular. Then an A-Module of finite length and finite projective dimension is the zero A-module.

ROUGH SKETCH OF PROOF. Let M be a non-zero, A-module of finite length and finite projective dimension. Since A is not Cohen-Macaulay, we have inequalities

$$\operatorname{Projdim}_A M \leq \operatorname{Projdim}_A M + \operatorname{depth}_A M = \operatorname{depth} A < 2\,.$$

Let

$$A^{\oplus n} \xrightarrow{\phi} A^{\oplus m} \to M \to 0$$

be a projective (= free) resolution of M. Then we can easily notice that $n = m$ and let us put $f = \det \phi$. Then, we can easily see that f is a non zero divisor. This implies that M_f is zero. By the assumption, we have Supp $M = \{\mathfrak{m}\}$. This is a contradiction. □

In this section, we assume that A is a Cohen-Macaulay ring from here.

DEFINITION 2.4. Let us denote the category of A-modules of weight p by $\mathbf{Wt}_A^p$. Since $\mathbf{Wt}_A^p$ is closed under extensions in $\mathcal{M}_A$, we can consider $\mathbf{Wt}_A^p$ as an exact category naturally.

Moreover, we assume that A is local ring of Krull dimension d from here.

PROPOSITION 2.5. *Let M be a non-zero pure weight p A-module. Then M is a Cohen-Macaulay module of dimension $d - p$.*

PROOF. We have two equalities:

$$\dim_A M + \operatorname{Codim}_A M = d \tag{1}$$

$$\operatorname{Projdim}_A M + \operatorname{depth}_A M = d \tag{2}$$

Therefore we have

$$d - p \leq \operatorname{depth}_A M \leq \dim_A M \leq d - p\,.$$

Hence we get $\operatorname{depth}_A M = \dim_A M = d - p$. □

COROLLARY 2.6. *For a non-zero pure weight module M, its associated prime ideal is minimal.*

PROOF. It is a general property of Cohen-Macaulay modules. □

3. Cubes

In this section, we fix a general notion of cubes. Let $\mathcal{C}$ be a category.

3.1. We write $\mathcal{P}(S)$ for the *power set of* S. Namely $\mathcal{P}(S)$ is the set of all subsets of S. We regard $\mathcal{P}(S)$ as a partially ordered set ordered by set inclusions, a fortiori, a category. For a set S, an *S-cube* or *$\#S$-cube* in a category $\mathcal{C}$ is a contravariant functor from $\mathcal{P}(S)$ to $\mathcal{C}$. We denote the category of S-cubes in a category $\mathcal{C}$ by $\mathbf{Cub}^S\,\mathcal{C}$ where the morphisms between cubes are just natural transformations. Let x be an S-cube in $\mathcal{C}$. For $T \in \mathcal{P}(S)$, we denote $x(T)$ by x_T and call it a *vertex of x (at T)*. For $k \in T$, we also write $d_T^{x,k}$ or shortly d_T^k for $x(T \smallsetminus \{k\} \hookrightarrow T)$ and call it a *(k-)boundary morphism of x (at T)*.

EXAMPLE 3.2. Let $f_1, \dots, f_n$ be elements in A. We define the n-cube $\mathrm{Typ}(f_1, \dots, f_n)$ (or shortly $\mathrm{Typ}(\mathfrak{f}_1^n)$) in $\mathcal{P}_A$ by $\mathrm{Typ}(\mathfrak{f}_1^n)_S := A$ for any $S \in \mathcal{P}_n$ and $d_S^j = f_j$ for any $S \in \mathcal{P}_n$ and $j \in S$.

From now on, let us fix an abelian category $\mathcal{A}$ and S a finite set. We start by considering a typical example.

3.3. An S-cube Let us fix an S-cube x in $\mathcal{A}$. For each $k \in S$, an S-cube the *k-direction 0-th homology* of x is the $S \smallsetminus \{k\}$-cube $\mathrm{H}_0^k(x)$ in $\mathcal{A}$ defined by $\mathrm{H}_0^k(x)_T := \operatorname{Coker} d_{T\cup\{k\}}^k$. For any $T \in \mathcal{P}(S)$ and $k \in S \smallsetminus T$, we denote the canonical projection morphism $x_T \twoheadrightarrow \mathrm{H}_0^k(x)_T$ by $\pi_T^{k,x}$ or simply π_T^k.

An A-regular sequence $f_1, \dots, f_n$ is said to be an *A-sequence* if $f_{\sigma(1)}, \dots, f_{\sigma(n)}$ is also a regular sequence for any bijection σ on $(n]$.

REMARK 3.4. For any regular sequence $f_1, \dots, f_n$, if either assumption (a) or (b) below is satisfied, then $f_1, \dots, f_n$ is an A-sequence. (See [Mat86, §16]).

(a) $f_1, \dots, f_n$ is contained in the Jacobson radical in A.

(b) For each i, $A/(f_1, \dots, f_i)A$ is complete for the $(f_1, \dots, f_n)$-adic topology.

EXAMPLE 3.5. For a sequence $f_1, \dots, f_n$ in A, it is an A-sequence if and only if for any k in $[n-1]$ and distinct numbers $i_1, \dots, i_k$ in $(n]$, a map $f_{i_{k+1}} : A/(f_{i_1}, \dots, f_{i_k}) \to A/(f_{i_1}, \dots, f_{i_k})$ is injective. This is equivalent to the n-cube $\mathrm{Typ}(\mathfrak{f}_1^n)$ (for definition, see 3.2) satisfies the following condition:

For any k in $[n-1]$ and distinct numbers $i_1, \dots, i_k$ in $(n]$, boundary maps of $\mathrm{H}_0^{i_1}(\cdots(\mathrm{H}_0^{i_k}(\mathrm{Typ}(\mathfrak{f}_1^n)))\cdots)$ are injections.

In the case, $\mathrm{Typ}(\mathfrak{f}_1^n)$ is said to be the *typical Koszul cubes* associated with $f_1, \dots, f_n$.

3.6 (Admissible cubes). Let x be an S-cube in an abelian category $\mathcal{A}$. When $\#S = 1$, we say that x is *admissible* if its unique boundary morphism is a monomorphism. For $\#S > 1$, we define the notion of an admissible cube inductively by saying that x is *admissible* if its boundary morphisms are monomorphisms and if for every k in S, $\mathrm{H}_0^k(x)$ is admissible. If x

is admissible, then for any distinct elements $i_1, \dots, i_k$ in S and for any automorphism σ of S, the identity morphism on $x|_{S\smallsetminus\{i_1,\dots,i_k\}}^{\emptyset}$ induces an isomorphism

$$\mathrm{H}_0^{i_1}(\mathrm{H}_0^{i_2}(\cdots(\mathrm{H}_0^{i_k}(x))\cdots)) \xrightarrow{\sim} \mathrm{H}_0^{i_{\sigma(1)}}(\mathrm{H}_0^{i_{\sigma(2)}}(\cdots(\mathrm{H}_0^{i_{\sigma(k)}}(x))\cdots))$$

(cf. [Moc13, 3.11]). For an admissible S-cube x and a subset $T = \{i_1, \dots, i_k\} \subset S$, we put $\mathrm{H}_0^T(x) := \mathrm{H}_0^{i_1}(\mathrm{H}_0^{i_2}(\cdots(\mathrm{H}_0^{i_k}(x))\cdots))$ and $\mathrm{H}_0^{\emptyset}(x) = x$. Notice that $\mathrm{H}_0^T(x)$ is an $S \smallsetminus T$-cube for any $T \in \mathcal{P}(S)$. Let us fix a bijection α from S to the set of integers $1 \leq k \leq n := \#S$ and we consider S as the set $\{1, \dots, n\}$ by α. For an admissible S-cube x, we define the *total complex* $\mathrm{Tot}_\alpha x = \mathrm{Tot}\, x$ as follows. $\mathrm{Tot}\, x$ is a chain complex in $\mathcal{A}$ concentrated in degrees $0, \dots, n$ whose component at degree k is given by

$$(\mathrm{Tot}\, x)_k := \bigoplus_{\substack{T\in\mathcal{P}(S)\\ \#T=k}} x_T$$

and whose boundary morphisms $d_k^{\mathrm{Tot}\, x} : (\mathrm{Tot}\, x)_k \to (\mathrm{Tot}\, x)_{k-1}$ are defined by

$$(-1)^{\sum_{t=j+1}^{n} \chi_T(t)} d_T^j : x_T \to x_{T\smallsetminus\{j\}}$$

by mapping its x_T component to $x_{T\smallsetminus\{j\}}$ component. Here χ_T is the characteristic function of T. Namely $\chi_T(k) = 1$ if k is in T and otherwise $\chi_T(k) = 0$. We have the isomorphisms

$$\mathrm{H}_p(\mathrm{Tot}(x)) \xrightarrow{\sim} \begin{cases} \mathrm{H}_0^S(x) & \text{for } p = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

See [Moc13, 3.13].

4. Generalized Koszul resolution

In this section, we introduce a generalization of Koszul resolution. Let us start by reviewing *Koszul complexes*.

REVIEW 4.1 (Koszul complex). Let $f : P \to Q$ be an A-module homomorphism between projective A-modules. The *n-th Koszul complex* associated with f is denoted by $\mathrm{Typ}^n(f)$ and defined as follows:

$$\mathrm{Typ}^n(f)_k = \Lambda^k(P) \otimes \mathrm{Sym}^{n-k}(Q)$$

and the Koszul differential $d_{k+1} : \mathrm{Typ}^n(f)_{k+1} \to \mathrm{Typ}^n(f)_k$ is given by

$$p_1 \wedge \cdots \wedge p_{k+1} \otimes q_{k+2} \cdots q_n \mapsto \sum_{i=1}^{k+1} (-1)^{k+1-i} p_1 \wedge \cdots \wedge \hat{p}_i \wedge \cdots \wedge p_{k+1} \otimes f(p_i) q_{k+2} \cdots q_n.$$

EXAMPLE 4.2. Let $f_1, \dots, f_n$ be elements in A. We consider the homomorphism $\mathfrak{f} = \begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix} : A^{\oplus n} \to A$. For simplicity, we denote the n-th Koszul complex associated with $\mathfrak{f}$ by $\mathrm{Typ}(\mathfrak{f})$. Notice that $\mathrm{Tot}^{[n]}(\mathrm{Typ}(\mathfrak{f}_1^n))$, the total complex of $\mathrm{Typ}(\mathfrak{f}_1^n)$ defined in 3.2, is isomorphic to $\mathrm{Typ}(\mathfrak{f})$ above.

It is well-known that if $f_1, \dots, f_n$ forms a regular sequence, then $\mathrm{H}_k(\mathrm{Typ}(\mathfrak{f}))$ is trivial for $k \geq 1$. In this section, we generalize this fact.

DEFINITION 4.3 (Generalized Koszul complexes). Let n, m be positive integers and $f_1, \dots, f_n$ elements in A. For a family of endomorphisms $\mathfrak{d} = \{d_S^j\}_{S \in \mathcal{P}_n,\ j \in S}$ on $A^{\oplus m}$ such that $\det d_S^j = f_j$ for any $S \in \mathcal{P}_n$ and $j \in S$, we define the *generalized Koszul complex* associated with $\mathfrak{d}$, $\mathrm{Kos}(\mathfrak{d})$ as follows.

$\mathrm{Kos}(\mathfrak{d})_k := \bigoplus_{\sharp S = k}^{S \in \mathcal{P}_n} F_S$ where $F_S := A^{\oplus m}$ and its boundary maps are defined by

$$(-1)^{\sum_{t=j+1}^{n} \chi_S(t)} d_S^j : F_S \to F_{S \smallsetminus \{j\}}$$

on its F_S component.

REMARK 4.4. The definition in 4.3 is equivalent to that $\mathrm{Kos}(\mathfrak{d})$ is the total complex of the following n-cube x:

$x_S := A^{\oplus m}$ and $d_S^s := d_S^s$ for any $S \in \mathcal{P}_n$ and $s \in S$.

Therefore we also denote x by $\mathrm{Kos}(\mathfrak{d})$.

Recall that a complex $E_\bullet$ on an abelian category is said to be *n-spherical* if $\mathrm{H}_k(E_\bullet) = 0$ unless $k \neq n$.

LEMMA 4.5 ([Moc13] Theorem 4.19). *For a family* $\mathfrak{d}$ *as in the notation* 4.3 *if* $f_1, \dots, f_n$ *forms an* A*-sequence*, *then* $\mathrm{Kos}(\mathfrak{d})$ *is admissible and in particular*, *it is* 0-*spherical.* □

DEFINITION 4.6 (Generalized Koszul resolutions). Let $f_1, \dots, f_n$ be an A-sequence.

A (*free*) *Koszul* (n-)*cube* associated with $f_1, \dots, f_n$ is a n-cube x in $\mathcal{M}_A$ satisfying the following conditions:

(1) each vertex of x is a finitely generated free A-modules and their rank is constant. Therefore we can consider the determinant of its boundary maps.

(2) there are positive integers $m_1, \dots, m_n$ and $\det d_S^s = f_s^{m_s}$ for each $S \in \mathcal{P}_n$ and $s \in S$.

A *generalized Koszul resolution* associated with $f_1, \dots, f_n$ is the total complex of the Koszul cube associated with $f_1, \dots, f_n$. Let us denote the category of Koszul cubes (resp. generalized Koszul resolution) associated with $f_1, \dots, f_n$ by $\mathbf{Kos}_A^{\mathfrak{f}_1^n}$ (resp. $\mathbf{GKos}_A^{\mathfrak{f}_1^n}$).

NOTATIONS 4.7 (Rank and determinants of Koszul cubes). For an n-Koszul cube x, we define the *rank of* x by $\operatorname{rank} x := \operatorname{rank} x_{\emptyset}$. We also define the *$j$-th determinant of* x by $\det_j x := \det d^j_{\{j\}}$ for any $j \in (n]$.

REMARK 4.8. By virtue of 4.5, generalized Koszul resolutions are 0-spherical and Koszul cubes are admissible. As in the notation 4.6, we have the total functor

$$\mathrm{Tot} : \mathrm{Typ}_A^{\mathfrak{f}_1^n} \to \mathbf{GKos}_A^{\mathfrak{f}_1^n}$$

which is exact, essentially surjective and faithful.

5. Weight two cases

In this section, we assume that A is a commutative noetherian ring with 1 and that f, g is an A-sequence. We give a proof for Theorem 1.3.

PROOF OF THEOREM 1.3. Obviously (2) implies (1) and (3) implies (2). Let us fix an A-module M in $\mathbf{Wt}_A^{(f,g)}$. By replacing f, g with f^α, g^β for some α, β, we may assume that $fM = gM = 0$. Then we have a surjection $(A/f)^{\oplus n} \to M$ with kernel L. Since $\mathbf{Wt}_A^{(f)}$ is closed under taking kernels of surjections, L is in $\mathbf{Wt}_A^{(f)}$. By considering resolutions of L and $(A/f)^{\oplus n}$, we get the following diagram:

$$\begin{array}{ccc} A^{\oplus m} & \longrightarrow & A^{\oplus n} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle f} \\ A^{\oplus m} & \underset{U}{\longrightarrow} & A^{\oplus n} \end{array} \tag{4}$$

CLAIM. $\det P = unit \times f^n$.

PROOF OF CLAIM. by localizing at f for the sequence below,

$$A^{\oplus m} \overset{P}{\to} A^{\oplus m} \to L \to 0,$$

we notice that $\frac{\det P}{f^n}$ is in $A_f^\times$. By localizing at g for the sequence above again, we get the following diagram:

$$\begin{array}{ccccc} A_g^{\oplus n} & \overset{f}{\longrightarrow} & A_g^{\oplus n} & \longrightarrow & (A_g/f)^{\oplus n} \\ \downarrow & & \downarrow & & \downarrow\wr \\ A_g^{\oplus m} & \underset{P_g}{\longrightarrow} & A_g^{\oplus m} & \longrightarrow & L_g \end{array}$$

where P_g is the localization of P at g. Take a prime ideal $\mathfrak{p}$ in $\operatorname{Spec} A_g$ and localize the diagram above at $\mathfrak{p}$. Since the top line is a minimal resolution, the vertical morphism is a split quasi-isomorphism. Therefore it turns out that

$$\frac{\det P}{f^n} \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A_g} (A_g)_{\mathfrak{p}}^{\times} = A_g^{\times}$$

where intersection is taken in the total quotient ring of A Since f, g forms A-sequence, we have the equality $A_f^{\times} \cap A_g^{\times} = A^{\times}$ in the total quotient ring of A. Therefore we obtain the result. □

To get a Koszul cube, we arrange the square above.

CLAIM. *There are $n \times m$ matrix X and $n \times n$ matrix V such that $UX = gE_n + fV$ where E_n is the n-th unit matrix. Here the matrix U is coming from the commutative diagram* (4).

PROOF OF CLAIM. Put $\mathfrak{e}_k$ the k-th unit vector in $A^{\oplus n}$. For each $k \in (n]$, let us denote one of a pull back of $g\,\mathfrak{e}_k$ in $(A/f)^{\oplus n}$ by the maps

$$A^{\oplus m} \twoheadrightarrow L \hookrightarrow (A/f)^{\oplus n}$$

by $\mathfrak{x}_k = \begin{pmatrix} x_{k1} \\ \vdots \\ x_{km} \end{pmatrix} \in A^{\oplus m}$. Then there is a vector $\mathfrak{v}_k = \begin{pmatrix} v_{k1} \\ \vdots \\ v_{kn} \end{pmatrix} \in A^{\oplus n}$ such that $U\,\mathfrak{x}_k = g\,\mathfrak{e}_k + f\,\mathfrak{v}_k$ We put $X = \begin{pmatrix} \mathfrak{x}_1 & \cdots & \mathfrak{x}_n \end{pmatrix}$ and $V = \begin{pmatrix} \mathfrak{v}_1 & \cdots & \mathfrak{v}_n \end{pmatrix}$. □

Put the matrix $\bar{U}$ as follows:

$$\bar{U} = \begin{pmatrix} fV & U \\ X & E_m \end{pmatrix}$$

where E_m is the m-th unit matrix. Since we have

$$\begin{pmatrix} -gE_n & 0 \\ 0 & E_m \end{pmatrix} = \begin{pmatrix} E_n & -U \\ 0 & E_m \end{pmatrix} \bar{U} \begin{pmatrix} E_n & 0 \\ -X & E_m \end{pmatrix}$$

it follows that $\det \bar{U} = (-g)^n$.
Now the following diagram is the Koszul cube x:

$$\begin{array}{ccc} A^{\oplus m+n} & \dashrightarrow & A^{\oplus m+n} \\ {\scriptstyle \begin{pmatrix} E_n & 0 \\ 0 & P \end{pmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{pmatrix} fE_n & 0 \\ 0 & E_m \end{pmatrix}} \\ A^{\oplus m+n} & \xrightarrow[\bar{U}]{} & A^{\oplus m+n} \end{array}$$

where dotted map is induced by the commutative diagram below:

$$\begin{array}{ccc} A^{\oplus m+n} & \xrightarrow{\bar{U}} & A^{\oplus m+n} \\ \downarrow & & \downarrow \\ L & \longrightarrow & (A/f)^{\oplus n}. \end{array}$$

□

COROLLARY 5.1. *For any non-zero M in $\mathbf{Wt}_A^{(f,g)}$, there are endomorphisms P, Q : $A^{\oplus n} \to A^{\oplus n}$ of A-modules and positive integer α such that M is isomorphic to $\frac{A^{\oplus n}}{<\operatorname{Im} P, \operatorname{Im} Q>}$ and P and Q are similar to the following matrices:*

$$P \sim \begin{pmatrix} f^\alpha E_m & 0 \\ 0 & E_{n-m} \end{pmatrix}, \quad Q \sim \begin{pmatrix} -g^\alpha E_m & 0 \\ 0 & E_{n-m} \end{pmatrix}$$

where f, g forms an A-sequence.

6. Application to Gersten conjecture

For a regular local ring A, in [Moc13], we gave the following result.

THEOREM 6.1 ([Moc13] Proposition 5.8). *For a regular commutative ring A, the canonical inclusion functor $\mathbf{Wt}_A^p \hookrightarrow \mathcal{M}_A^p$ induces a homotopy equivalence of spectra on K-theory*

$$K(\mathbf{Wt}_A^p) \xrightarrow{\simeq} K(\mathcal{M}_A^p).$$

□

COROLLARY 6.2. *For a regular local ring A, Gersten conjecture is equivalent to the following assertion*:

The canonical inclusion $\mathbf{Wt}_A^{p+1} \hookrightarrow \mathcal{M}_A^p$ induces the zero maps on K-groups

$$K_n(\mathbf{Wt}_A^{p+1}) \to K_n(\mathcal{M}_A^p)$$

for any $0 \leq p \leq \dim A - 1$. □

In this paper, we mainly consider $\operatorname{Ker}(K_0(\mathbf{Wt}_A^{p+1}) \to K_0(\mathcal{M}_A^p))$ for a Cohen-Macaulay ring A.

EXAMPLE 6.3 (Weight one case). For any A such that every finitely generated projective modules are free, the canonical inclusion $\mathbf{Wt}_A^1 \hookrightarrow \mathcal{M}_A$ induces the zero map on K_0-groups.

PROOF. Since for every A-module M in $\mathbf{Wt}^1_A$, its projective dimension is one, we have a resolution

$$0 \to A^{\oplus m} \to A^{\oplus n} \to M \to 0\,.$$

There is a non-zero divisor f such that $\operatorname{Supp} M = V(f)$. By localizing the short exact sequence above by f, it turns out that $n = m$. Therefore we have the following equality in $K_0(\mathcal{M}_A)$.

$$[M] = [A^{\oplus n}] - [A^{\oplus n}] = 0\,.$$

□

EXAMPLE 6.4 (Highest weight case). Let $(A, \mathfrak{m})$ be a regular local ring of Krull dimension d. Then the canonical inclusion $\mathbf{Wt}^d_A \hookrightarrow \mathcal{M}^{d-1}_A$ induces the zero map on K_0-groups. For $A = \mathbb{C}[[x, y, z, w]]/(xy - zw)$, there is an A-module M in $\mathbf{Wt}^3_A$ and the class of M in $K_0(\mathcal{M}^2_A)$ is not zero. (See [Bal09], [DHM85]).

PROOF. Since A is regular, for each non-negative integer r, $A/\mathfrak{m}^r$ is finite projective dimensional A-Module. Therefore by dévissage argument, it turns out that $K_0(\mathbf{Wt}^d_A)$ is generated by $A/\mathfrak{m}$. Let $f_1, \ldots, f_d$ be a regular system of parameter of A. Then we have a resolution

$$0 \to A/(f_1, \ldots, f_{d-1}) \overset{f_d}{\to} A/(f_1, \ldots, f_{d-1}) \to A/\mathfrak{m} \to 0\,.$$

Therefore we have the following identity in $K_0(\mathcal{M}^{d-1}_A)$.

$$[A/\mathfrak{m}] = [A/(f_1, \ldots, f_{d-1})] - [A/(f_1, \ldots, f_{d-1})] = 0\,.$$

□

Therefore the first non-trivial part of this conjecture is weight two case. This was proven in [Smo87]. But as a consequence of previous section, we have the following corollaries.

NOTATIONS 6.5. For a scheme X and closed subset Y, let us denote the category of perfect complexes on X whose cohomological support are in Y by Perf^Y_X and put $\operatorname{Perf}^p_X := \bigcup_{\operatorname{Codim} Y \geq p} \operatorname{Perf}^Y_X$.

COROLLARY 6.6. (1) *For an A-sequence f, g, $K_0(\mathbf{Wt}^{(f,g)}_A) \overset{\sim}{\to} K_0(\operatorname{Perf}^{V(f,g)}_A; \text{qis})$ is generated by* $\left[\frac{A^{\oplus n}}{<\operatorname{Im} P, \operatorname{Im} Q>}\right]$ *where P and Q are endomorphisms on $A^{\oplus n}$ such that P and Q are similar to the following type matrices*:

$$P \sim \begin{pmatrix} f^{\alpha} E_m & 0 \\ 0 & E_{n-m} \end{pmatrix}, \quad Q \sim \begin{pmatrix} g^{\beta} E_m & 0 \\ 0 & E_{n-m} \end{pmatrix}$$

where α and β are greater than 0.

(2) *If moreover A is a Cohen-Macaulay ring*, $K_0(\mathbf{Wt}^2_A) \overset{\sim}{\to} K_0(\mathrm{Perf}^2_A; \mathrm{qis})$ *is generated by* $\left[\frac{A^{\oplus n}}{<\mathrm{Im}\,P, \mathrm{Im}\,Q>}\right]$ *where P and Q are endomorphisms on* $A^{\oplus n}$ *such that P and Q are similar to the following type matrices*:

$$P \sim \begin{pmatrix} fE_m & 0 \\ 0 & E_{n-m} \end{pmatrix}, \quad Q \sim \begin{pmatrix} gE_m & 0 \\ 0 & E_{n-m} \end{pmatrix}$$

where f and g forms A-sequences.

(3) *Moreover if A is regular noetherian*, $K_0(\mathcal{M}^2_A)$ *is generated by the modules in* (2).

(4) *Moreover*, *if P and Q are commutative*, $\left[\frac{A^{\oplus n}}{<\mathrm{Im}\,P, \mathrm{Im}\,Q>}\right]$ *is in* $\mathrm{Ker}(K_0(\mathbf{Wt}^2_A) \to K_0(\mathcal{M}^1_A))$.

PROOF. (1), (2) and (3) are direct corollaries of previous section and [HM10]. Assume that P and Q are commutative. Then

$$\begin{array}{ccc} A^{\oplus n} & \xrightarrow{P} & A^{\oplus n} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ A^{\oplus n} & \xrightarrow[P]{} & A^{\oplus n} \end{array}$$

is a Koszul cube resolution of $\frac{A^{\oplus n}}{<\mathrm{Im}\,P, \mathrm{Im}\,Q>}$. (We call such a Koszul cube commutative.) Therefore we have the following short exact sequence

$$0 \to A^{\oplus n}/\,\mathrm{Im}\,P \overset{\bar{Q}}{\to} A^{\oplus n}/\,\mathrm{Im}\,P \to \frac{A^{\oplus n}}{< \mathrm{Im}\,P, \mathrm{Im}\,Q >} \to 0\,.$$

Therefore we have the following equalities in $K_0(\mathcal{M}^1_A)$.

$$\left[\frac{A^{\oplus n}}{< \mathrm{Im}\,P, \mathrm{Im}\,Q >}\right] = [A^{\oplus n}/\,\mathrm{Im}\,P] - [A^{\oplus n}/\,\mathrm{Im}\,P] = 0\,.$$

□

COROLLARY 6.7. *For a* 3*-dimensional regular local ring*, *Gersten's conjecture for* K_0*-groups is true if and only if* $K_0(\mathbf{Wt}^2(A))$ *is generated by the class of modules* $\left[\frac{A^{\oplus n}}{<\mathrm{Im}\,P, \mathrm{Im}\,Q>}\right]$ *where P and Q are commutative.*

PROOF. It is just a direct corollary of 6.2, 6.3, 6.4 and 6.6. □

The second author propose the following conjecture.

CONJECTURE 6.8. *For any two dimensional Koszul cube x*, *there is a commutative two dimensional Koszul cube y and a morphism of cubes $f : x \to y$ such that it forms three dimensional Koszul cubes.*

PROPOSITION 6.9. *The conjecture* 6.8 *implies Gersten's conjecture for K_0-groups for a three dimensional regular local ring.*

PROOF. It is enough to show that π_1^2 is zero map by 6.3 and 6.4.
For any weight two module M, there exist a Koszul cube resolution x of M by Theorem 1.3. Then by the conjecture 6.8, there exist a commutative Koszul cube y and a map $x \overset{f}{\to} y$ such that $z := [x \overset{f}{\to} y]$ is a three dimensional Koszul cube. Then we have a short exact sequence:

$$0 \to M \to \mathrm{H}_0(\mathrm{Tot}(y)) \to \mathrm{H}_0(\mathrm{Tot}(z)) \to 0\,.$$

By 6.4, the class of $\mathrm{H}_0(\mathrm{Tot}(y))$ is zero in $K_0(\mathcal{M}_A^2)$. Therefore in $K_0(\mathcal{M}_A^1)$ we have an equalities:

$$[M] = [\mathrm{H}_0(\mathrm{Tot}(z))] = 0$$

where the last equality coming from 6.6. □

References

[Bal09] P. BALMER, Niveau spectral sequences on singular schemes and failure of generalized Gersten conjecture, Proc. Amer. Math. Soc. **137** (2009), 99–106.

[Blo86] S. BLOCH, A note on Gersten's conjecture in the mixed characteristic case, Contemporary Math. **55**, Part I (1986), 75–78.

[CF68] L. CLABORN and R. FOSSUM, Generalizations of the notion of class group, Ill. Jour. Math. **12** (1968), 228–253.

[DHM85] S. P. DUTTA, M. HOCHSTER and J. E. MCLAUGHLIN, Modules of finite projective dimension with negative intersection multiplicities, Invent. Math. **79** (1985), 253–291.

[Dut93] S. P. DUTTA, A note on Chow groups and intersection multiplicity of Modules, Journal of Algebra **161** (1993), 186–198.

[Dut95] S. P. DUTTA, On Chow groups and intersection multiplicity of Modules, Journal of Algebra **171** (1995), 370–381.

[GL00] T. GEISSER and M. LEVINE, The K-theory of fields in characteristic p, Invent. Math. **139** (2000), 459–493.

[Ger73] S. GERSTEN, Some exact sequences in the higher K-theory of rings, In *Higher K-theories*, Lecture Notes in Math. **341**, Springer (1973), 211–243.

[Ger74] S. GERSTEN, The localization theorem for projective modules, Comm. Alg. **2** (1974), 307–350.

[Gil86] H. GILLET, Gersten's conjecture for the K-theory with torsion coefficients of a discrete valuation ring, Journal of Algebra **103** (1986), 377–380.

[GL87] H. GILLET and M. LEVINE, The relative form of Gersten's conjecture over a discrete valuation ring: The smooth case, Jour. Pure. App. Alg. **46** (1987), 59–71.

[GS87] H. GILLET and C. SOULÉ, Intersection theory using Adams operations, Invent. Math. **90** (1987), 243–277.

[HM10] T. HIRANOUCHI and S. MOCHIZUKI, Pure weight perfect Modules on divisorial schemes, in *Deformation Spaces: Perspectives on Algebro-geometric Moduli* (2010), 75–89.

[Moc13] S. MOCHIZUKI, Higher K-theory of Koszul cubes, available at arXiv:1303.1239 (2013).

[Lev85] M. LEVINE, A K-theoretic approach to multiplicities, Math. Ann. **271** (1985), 451–458.

[Lev88] M. LEVINE, Localization on singular varieties, Invent. Math. **91** (1988), 423–464.

[Mat86] H. MATSUMURA, *Commutative ring theory*, Cambridge studies in advanced mathematics **8**, (1986).
[Pan03] I. PANIN, The equi-characteristic case of the Gersten conjecture, Proc. Steklov. Inst. Math. **241** (2003), 154–163.
[Qui73] D. QUILLEN, Higher algebraic K-theory I, in *Higher K-theories*, Lecture Notes in Math. **341**, Springer (1973), 85–147.
[RS90] L. REID and C. SHERMAN, The relative form of Gersten's conjecture for power series over a complete discrete valuation ring, Proc. Amer. Math. Soc. **109** (1990), 611–613.
[Ser65] J. P. SERRE, *Algèbre locale: Multiplicites*, Lecture Notes in Math. **11**, Springer (1965).
[She82] C. SHERMAN, Group representations and algebraic K-theory, in *Algebraic K-theory, Part I*, Lecture Notes in Math. **966**, Springer, Berlin (1982), pp. 208–243.
[Smo87] W. SMOKE, Perfect Modules over Cohen-Macaulay local rings, Journal of Algebra **106** (1987), 367–375.
[TT90] R. W. THOMASON and T. TROBAUGH, Higher K-theory of schemes and of derived categories, in *The Grothendieck Festscrift, Vol III* (1990), 247–435.
[Wei94] C. A. WEIBEL, *An introduction to homological algebra*, Cambridge studies in advanced mathematics **38** (1994).

Present Addresses:
SATOSHI MOCHIZUKI
DEPARTMENT OF MATHEMATICS,
CHUO UNIVERSITY,
BUNKYO-KU, TOKYO, JAPAN.
e-mail: mochi@gug.math.chuo-u.ac.jp

AKIYOSHI SANNAI
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES,
UNIVERSITY OF TOKYO,
3–8–1 KOMABA, MEGURO-KU, TOKYO, JAPAN.
e-mail: sannai@ms.u-tokyo.ac.jp