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Pseudo-Anosov Maps and Pairs of Filling Simple Closed Geodesics on Riemann Surfaces, II

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Abstract. Let S be a Riemann surface containing at least two punctures z and z_0. Let $\mathscr{F}(S)$ be the set of pseudo-Anosov maps of S that are isotopic to the identity on $S \cup \{z\}$. We show that for any $f \in \mathscr{F}(S)$ and any twice punctured disk Δ enclosing z and z_0, the pair $(\partial\Delta, f(\partial\Delta))$ fills S, where $\partial\Delta$ denotes the boundary of Δ. Fix such a Δ, and denote by $\mathscr{T}(\Delta)$ the set of twice punctured disks Δ' on S enclosing z and z_0 with the property that $(\partial\Delta, \partial\Delta')$ fills S. Let $\Delta_0 \in \mathscr{T}(\Delta)$. We describe all possible pseudo-Anosov maps f in $\mathscr{F}(S)$ sending Δ to Δ_0, and classify elements of $\mathscr{F}(S)$ in terms of $\mathscr{T}(\Delta)$. We also show that there are infinitely many elements $f_k \in \mathscr{F}(S)$ with $f_k(\Delta) = \Delta_0$ such that their dilatations $\lambda(f_k) \to +\infty$ as $k \to +\infty$.

1. Introduction and statement of results

Let S be an analytically finite Riemann surface of type (p, n) with $3p+n > 3$, where p is the genus and n is the number of punctures of S. For any pseudo-Anosov map $f : S \to S$, and any simple closed geodesic $a \subset S$ (with respect to a hyperbolic metric on S, of course), the set $\mathscr{S} = \{a, f(a), f^2(a), \dots\}$ fills S in the sense that each closed geodesic on S intersects one of the elements in $\mathscr{S}$ (see [6, 7]), where and below $f^i(a)$ denotes the geodesic representative in the homotopy class of the image curve of a under f^i. In [11] Masur–Minsky showed that $(a, f^k(a))$ fills S for all sufficiently large integers k.

Consider the case where $3p + n > 4$ and $n \geq 1$. Let z denote a puncture of S. Write $\tilde{S} = S \cup \{z\}$. Let $c \subset S$ be a simple closed geodesic. Then c can also be viewed as a curve $\tilde{c}$ on $\tilde{S}$. Note that $\tilde{c}$ could be trivial, that is, $\tilde{c}$ could be homotopic to a puncture of $\tilde{S}$. If this occurs, then c bounds a (topological) twice punctured disk on S enclosing z and another puncture of S. See Fig. 1 (a) and (b) for examples of twice punctured disks. It is clear that no such geodesic exists when $n = 1$. If $n \geq 2$, there are infinitely many non-trivial geodesics on S that are trivial on $\tilde{S}$. When $\tilde{c}$ is non-trivial, there is a unique geodesic representative in the homotopy class of $\tilde{c}$. For simplicity, we call this geodesic representative $\tilde{c}$ also.

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Let $\mathscr{F}_0(S)$ be the set consisting of mapping classes on S that fix z and are isotopic to the identity on $\tilde{S}$ as z is filled in. Let $\mathscr{F}(S)$ be the subset of $\mathscr{F}_0(S)$ consisting of pseudo-Anosov elements. It was shown in [21] that for any $f \in \mathscr{F}(S)$, and any simple closed geodesic a with $\tilde{a}$ being non-trivial on $\tilde{S}$, $(a, f^k(a))$ fills S for all $k \geq 3$. In this article, we consider the set of geodesics that are boundaries of twice punctured disks, which is identified with the set of geodesics b with $\tilde{b}$ being trivial on $\tilde{S}$.

Throughout the article we assume that S contains at least two punctures z and z_0. We first prove the following result.

THEOREM 1.1. *With the above assumptions, let Δ be a twice punctured disk on S that encloses z and z_0. Then for any $f \in \mathscr{F}(S)$, $(\partial\Delta, \partial f^k(\Delta))$ fills S for all $k \geq 1$.*

The converse is not true. Let t_c denote the positive Dehn twist along a simple closed geodesic c. We know that there is a geodesic c and thus a Dehn twist t_c such that both pairs $(\partial\Delta, c)$ and $(\partial\Delta, t_c(\partial\Delta))$ fill S (See [16] for constructions). In Section 7, we will acquire some spin maps $t_c \circ t_{c_0}^{-1}$ on S such that $(\partial\Delta, t_c \circ t_{c_0}^{-1}(\partial\Delta))$ fill S. Theorem 1.1 can be extended to the following corollary.

COROLLARY 1.1. *Let $\alpha \subset S$ be a simple closed geodesic which bounds a planar region D_α enclosing z and at least one more puncture of $\tilde{S}$. Then for every $f \in \mathscr{F}(S)$, $(\alpha, f(\alpha))$ fills S.*

Let Δ be a fixed twice punctured disk that encloses z and z_0. Note that z_0 is also a puncture on $\tilde{S}$. Let $\mathscr{T}(\Delta)$ be the set of twice punctured disks Δ_0 enclosing z and z_0 with geodesic boundaries such that $(\partial\Delta, \partial\Delta_0)$ fills S. There are infinitely many elements in $\mathscr{T}(\Delta)$ (see [18]).

THEOREM 1.2. *Let S be as above. Then for any $\Delta_0 \in \mathscr{T}(\Delta)$, there is $F \in \mathscr{F}_0(S)$ such that $F(\Delta) = \Delta_0$. Furthermore, by suitably choosing $\varepsilon = 1$ or -1, the maps $F \circ t_{\partial\Delta}^{\varepsilon k}$ are pseudo-Anosov for any $k > 0$ and send Δ to Δ_0.*

It should be noted that in Theorem 1.2 we do not assume F is pseudo-Anosov, and only assume that the image $F(\partial\Delta)$ along with $\partial\Delta$ fills S (if F is pseudo-Anosov, the theorem was proved in [22]).

For a general pseudo-Anosov map f and a Dehn twist t_c for a simple geodesic c, Long–Morton [12] proved that $f \circ t_c^k$ are pseudo-Anosov except for at most $N(< \infty)$ consecutive integer values of k. Fathi [6] showed that $N \leq 7$, and later Boyer *et al.* [5] showed that $N \leq 6$. During the course of the proof of Theorem 1.2, we describe the condition which guarantees that $F \circ t_{\partial\Delta}^k$ are pseudo-Anosov for all $k > 0$ or $k < 0$. Of course, our method is different from those used in [5, 6, 12].

Let $\mathbf{D}$ denote the unit disk equipped with the hyperbolic metric $2\,|dz|\,/(1-|z|^2)$, and let $\varrho : \mathbf{D} \to \tilde{S}$ denote the universal covering map with a covering group G which is isomorphic to the fundamental group $\pi_1(\tilde{S}, z)$. It is well known [10] that for each $\Delta' \in \mathscr{T}(\Delta)$, there are parabolic elements $T, T' \in G$ that correspond to $t_{\partial\Delta}$ and $t_{\partial\Delta'}$, respectively, under the Bers isomorphism φ (see Section 2 for expositions). Note that Δ and Δ' enclose the same punctures z and z_0. Hence T is conjugate to T' in G, which means that there is an element $h \in G$ that sends the fixed point of T to the fixed point of T'. Let h^* be the corresponding element in $\mathscr{F}_0(S)$. By combining Theorem 1.2 we can obtain the following corollary.

COROLLARY 1.2. *Let Δ, Δ' be any twice punctured disks enclosing z. Then there is $f \in \mathscr{F}(S)$ sending Δ to Δ' if and only if $\Delta' \in \mathscr{T}(\Delta)$.*

It is well known [2, 4] that $\mathscr{F}_0(S)$ is isomorphic to $\pi_1(\tilde{S}, z)$ and that there is a bijection between $\mathscr{F}(S)$ and the set of essential hyperbolic elements of G, where an element $g \in G$ is called an essential hyperbolic if it is hyperbolic and its axis $\mathrm{axis}(g)$ projects to a filling closed geodesic $\tilde{\gamma}$ in the sense that $\tilde{\gamma}$ intersects every simple closed geodesic on $\tilde{S}$. Moreover, the set of conjugacy classes of elements of $\mathscr{F}(S)$ in $\mathscr{F}_0(S)$ is one-to-one correspondent with the set of oriented primitive filling closed geodesics on $\tilde{S}$.

Two elements $f, f' \in \mathscr{F}(S)$ are said to be Δ-equivalent (denoted by $f \sim f'$) if $f = f' \circ t_{\partial\Delta}^k$ for an integer k. It is obvious that "$\sim$" is an equivalent relation. Our next result gives a new characterization of equivalence classes of elements of $\mathscr{F}(S)$ by means of twice punctured disks on S.

THEOREM 1.3. *There is a bijection between $\mathscr{F}(S)/\sim$ and $\mathscr{T}(\Delta)$.*

In [9], Harvey introduced a complex $\mathcal{C}(S)$ of curves on S. A k-th dimensional simplex of $\mathcal{C}(S)$ is a collection of $k+1$ disjoint simple closed geodesics on S. In particular, the vertices $\mathcal{C}_0$ of $\mathcal{C}(S)$ are collections of simple closed geodesics on S. We define the length of each edge in $\mathcal{C}_1$ is one, and define the distance $d_{\mathcal{C}}(a, b)$ between two vertices $a, b \in \mathcal{C}_0$ to be the least number of edges in $\mathcal{C}_1$ joining a and b. By the definition, we know that $d_{\mathcal{C}}(a, b) \geq 3$ if and only if (a, b) fills S. Also, $d_{\mathcal{C}}(a, b) = 1$ if and only if a and b are disjoint. Thus, for any $\Delta_0, \Delta_1 \in \mathscr{T}(\Delta)$, $d_{\mathcal{C}}(\partial\Delta_0, \partial\Delta_1) > 1$ and Theorem 1.1 says that $d_{\mathcal{C}}(\partial\Delta, f(\partial\Delta)) \geq 3$ for any $f \in \mathscr{F}(S)$.

In [23], we considered vertices $a_1, a_2 \in \mathcal{C}_0$ that are non-trivial and are homotopic to each other on $\tilde{S}$, and proved that if $d_{\mathcal{C}}(a_1, a_2) \geq 3$, there is a sequence $f_k \in \mathscr{F}(S)$ such that $f_k(a_1) = a_2$ while their dilatations $\lambda(f_k)$ tend to infinity. Here we treat the case in which $a_1, a_2 \in \mathscr{T}(\Delta)$:

THEOREM 1.4. *Let Δ be a twice punctured disk on S enclosing z and another puncture z_0 of S.*

(1) *For any $\Delta_0 \in \mathscr{T}(\Delta)$, any large integer M, there are $f \in \mathscr{F}(S)$ such that $f(\Delta) = \Delta_0$ and $\lambda(f) > M$.*

(2) *Let $\Delta_k \in \mathscr{T}(\Delta)$ be such that $d_{\mathcal{C}}(\partial\Delta, \partial\Delta_k) \to +\infty$ as $k \to +\infty$. Then for any elements $f_k : \Delta \to \Delta_k$ of $\mathscr{F}(S)$, the sequence $\{\lambda(f_k)\}$ is unbounded.*

This article is organized as follows. In Section 2, we collect background materials on z-pointed mapping class group Mod_S^z. Some special elements in Mod_S^z and their combinations are investigated. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we classify elements of $\mathscr{F}(S)$ in terms of $\mathscr{T}(\Delta)$ and prove Theorem 1.3. In Section 6, we study the relationship between the path distance $d_{\mathcal{C}}(\partial\Delta, \partial\Delta_k)$ for any $\Delta_k \in \mathscr{T}(\Delta)$ and the dilatation of any associated pseudo-Anosov maps obtained from Theorem 1.2, and prove Theorem 1.4. In Section 7, we illustrate that for a filling pair $(\partial\Delta, \partial\Delta_0)$ with $\partial\Delta_0 = \partial f(\Delta)$, the maps f may not be pseudo-Anosov. We give some examples showing that f could stem from parabolic or simple hyperbolic elements of G.

2. Background and some preliminary results

Let G be the covering group of a holomorphic universal covering map $\varrho : \mathbf{D} \to \tilde{S}$. Then G is a torsion free finitely generated Fuchsian group of the first kind. Elements of G are either parabolic or hyperbolic and are isometric motions on $\mathbf{D}$ with respect to the hyperbolic metric on $\mathbf{D}$. Let $Q(G)$ be the group of quasiconformal automorphisms w of $\mathbf{D}$ such that $wGw^{-1} = G$. Two maps $w, w_0 \in Q(G)$ are said to be equivalent (denoted by $w \sim w_0$) if $w|_{\mathbf{S}^1} = w_0|_{\mathbf{S}^1}$. Denote by $[w]$ the equivalence class of w. Thus the restriction $[w]|_{\mathbf{S}^1}$ is well defined and is a quasisymmetric map on the unit circle $\mathbf{S}^1$. By the Bers isomorphism theorem [2], the quotient group $Q(G)/\sim$ is isomorphic to the z-pointed mapping class group Mod_S^z that consists of mapping classes f with $f(z) = z$.

According to the Nielsen–Thurston classification for surface homeomorphisms [14], every non-periodic element of Mod_S^z is either reducible or pseudo-Anosov, where by a reducible mapping class f we mean that there is a representative of the mapping class (also denoted by f) and a curve simplex

$$\Gamma = \{u_1, \ldots, u_s\}, \quad s \geq 1, \tag{2.1}$$

such that $f(\{u_1, \ldots, u_s\}) = \{u_1, \ldots, u_s\}$; and by a pseudo-Anosov mapping class f we mean that there is a representative (denoted by f also), a pair $(\mathcal{F}_+, \mathcal{F}_-)$ of transverse measured foliations and a real number $\lambda > 1$ such that $f(\mathcal{F}_+) = \lambda\mathcal{F}_+$ and $f(\mathcal{F}_-) = (1/\lambda)\mathcal{F}_-$. The number $\lambda = \lambda(f)$ is called the dilatation of f.

Let $w \in Q(G)$ be such that $[w]$ corresponds to f under the Bers isomorphism. As all elements of Mod_S^z fix z, it is clear that there defines a group homomorphism of Mod_S^z onto the ordinary mapping class group $\mathrm{Mod}(\tilde{S})$ by sending every element $f \in \mathrm{Mod}_S^z$ to an element of $\mathrm{Mod}(\tilde{S})$ induced by a homeomorphism $\tilde{f}$ of $\tilde{S}$, where $\tilde{f}$ can also be obtained from the projection of the map w via the universal covering map ϱ.

In what follows, for each $w \in Q(G)$, we denote by $[w]^* \in \mathrm{Mod}_S^z$ the corresponding element under the Bers isomorphism. In particular, as G is considered a normal subgroup of $Q(G)/\sim$, we use the symbol h^*, where $h \in G$, to denote the mapping class in Mod_S^z as well as a homeomorphism representing h^*.

We proceed to investigate mapping classes h^* for elements $h \in G$. Details can be found in Kra [10]. In the case where h is parabolic, h^* is the Dehn twist $t_{\partial\Delta}$ or its inverse $t_{\partial\Delta}^{-1}$ along

$\partial\varDelta$ for a twice punctured disk $\varDelta$ enclosing z, by which we mean a planar region on S that contains the puncture z and another puncture of $\tilde{S}$. Fig. 1 (a) exhibits an "obvious" twice punctured disk on a surface of type $(2, 4)$, which encloses z and z_0, while Fig. 1 (b) is a highly complicated twice punctured disk on the same surface; it also encloses z and z_0.

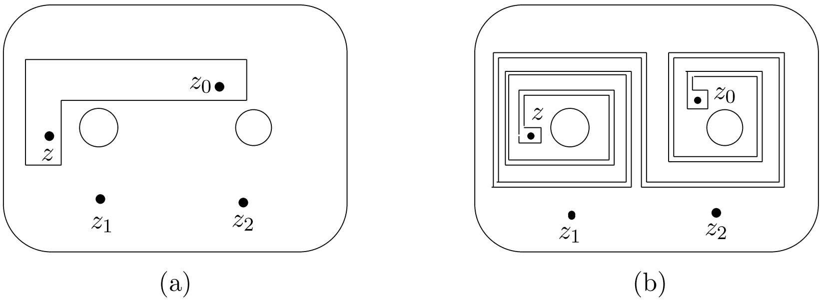

FIG. 1

Conversely, for any Dehn twist $t_{\partial\varDelta}$ along the boundary $\partial\varDelta$ of a twice punctured disk $\varDelta$ enclosing z, there exists a parabolic element $h \in G$ such that $h^* = t_{\partial\varDelta}$.

If h is simple hyperbolic; that is, its axis $\mathrm{axis}(h) \subset \mathbf{D}$ projects to a simple closed geodesic $\varrho(\mathrm{axis}(h)) \subset \tilde{S}$, then there is a pair of simple closed geodesics $\{c_0, c\} \subset S$ that bounds a z-punctured cylinder, such that $h^* = t_{c_0} \circ t_c^{-1}$, and that $\varrho(\mathrm{axis}(h)) = \tilde{c} = \tilde{c}_0$, where we recall that $\tilde{c}$ is the geodesic representative in the homotopy class of c if c is regarded as a curve on $\tilde{S}$. Conversely, for any z-punctured cylinder $\mathscr{P}$ on S, there is a simple hyperbolic element $h \in G$ such that $h^* = t_{c_0} \circ t_c^{-1}$ for $\{c_0, c\} = \partial\mathscr{P}$ and $\mathrm{axis}(h)$ projects to $\varrho(\mathrm{axis}(h)) = \tilde{c} = \tilde{c}_0$.

If h is essential hyperbolic; that is, $\mathrm{axis}(h)$ projects to a filling closed geodesic $\varrho(\mathrm{axis}(h))$ on $\tilde{S}$, then h^* is pseudo-Anosov and hence $h^* \in \mathscr{F}(S)$. By Theorem 2 of [10], all elements of $\mathscr{F}(S)$ can be obtained in this way.

Finally, if $h \in G$ is non-simple and non-essential, i.e., $\varrho(\mathrm{axis}(h))$ is a non-filling self-intersecting closed geodesic on $\tilde{S}$, then $h^* \in \mathscr{F}_0(S)$ is reducible by a maximal reduced curve simplex (call it $\varGamma$ also). Let P be the component of $S\backslash\varGamma$ that contains the puncture z. Then by Theorem 2 of [10], we know that $h^*|_{S\backslash P}$ is the identity and $h^*|_P$ is pseudo-Anosov. In what follows P is called the pseudo-Anosov component for h^*.

We also need to explore some special elements in $Q(G)/\sim$ that are different from elements of G. Let $u \subset S$ be a simple closed geodesic such that $\tilde{u} \subset \tilde{S}$ is also non-trivial. Let $\{\varrho^{-1}(\tilde{u})\}$ be the collection of all geodesics $\hat{u}$ in $\mathbf{D}$ such that $\varrho(\hat{u}) = \tilde{u}$. Since $\tilde{u}$ is simple, all geodesics in $\{\varrho^{-1}(\tilde{u})\}$ are mutually disjoint.

Fix a geodesic $\hat{u} \in \{\varrho^{-1}(\tilde{u})\}$ and fix a component U of $\mathbf{D}\backslash\hat{u}$, there is a lift $\tau_{\hat{u}}$ of the Dehn twist $t_{\tilde{u}}$ with respect to U. See [17, 19] for more information on the lift $\tau_{\hat{u}}$. It is known that

$\tau_{\hat{u}} \in Q(G)$ determines a maximal convex region $\Omega_{\hat{u}}$ in $\mathbf{D}\backslash U$ with geodesic boundaries, and that the restriction $\tau_{\hat{u}}|_{\Omega_{\hat{u}}}$ is the identity. By Lemma 3.2 of [17], we know that $\hat{u}$ (and hence U) can be properly chosen so that $[\tau_{\hat{u}}]^* = t_u$. Therefore, the pair $(\hat{u}, U)$ completely determines the geodesic u.

In fact, $\Omega_{\hat{u}}$ is a component of $\mathbf{D}\backslash\{\varrho^{-1}(\tilde{u})\}$ that takes $\hat{u}$ as a component of the boundary $\partial\Omega_{\hat{u}}$. The complement of the closure of $\Omega_{\hat{u}}$ are the disjoint union of half-spaces in $\mathbf{D}$, where by a half-space we mean one of the components of a geodesic in $\{\varrho^{-1}(\tilde{u})\}$ which is disjoint from $\Omega_{\hat{u}}$. By our convention, a half-space D includes the open arc $D \cap \mathbf{S}^1$. Note that the end-points of this arc are the fixed points of a simple hyperbolic element of G. Let $\mathscr{U}_{\hat{u}}$ denote the collection of all half-spaces in $\mathbf{D}$ defined by the geodesics in $\{\varrho^{-1}(\tilde{u})\}$. We see that $\mathscr{U}_{\hat{u}}$ forms a partially ordered set whose order is defined by inclusion. Each component in the complement of the closure of $\Omega_{\hat{u}}$ is called a maximal element of $\mathscr{U}_{\hat{u}}$. Observe that $\{\varrho^{-1}(\tilde{u})\}$ contains infinitely many mutually disjoint geodesics and $\Omega_{\hat{u}}$ contains no geodesics in $\{\varrho^{-1}(\tilde{u})\}$. Every geodesic in $\{\varrho^{-1}(\tilde{u})\}$, if not the boundary of any maximal element, is included in a maximal element of $\mathscr{U}_{\hat{u}}$. As such, each maximal element contains infinitely many elements of $\mathscr{U}_{\hat{u}}$ of higher orders. Notice that the map $\tau_{\hat{u}}$ leaves invariant each maximal element of $\mathscr{U}_{\hat{u}}$ and sends each element of $\mathscr{U}_{\hat{u}}$ to an element of $\mathscr{U}_{\hat{u}}$ with the same order. In what follows, the triple $(\tau_{\hat{u}}, \Omega_{\hat{u}}, \mathscr{U}_{\hat{u}})$ is called a configuration corresponding to the geodesic u.

LEMMA 2.1. *Let $h \in G$. Then h sends every maximal element of $\mathscr{U}_{\hat{u}}$ to a different maximal element if and only if the fixed point(s) of h lies in $\Omega_{\hat{u}} \cap \mathbf{S}^1$.*

PROOF. We only prove the case that h is parabolic (the hyperbolic case can be handled similarly). Let x be the fixed point of h. If $x \in \Omega_{\hat{u}} \cap \mathbf{S}^1$, that is, x lies outside of all maximal elements of $\mathscr{U}_{\hat{u}}$, then by construction, $\tau_{\hat{u}}(x) = x$. Since $\tau_{\hat{u}} \in Q(G)$, $\tau_{\hat{u}} h \tau_{\hat{u}}^{-1}$ is also a primitive parabolic element of G with fixed point x. It follows that $\tau_{\hat{u}} h \tau_{\hat{u}}^{-1} = h$, i.e., $\tau_{\hat{u}} h = h \tau_{\hat{u}}$. Hence for each maximal element $U \in \mathscr{U}_{\hat{u}}$, $h(U)$ is also a maximal element. Conversely, if $x \in U$ for a maximal element U of $\mathscr{U}_{\hat{u}}$, then x lies outside of $\mathbf{D}\backslash U$. By examining the action of h on $\mathbf{D}$, $h(\mathbf{D}\backslash U)$ is disjoint from $\mathbf{D}\backslash U$. Hence $h(\mathbf{D}\backslash U) \subset U$ and thus U intersects $h(U)$. It follows from Lemma 4.3 of [19] that $h(U)$ is not a maximal element of $\mathscr{U}_{\hat{u}}$. □

LEMMA 2.2. *Let $(\tau_{\hat{u}}, \Omega_{\hat{u}}, \mathscr{U}_{\hat{u}})$ be the configuration corresponding to u. Let x be the parabolic fixed point of G that corresponds to a simple closed geodesic $a = \partial\Delta$. Then the geodesic u intersects a if and only if x is covered by a maximal element of $\mathscr{U}_{\hat{u}}$.*

PROOF. If x lies outside of any maximal element of $\mathscr{U}_{\hat{u}}$, i.e., $x \in \Omega_{\hat{u}} \cap \mathbf{S}^1$, then $\tau_{\hat{u}}(x) = x$. Let $T \in G$ be the primitive parabolic element with the fixed point x. By the same argument of Lemma 2.1, $\tau_{\hat{u}} T = T \tau_{\hat{u}}$. Via the Bers isomorphism, we obtain $t_u \circ t_a = t_a \circ t_u$, which implies that u and a are disjoint. Conversely, if u is disjoint from a, then $\tau_{\hat{u}}$ fixes x. So $x \in \Omega_{\hat{u}} \cap \mathbf{S}^1$. □

From Lemma 2.2, we know that u is disjoint from $\partial\Delta$ if x stays outside of all maximal elements of $\mathscr{U}_{\hat{u}}$. Similar situation occurs when h is non-essential hyperbolic with axis axis(h). In this case, there exists a simple closed geodesic $v \subset S$, with $\tilde{v}$ being non-trivial, such that v is disjoint from the pseudo-Anosov component P of h^*. It follows from Lemma 2.1 that both fixed points of h lie in $\Omega_{\hat{v}} \cap \mathbf{S}^1$. As $\Omega_{\hat{v}}$ is convex with geodesic boundary, it is clear that the axis axis(h) of h lies outside of any maximal element of $\mathscr{U}_{\hat{v}}$.

Let $w \in Q(G)$ be such that $[w]^* \in \mathrm{Mod}_S^z$ is a reducible mapping class by a reduced curve simplex Γ as defined in (2.1). Note that if $\tilde{u}_j$ is a non-trivial geodesic for some $j \in \{1, 2, \ldots, s\}$, that is, u_j, if viewed as a curve on $\tilde{S}$, is homotopic to neither a point nor a puncture of $\tilde{S}$, then, as discussed earlier, there defines a configuration $(\tau_{\hat{u}_j}, \Omega_{\hat{u}_j}, \mathscr{U}_{\hat{u}_j})$ that corresponds to u_j (in the sense that we can choose the lift $\hat{u}_j$ of $\tilde{u}_j$ and the component U of $\mathbf{D}\backslash\hat{u}_j$ on which the lift $\tau_{\hat{u}_j}$ of $t_{\tilde{u}_j}$ is constructed). See the discussion above Lemma 2.1.

Note also that any two twice punctured disks, if both enclose z, must intersect. Since all elements of Γ are disjoint, there is at most one geodesic u in Γ such that $\tilde{u}$ is trivial.

LEMMA 2.3. *With the above conditions*:

(1) *If there is a $u_i \in \Gamma$ with $\tilde{u}_i$ being non-trivial such that $[w]^*(u_i) = u_i$, Then there is $w_0 \in Q(G)$ with $w_0 \sim w$ such that w_0 sends every maximal element of $\mathscr{U}_{\hat{u}_i}$ to a maximal element of $\mathscr{U}_{\hat{u}_i}$.*

(2) *If there are $u_i, u_j \in \Gamma$, $i \neq j$, such that $[w]^*(u_i) = u_j$, then $\tilde{u}_i$ and $\tilde{u}_j$ are non-trivial and there is $w_0 \in Q(G)$ with $w_0 \sim w$ such that w_0 sends every maximal element of $\mathscr{U}_{\hat{u}_i}$ to a maximal element of $\mathscr{U}_{\hat{u}_j}$.*

PROOF. (1) is proved in [19]. For (2), we notice that if Γ contains a geodesic u so that $\tilde{u}$ is trivial on $\tilde{S}$, then such a curve u is unique. This tells us that $[w]^*(u) = u$. In other words, if $[w]^*(u_i) = u_j$ for some $u_i, u_j \in \Gamma$ with $u_i \neq u_j$, then both $\tilde{u}_i$ and $\tilde{u}_j$ are non-trivial. Thus the configurations $(\tau_{\hat{u}_i}, \Omega_{\hat{u}_i}, \mathscr{U}_{\hat{u}_i})$ and $(\tau_{\hat{u}_j}, \Omega_{\hat{u}_j}, \mathscr{U}_{\hat{u}_j})$, which correspond to u_i and u_j, respectively, are defined. The rest of the proof is similar to (1) which was given in [19]. □

More generally, we have

LEMMA 2.4. *Assume that $[w]^* \in \mathrm{Mod}_S^z$ is a reducible mapping class with the reduced curve simplex* (2.1). *Also assume that each $\tilde{u}_i$, $u_i \in \Gamma$, is non-trivial. Then for every maximal element $U_1 \in \mathscr{U}_{\hat{u}_1}$, $w^k(U_1) \cup U_1 \neq \mathbf{D}$ for all integers k.*

PROOF. Suppose that for an integer k_0 and a maximal element $U_1 \in \mathscr{U}_{\hat{u}_1}$, we have $w^{k_0}(U_1) \cup U_1 = \mathbf{D}$. If $s = 1$, then $[w]^*(u_1) = u_1$. By Lemma 2.3 (1), w sends every maximal element $U_1 \in \mathscr{U}_{\hat{u}_1}$ to a maximal element. Since all maximal elements of $\mathscr{U}_{\hat{u}_1}$ are disjoint and $\Omega_{\hat{u}_1}$ is not empty, we see that $w^k(U_1) \cup U_1 \neq \mathbf{D}$. Thus we assume that $s \geq 2$ and that $([w]^*)^{k_0}(u_1) = u_2$, where $u_1, u_2 \in \Gamma$. Then $t_{u_2} = ([w]^*)^{k_0} \circ t_{u_1} \circ ([w]^*)^{-k_0}$, which says that $w^{k_0}\tau_{\hat{u}_1}w^{-k_0} = \tau_{\hat{u}_2}$. It follows that $w^{k_0}(\mathscr{U}_{\hat{u}_1})$ is the collection of half-spaces defined by $\tau_{\hat{u}_2}$ and that $w^{k_0}(U_1) \in \mathscr{U}_{\hat{u}_2}$ is a maximal element. Since $w^{k_0}(U_1) \cup U_1 = \mathbf{D}$, by Lemma

4 of [15], we have $[\tau_{\hat{u}_2}][\tau_{\hat{u}_1}] \neq [\tau_{\hat{u}_1}][\tau_{\hat{u}_2}]$. Thus $t_{u_1} \circ t_{u_2} \neq t_{u_2} \circ t_{u_1}$. Hence u_1 intersects u_2, contradicting that $u_1, u_2 \in \Gamma$. □

3. Proof of Theorem 1.1

For simplicity, write $a = \partial \Delta$, $b = f(\partial \Delta)$ and $f = g^*$ for some essential hyperbolic element $g \in G$. Suppose that (a, b) does not fill S. There exists a simple closed geodesic u such that

$$t_a^r \circ t_b^{-s}(u) = u \tag{3.1}$$

for all positive integers r and s.

Case 1. The geodesic $\tilde{u}$ is trivial on $\tilde{S}$. In this case, $u = \partial \Delta_1$ for a twice punctured disk Δ_1 enclosing z. From Theorem 2 of [10] and Theorem 2 of [13], there are parabolic elements $h_1, h_2 \in G$ such that

$$h_1^* = t_a \quad \text{and} \quad h_2^* = t_b\,. \tag{3.2}$$

Note that Δ_1 is also a twice punctured disk enclosing z. There is a parabolic element $T_1 \in G$ that corresponds to the Dehn twist $t_{\partial \Delta_1}$, i.e., $T_1^* = t_{\partial \Delta_1} = t_u$. But we know that $t_b = t_{f(a)} = f \circ t_a \circ f^{-1}$. Hence $h_2 = g h_1 g^{-1}$. From (3.1) (by setting $r = s = 1$) we obtain $t_a \circ f \circ t_a^{-1} \circ f^{-1}(u) = u$, which tells us that the commutator $[h_1, g] = h_1 g h_1^{-1} g^{-1}$ commutes with T_1. From Lemma 5.2 of [20], $[h_1, g]$ also fixes the fixed point of T, which says that $[h_1, g]$ and T share a common fixed point. Clearly, $[h_1, g]$ is non-trivial (otherwise, h_1 commutes with g, a contradiction). Since G is discrete, by Theorem 5.1.2 of [1], $[h_1, g]$ cannot be hyperbolic. But on the other hand, by Theorem 7.39.1 of Beardon [1], for a parabolic element h_1, and a hyperbolic element g, the commutator $[h_1, g] \in G$ is always hyperbolic. This is a contradiction.

Case 2. The geodesic $\tilde{u}$ is non-trivial on $\tilde{S}$. Note that $\tilde{u}$ denotes the geodesic representative on $\tilde{S}$ homotopic to u when u is viewed as a curve on $\tilde{S}$. As discussed in Section 2, we denote by $(\tau_{\hat{u}}, \Omega_{\hat{u}}, \mathscr{U}_{\hat{u}})$ the configuration corresponding to u. The equality (3.1) yields that $\left(t_a^r \circ t_b^{-s}\right) \circ t_u = t_u \circ \left(t_a^r \circ t_b^{-s}\right)$. It follows from (3.1) and Lemma 2.3 that both $[h_1, g]$ and $h_1^r h_2^{-s}$ send each maximal element of $\mathscr{U}_{\hat{u}}$ to a maximal element. By the assumption, g is an essential hyperbolic element of G whose axis axis(g) intersects one (and hence infinitely many) of the preimages $\{\varrho^{-1}(\tilde{u})\}$, say $\hat{u}_0$. Note that $\hat{u}_0$ could be the boundary of a maximal element of $\mathscr{U}_{\hat{u}}$, but $\hat{u}_0$ could also be a boundary of an element of $\mathscr{U}_{\hat{u}}$ of higher order. If the later occurs, axis(g) is contained in a maximal element of $\mathscr{U}_{\hat{u}}$.

In each of the following cases we will show there is a maximal element U_0 of $\mathscr{U}_{\hat{u}}$ such that $h_1^r h_2^{-s}$ or its inverse $h_2^s h_1^{-r}$ does not send U_0 to a maximal element of $\mathscr{U}_{\hat{u}}$. But from (3.2), (3.1) and Lemma 2.3, $h_1^r h_2^{-s}$ and $h_2^s h_1^{-r}$ send every maximal element of $\mathscr{U}_{\hat{u}}$ to a maximal element of $\mathscr{U}_{\hat{u}}$, which will lead to a contradiction.

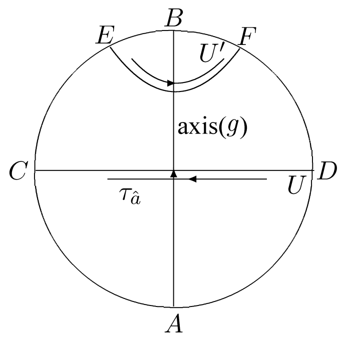

FIG. 2

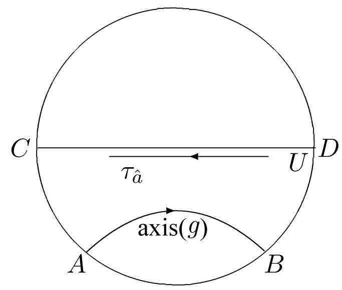

FIG. 3

Subcase 1. The geodesic $\hat{u}_0$ is the boundary of a maximal element $U \in \mathscr{U}_{\hat{u}}$. In this case, we may assume that $\hat{u}_0 = \hat{u} = \partial U$ and that U covers the repelling fixed point A of g, as shown in Fig. 2.

In the rest of the article we use (AC), for example, to denote the unoriented arc in $\mathbf{S}^1$ connecting the two labeling points A and C without passing through any other labeling points. Denote by $U' \in \mathscr{U}_{\hat{u}}$ the other maximal element containing $g(\mathbf{D}\backslash U)$. Then U' covers the attracting fixed point B of g. Let x denote the fixed point of h_1. Then $g(x)$ is the fixed point of $h_2 = gh_1g^{-1}$. We assume that h_1 points in the counterclockwise direction, and thus h_2^{-1} points in the clockwise direction.

If $x \in (CE)$, then $g(x) \in (BE)$. As $\tilde{u}$ is simple, for sufficiently large integers r and s, $h_2^{-s}(\mathbf{D}\backslash U') \cap \mathbf{S}^1 \subset (EB)$, and thus $h_1^r h_2^{-s}(\mathbf{D}\backslash U') \subset \mathbf{D}\backslash U'$. It follows from Lemma 4.3 of [19] that $h_1^r h_2^{-s}(U')$ is not a maximal element of $\mathscr{U}_{\hat{u}}$. This contradicts Lemma 2.3. The same argument applies to the case of $x \in (FD)$. If $x \in (EB)$, then since B is the attracting fixed point of g, $g(x) \in (EB)$ is closer to B than x. For large r and s, $h_2^{-s}(\mathbf{D}\backslash U') \cap \mathbf{S}^1 \subset (xg(x))$ is disjoint from $\mathbf{D}\backslash U'$. It follows that $h_1^r h_2^{-s}(\mathbf{D}\backslash U')$ is disjoint from $\mathbf{D}\backslash U'$. Hence by Lemma 4.3 of [19], $h_1^r h_2^{-s}$ and hence its inverse $h_2^s h_1^{-r}$ does not send U' to any maximal element of $\mathscr{U}_{\hat{u}}$. This again contradicts Lemma 2.3. The same is true when $x \in (BF)$.

If $x \in (CA)$, then since A is repelling fixed point of g, either $g(x) \in (CA)$, or $g(x) \in (CB)$. If $g(x) \in (CA)$, then $h_2^{-s}(\mathbf{D}\backslash U) \cap \mathbf{S}^1 \subset (g(x)x)$, and thus $h_1^r h_2^{-s}(\mathbf{D}\backslash U) \subset (g(x)x)$. It follows that $\mathbf{D}\backslash U$ is disjoint from $h_1^r h_2^{-s}(\mathbf{D}\backslash U)$. Hence by Lemma 4.3 of [19], $h_2^s h_1^{-r}(U)$ is not a maximal element of $\mathscr{U}_{\hat{u}}$. This contradicts Lemma 2.3. If $g(x) \in (CE)$, then $h_1^{-r}(\mathbf{D}\backslash U) \cap \mathbf{S}^1 \subset (CA)$ and thus $h_2^s h_1^{-r}(\mathbf{D}\backslash U) \cap \mathbf{S}^1 \subset (CE)$. We see that $h_2^s h_1^{-r}(\mathbf{D}\backslash U) \subset \mathbf{D}\backslash U$. This implies $U \subset h_2^s h_1^{-r}(U)$. In particular, U is not a maximal element of $\mathscr{U}_{\hat{u}}$. If $g(x) \in (EB)$, then $h_1^r h_2^{-s}(U) \subset U$. This is also impossible. The same argument applies to the case of $x \in (AD)$.

Subcase 2. The axis axis(g) is contained in a maximal element $U \in \mathscr{U}_{\hat{u}}$. See Fig. 3. If

$x \in (AC)$, then $g(x)$ lies in (AC), (CD) or (BD). If $g(x)$ is in (AC), $g(x)$ is closer to C than x. One sees that $h_2^s h_1^{-r}(\mathbf{D}\backslash U)$ is disjoint from $\mathbf{D}\backslash U$ for large r and s. So $h_2^s h_1^{-r}$ and hence $h_1^r h_2^{-s}(U)$ is not a maximal element. If $g(x)$ is in (CD), one checks that $h_1^r h_2^{-s}(U) \subset U$ for large r and s, and this would imply that $h_1^r h_2^{-s}(U)$ is not a maximal element. If $g(x)$ is in (BD), $h_2^s h_1^{-r}(\mathbf{D}\backslash U)$ is disjoint from $\mathbf{D}\backslash U$ for large r and s, which says that U is not a maximal element of $\mathscr{U}_{\hat{u}}$.

If $x \in (BD)$ (resp. $x \in (AB)$), then since B is the attracting fixed point of g, $g(x) \in (BD)$ (resp. $g(x) \in (AB)$) is closer to B than x. As one can see, $h_1^r h_2^{-s}(\mathbf{D}\backslash U)$ is disjoint from $\mathbf{D}\backslash U$. It follows that $h_1^r h_2^{-s}(U)$ is not a maximal element of $\mathscr{U}_{\hat{u}}$. Finally, if $x \in (CD)$, then $g(x) \in (BD)$. For large r and s, $h_2^s h_1^{-r}(U) \subset U$. This tells us that $h_2^s h_1^{-r}(U)$ is not a maximal element of $\mathscr{U}_{\hat{u}}$.

This case-by-case argument finishes the proof of Theorem 1.1. □

PROOF OF COROLLARY 1.1: Let $z, z_1, \dots, z_k$ denote all the punctures contained in D_α. Let Λ_α be the corresponding path connecting $z, z_1, \dots, z_k$ in this order. Let Λ'_α be the sub-path of Λ_α connecting z and z_1. Let Δ_α be a fattening of Λ'_α. Then Δ_α is a twice punctured disk enclosing z. From Theorem 1.1, $(\partial\Delta_\alpha, f^k(\partial\Delta_\alpha))$ fills S for all $k \geq 1$. It is clear that $(\partial\Delta_\alpha, f(\partial\Delta_\alpha))$ fills $\tilde{S}$ if and only if $(\Lambda'_\alpha, f(\Lambda'_\alpha))$ fills S. Since $\Lambda'_\alpha \subset \Lambda_\alpha$ and $f(\partial\Delta_\alpha) = \partial f(\Delta_\alpha)$, we see that $(\Lambda_\alpha, f(\Lambda_\alpha))$ fills $\tilde{S}$. From the construction, D_α is a fattening of Λ_α. We conclude that $(\alpha, f(\alpha))$ fills S, as asserted. □

4. Proof of Theorem 1.2

By the assumption, we know that Δ and Δ_0 enclose the same punctures z and z_0 and that $(\partial\Delta, \partial\Delta_0)$ fills S. As $\partial\Delta$ and $\partial\Delta_0$ are loops around the same puncture z_0 of $\tilde{S}$ as z is filled in, it is clear that the primitive parabolic elements T and T_0 of G corresponding to $\partial\Delta$ and $\partial\Delta_0$ are conjugate to each other in G. It follows that there is an element $h \in G$ sending the fixed point x of T to the fixed point x_0 of T_0. As such, $F = h^*$ sends $\partial\Delta$ to $\partial\Delta_0$. Of course, $F \in \mathscr{F}_0(S)$. We need to prove that $F \circ t_{\partial\Delta}^{-k}$ are pseudo-Anosov for either all $k > 0$ or all $k < 0$.

If h is essential hyperbolic, then F is pseudo-Anosov. Hence $F \in \mathscr{F}(S)$ and by Lemma 3.1 of [22], we conclude that $F \circ t_{\partial\Delta}^{-k}$ are pseudo-Anosov for all $k \geq 0$ or $k \leq 0$.

If h is parabolic, then by Theorem 2 of [10, 13], $F = t_c$ or t_c^{-1}, where c is a simple closed geodesic that is also trivial on $\tilde{S}$, i.e., $c = \partial\Delta'$ for some twice punctured disk Δ' enclosing z. Assume that $F = t_c$. By the definition, $\partial\Delta_0 = t_c(\partial\Delta)$. We see that

$$t_c \circ t_{\partial\Delta} \circ t_c^{-1} = t_{\partial\Delta_0}\,. \tag{4.1}$$

Since $(\partial\Delta, \partial\Delta_0)$ fills S, from (4.1), c intersects $\partial\Delta$. We claim that $(\partial\Delta, c)$ also fills S. In fact, the geodesic $t_c(\partial\Delta) = \partial\Delta_0$ is homotopic to a closed curve that stays in an arbitrary small neighborhood $\mathscr{N}$ of $\partial\Delta \cup c$. If $(\partial\Delta, c)$ does not fill S, then there is a non-trivial loop

e that is disjoint from $\partial\Delta \cup c$. So e is also disjoint from $\mathscr{N}$ if $\mathscr{N}$ is made to be sufficiently small. It follows that e is disjoint from both $\partial\Delta$ and $\partial\Delta_0$, contradicting that $(\partial\Delta, \partial\Delta_0)$ fills S.

Hence by Thurston's theorem [14], $t_c \circ t_{\partial\Delta}^{-k}$ for all $k > 0$ are pseudo-Anosov maps. Note also that both c and $\partial\Delta$ are trivial on $\tilde{S}$ (that is, they are freely homotopic to a puncture of $\tilde{S}$) and $t_c \circ t_{\partial\Delta}^{-k}(\partial\Delta) = t_c(\partial\Delta) = \partial\Delta_0$, we see that $t_c \circ t_{\partial\Delta}^{-k} \in \mathscr{F}(S)$ sends $\partial\Delta$ to $\partial\Delta_0$.

It remains to consider the case where h is non-essential hyperbolic and non-parabolic element of G. Recall that h possesses the property that $(\partial\Delta, h^*(\partial\Delta))$ fills S. Our aim is to show that $h^* \circ t_{\partial\Delta}^{-k}$ is pseudo-Anosov for either all $k > 0$ or all $k < 0$.

Suppose that for some $k > 0$ and some $k < 0$, there is a system Γ (which depends on k and is defined as in (2.1)) such that

$$h^* \circ t_{\partial\Delta}^{-k}(\{u_1, \ldots, u_s\}) = \{u_1, \ldots, u_s\}.$$

This tells us that

$$\left(h^* \circ t_{\partial\Delta}^{-k}\right) \circ \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) = \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) \circ \left(h^* \circ t_{\partial\Delta}^{-k}\right). \tag{4.2}$$

There are two cases to consider.

Case 1. All $\tilde{u}_i$, $u_i \in \Gamma$, are non-trivial. Our first claim is that there is at least one $u = u_i \in \Gamma$, say, such that u intersects $\partial\Delta$. Suppose to the contrary. That is, all u_i are disjoint from $\partial\Delta$. Hence $t_{u_1} \circ \cdots \circ t_{u_s}$ commutes with $t_{\partial\Delta}$. From (4.2) we see that h^* commutes with $t_{u_1} \circ \cdots \circ t_{u_s}$ and thus that

$$\left(h^* \circ t_{\partial\Delta}^{-k} \circ (h^*)^{-1}\right) \circ \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) = \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) \circ \left(h^* \circ t_{\partial\Delta}^{-k} \circ (h^*)^{-1}\right).$$

On the other hand, since $(\partial\Delta, \partial\Delta_0)$ fills S, every u_i must intersect $\partial\Delta_0$. This implies

$$t_{\partial\Delta_0}^{-k} \circ \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) \neq \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) \circ t_{\partial\Delta_0}^{-k}.$$

But $t_{\partial\Delta_0}^{-k} = h^* \circ t_{\partial\Delta}^{-k} \circ (h^*)^{-1}$. We see that

$$\left(h^* \circ t_{\partial\Delta}^{-k} \circ (h^*)^{-1}\right) \circ \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) \neq \left(t_{u_1} \circ \cdots \circ t_{u_s}\right) \circ \left(h^* \circ t_{\partial\Delta}^{-k} \circ (h^*)^{-1}\right).$$

This is absurd. We conclude that there is a geodesic $u \in \Gamma$ such that u intersects $\partial\Delta$. Note that $h^* \circ t_{\partial\Delta}^{-k} \in \mathscr{F}_0(S)$.

Our next claim is that for any integer m,

$$\left(h^* \circ t_{\partial\Delta}^{-k}\right)^m (u) = u. \tag{4.3}$$

This assertion was implicitly proved in [10]. For completeness, however, the proof of (4.3) is included as follows. Since $h^* \circ t_{\partial\Delta}^{-k} \in \mathscr{F}_0(S)$, we let $h_1 \in G$ be such that $h_1^* = h^* \circ t_{\partial\Delta}^{-k}$. From Theorem 2 of [13] and Theorem 2 of [10], we know that if h_1 is parabolic, then h_1^* is represented by a power of a Dehn twist t_c for a simple closed geodesic c on $\dot{S}$. In this case, $h_1^*(u_i) = u_i$ for each $u_i \in \Gamma$. If h_1 is simple hyperbolic, then h_1^* is represented by a power of a spin map $t_\alpha^{-1} \circ t_\beta$, where $\{\alpha, \beta\}$ forms the boundary of an z-punctured cylinder on S. In this case, we also see that $h_1^*(u_i) = u_i$ for each $u_i \in \Gamma$. If h_1 is non-simple and non-essential, then by Theorem 2 of [10], there is a unique pseudo-Anosov component $\mathscr{P} \subset S$ for h_1^* that contains z. As it turns out, any curve u_i in Γ cannot meet $\mathscr{P}$ in a non-trivial way, which means that all $u_i \in \Gamma$ stays outside of $\mathscr{P}$. It follows that $h_1^*(u_i) = u_i$. Finally, if h_1 is essential hyperbolic, then by Theorem 2 of [10] again, h_1^* is pseudo-Anosov. By the assumption, this case does not occur. We thus conclude that (4.3) holds for every $u \in \Gamma$.

Now let $(\tau_{\hat{u}}, \Omega_{\hat{u}}, \mathscr{U}_{\hat{u}})$ be the configuration corresponding to u. From Lemma 2.2, there exists a maximal element $U \in \mathscr{U}_{\hat{u}}$ that covers x. Recall that x is the fixed point of the parabolic element $T \in G$ that corresponds to $\partial \Delta$. If the axis axis(h) is disjoint from U (Fig. 4), then for any $k \neq 0$, $hT^{-k}(\mathbf{D}\backslash U) \subset \mathbf{D}\backslash U$. Hence $\left(hT^{-k}\right)^m (\mathbf{D}\backslash U) \subset \mathbf{D}\backslash U$. That is, $\left(hT^{-k}\right)^m (U)$ is not a maximal element of $\mathscr{U}_{\hat{u}}$. This contradicts Lemma 2.3.

Consider the case where axis(h) crosses U. Let $U' \in \mathscr{U}_{\hat{u}}$ be the other maximal element intersecting axis(h) (Lemma 2.1 of [21]). See Fig. 5. If the attracting fixed point of h is in U, that is, A is the attracting fixed point of h, then $T^{-k}(\mathbf{D}\backslash U) \subset U$ and thus $hT^{-k}(\mathbf{D}\backslash U) \subset U$, which says $hT^{-k}(U) \cup U = \mathbf{D}$, contradicting Lemma 2.4.

Now we assume that the attracting fixed point of h is in U'. In this case, U covers the repelling fixed point of h, denoted by A. Recall that the motion T points in the counterclockwise direction (as shown in Fig. 5). Now the relative position between x and A determines whether we choose $k > 0$ or $k < 0$. We assume without loss of generality that x is on the left side of A, as shown also in Fig. 5. By examining the action of T^{-k} for any $k < 0$, we find that the motion of T^{-k} (for any $k < 0$) and h have the same relative motion direction. It turns out that

$$T^{-k}(\mathbf{D}\backslash U) \cap \mathbf{S}^1 \subset (Cx)\,.$$

Since A is the repelling fixed point of h, $hT^{-k}(\mathbf{D}\backslash U)$ lies in either (i) U, or (ii) U', or

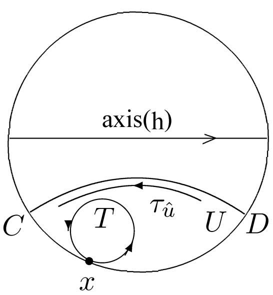

FIG. 4

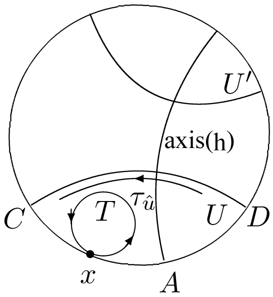

FIG. 5

(iii) $\mathbf{D}\backslash (U \cup U')$. Notice that (i) implies that $hT^{-k}(U) \cup U = \mathbf{D}$, which would contradict Lemma 2.4. (ii) implies that $(hT^{-k})^i (U') \subset U'$ for all $i > 0$, which says that $(hT^{-k})^i (U')$ are never maximal elements of $\mathscr{U}_{\hat{u}}$. If (iii) holds, then one easily checks that for all $i > 0$, $(hT^{-k})^i (\mathbf{D}\backslash U) \subset \mathbf{D}\backslash (U \cup U')$. That is, $U \cup U' \subset (hT^{-k})^i (U)$. In other words, $(hT^{-k})^i (U)$ never becomes a maximal element of $\mathscr{U}_{\hat{u}}$. From Lemma 2.3, we see that (4.3) never occurs.

REMARK. In the case where $k > 0$, we observe that $T^{-k}(\mathbf{D}\backslash U)$ could possibly cover the repelling fixed point A of h. If this occurs, then $hT^{-k}(\mathbf{D}\backslash U) \subset \mathbf{D}\backslash U'$ and there is no guarantee that $hT^{-k}(U) \neq U, U'$. Thus no contradiction can be found. Nevertheless, the above argument tells us that for all sufficiently large positive integers k, $hT^{-k}(U)$ are not maximal elements of $\mathscr{U}_{\hat{u}}$, which will lead to that $hT^{-k} \in \mathscr{F}(S)$ for sufficiently large k.

Case 2. There is one $u \in \Gamma$ such that $\tilde{u}$ is trivial. In this case, $u = \partial\Delta'$ for some twice punctured disk enclosing z and u is the only one element in Γ with $\tilde{u}$ being trivial. We have

$$h^* \circ t_{\partial\Delta}^{-k}(u) = u\,. \tag{4.4}$$

Let $y \in \mathbf{S}^1$ be the fixed point of the parabolic element corresponding to u. (4.4) then yields

$$hT^{-k}(y) = y\,. \tag{4.5}$$

This means that hT^{-k} is also a parabolic element. Write $T_u = hT^{-k}$. From (4.5), we have $T_u^* = t_u$ or t_u^{-1}. Assume that $T_u^* = t_u$ (the case where $T_u^* = t_u^{-1}$ can be handled similarly and is omitted). Then

$$h = T_u T^k\,, \quad \text{or} \quad h^* = t_u \circ t_{\partial\Delta}^k\,. \tag{4.6}$$

Now consider the pair $(\partial\Delta, u)$. It is clear that $(\partial\Delta, u)$ does not fill S. Otherwise, by Thurston [14], $t_u \circ t_{\partial\Delta}^k$ for each $k < 0$ would be a pseudo-Anosov map. It follows from (4.6) that h^* is pseudo-Anosov. Hence by Theorem 2 of [10], h is an essential hyperbolic element, contradicting the hypothesis.

We also know that u must intersect $\partial\Delta$. Since $(\partial\Delta, u)$ does not fill S, there is a simple closed geodesic v disjoint from $\partial\Delta \cup u$. The geodesic $\partial\Delta_0 = h^*(\partial\Delta)$ is homotopic to the image curve $t_u \circ t_{\partial\Delta}^k(\partial\Delta) = t_u(\partial\Delta)$ that is defined in a neighborhood $\mathscr{N}$ of $\partial\Delta \cup u$, where $\mathscr{N}$ is chosen to be so small that v is disjoint from $\mathscr{N}$. We conclude that v does not intersect $\partial\Delta \cup \partial\Delta_0$. That is, $(\partial\Delta, \partial\Delta_0)$ does not fill S. This contradicts the hypothesis.

We conclude that $h^* \circ t_{\partial\Delta}^{-k}$, which sends $\partial\Delta$ to $\partial\Delta_0$, is pseudo-Anosov for either all $k > 0$ or all $k < 0$. This completes the proof of Theorem 1.2. □

PROOF OF COROLLARY 1.2: Let $\{z, z_0\}$ and $\{z, z'\}$ denote the punctures in Δ and Δ', respectively. Suppose that such an f exists and that $z_0 \neq z'$. As f projects to a map $\tilde{f}$ on $\tilde{S}$, it is obvious that f fixes the puncture z and so $\tilde{f}(z_0) = z'$, contradicting the fact that $\tilde{f}$ is isotopic to the identity on $\tilde{S}$. From Theorem 1.1, $f(\partial\Delta) = \partial\Delta'$ and $(\partial\Delta, \partial\Delta')$ fills S. Hence $\Delta' \in \mathscr{T}(\Delta)$.

Conversely, if $\Delta' \in \mathscr{T}(\Delta)$, then by Theorem 1.2, there is an element $f \in \mathscr{F}(S)$ sending Δ to Δ', as claimed. □

5. A classification of elements of $\mathscr{F}(S)$ in terms of $\mathscr{T}(\Delta)$

To prove Theorem 1.3, we need the following lemma.

LEMMA 5.1. *Let $F : S \to S$ be obtained from Theorem* 1.2*. Then every element $\mathscr{F}(S)$ that sends $\partial\Delta$ to $\partial\Delta_0$ is of the form $F \circ t_{\partial\Delta}^{-k}$ for some integer k.*

PROOF. Let $f \in \mathscr{F}(S)$ be such that $f(\partial\Delta) = \partial\Delta_0$. Note that $\mathscr{F}_0(S)$ is the kernel of the group homomorphism of Mod_S^z onto $\mathrm{Mod}(\tilde{S})$. There is an essential hyperbolic element $g \in G$ so that $g^* = f$. Also, as mentioned earlier, the parabolic elements T and T_0 of G that correspond to $\partial\Delta$ and $\partial\Delta_0$ are conjugate to each other. Hence there is an element $F \in \mathscr{F}_0(S)$ sending $\partial\Delta$ to $\partial\Delta_0$. Recall that $F = h^*$ for some element $h \in G$.

Observe that $h(x) = x_0$ is the fixed point of $T_0 = hTh^{-1}$. On the other hand, since $f(\partial\Delta) = \partial\Delta_0$, $f \circ t_{\partial\Delta} \circ f^{-1} = t_{\partial\Delta_0}$. It follows that $gTg^{-1} = T_0$, which implies $g(x)$ is the fixed point of T_0. But T_0 is parabolic, it has unique fixed point x_0 on $\mathbf{S}^1$. We conclude that $g(x) = h(x)$ or $g^{-1}h(x) = x$. If $g = h$, then h is essential hyperbolic and thus F is pseudo-Anosov. Otherwise, $g^{-1}h$ is non-trivial. Since T is parabolic, it also has a unique fixed point x on $\mathbf{S}^1$. Hence $g^{-1}h$ and T share the same fixed point x. In particular, $g^{-1}h$ cannot be hyperbolic (otherwise, G would not be discrete) and the only possibility is that $g^{-1}h$ is also parabolic (if it is non-trivial) and so there is an integer k such that $g^{-1}h = T^k$ or $g = hT^{-k}$. That is, $f = h^* \circ t_{\partial\Delta}^{-k}$. □

PROOF OF THEOREM 1.3: Let $f \in \mathscr{F}(S)$. By Theorem 1.1, $(\partial\Delta, \partial f(\Delta))$ fills S. Note that f is isotopic to the identity on $\tilde{S}$, Δ and $f(\Delta)$ both enclose z and z_0. Thus $f(\Delta) \in \mathscr{T}(\Delta)$. Since $f \circ t_{\partial\Delta}^k(\partial\Delta) = f(\partial\Delta)$ for any k, we obtain a map $\omega : \mathscr{F}(S)/\sim \to \mathscr{T}(\Delta)$.

Conversely, let $\Delta_0 \in \mathscr{T}(\Delta)$. Then by the definition of $\mathscr{T}(\Delta)$, $(\partial\Delta_0, \partial\Delta)$ fills S. By Theorem 1.2, there is $F \in \mathscr{F}_0(S)$ such that $F(\partial\Delta) = \partial\Delta_0$. Let $\chi(\Delta_0)$ be the Δ-equivalence class of $F \circ t_{\partial\Delta}^k$. By Theorem 1.2, $F \circ t_{\partial\Delta}^k$ are pseudo-Anosov for either all $k > 0$ or $k < 0$. We thus obtain the map $\chi : \mathscr{T}(\Delta) \to \mathscr{F}(S)/\sim$.

We claim that $\chi \circ \omega = \mathrm{id}$ (which says that ω is injective). Indeed, for any $f \in \mathscr{F}(S)$, let $[f]_\Delta$ denote the Δ-equivalence class of f in $\mathscr{F}(S)/\sim$. By Theorem 1.1, $(\partial\Delta, f(\partial\Delta))$ fills S. By Theorem 1.2, there is F sending Δ to $f(\Delta)$. From Lemma 5.1, $f = F \circ t_{\partial\Delta}^k$ for some k, which says that $\chi \circ \omega(f)$ is Δ-equivalent to f. It follows that $\chi \circ \omega = \mathrm{id}$.

Finally, we prove that $\omega \circ \chi = \mathrm{id}$ (which says that ω is surjective). Let $\Delta_0 \in \mathscr{T}(\Delta)$. Then $(\partial\Delta, \partial\Delta_0)$ fills S. By Theorem 1.2 again, there is $F \in \mathscr{F}_0(S)$ such that $F(\Delta) = \Delta_0$ and that $f := F \circ t_{\partial\Delta}^k$ are pseudo-Anosov for all $k > 0$ or $k < 0$. This implies that $[f]_\Delta = \chi(\Delta_0)$.

But since $f \circ t_{\partial\Delta}^k(\partial\Delta) = \partial\Delta_0$ for any k, we have $\omega \circ \chi(\Delta_0) = \Delta_0$, and thus $\omega \circ \chi = \mathrm{id}$, as claimed. □

6. Distances between elements of $\mathscr{T}(\Delta)$ and dilatations of associated pseudo-Anosov maps

PROOF OF THEOREM 1.4: (1) From Theorem 1.3, we know that there is $f \in \mathscr{F}(S)$ such that $f(\Delta) = \Delta_0$. By Theorem 1.2, $f \circ t_{\partial\Delta}^k$ are pseudo-Anosov for either $k > 0$ or $k < 0$. We assume that $k > 0$. It is clear that for all $k > 0$, $f \circ t_{\partial\Delta}^k(\Delta) = \Delta_0$. We need to show that $\lambda(f \circ t_{\partial\Delta}^k) \to +\infty$ as $k \to +\infty$.

Note that $f \circ t_{\partial\Delta}^k \in \mathscr{F}(S)$ for any k. Let γ, γ_k denote the filling closed geodesics on $\tilde{S}$ corresponding to f and $f \circ t_{\partial\Delta}^k$, and let i_γ and i_{γ_k} denote the number of self-intersection points of γ and γ_k, respectively. Assume that $z \in \gamma$. As Δ determines a path Λ joining z and z_0, Δ in turn determines a parabolic element $\delta \in \pi_1(\tilde{S}, z)$ around z_0.

By the same argument of Theorem 1.1 of [22], the curve concatenation $\delta^k \cdot \gamma$ is freely homotopic to γ_k, where we note that γ_k is a filling closed geodesic. The associated homotopy is denoted by $\delta^k \cdot \gamma \sim \gamma_k$. Observe that the k-th power of δ repeats δ k times. During the deformation $\delta^k \cdot \gamma \sim \gamma_k$, a new set I_k of self-intersection points of δ^4 emerges. Fig. 6 below illustrates this process.

Fig. 6 (a) shows the multi-curve δ^4 as a portion in the curve concatenation $\delta^4 \cdot \gamma$. As we see, the multiplicity of δ is 4. Fig. 6 (b) shows what δ^4 looks like as a portion of γ_4, after the deformation $\delta^4 \cdot \gamma \sim \gamma_4$ is performed. We see that the set of self-intersection points $I_4 = \{p_1, p_2, p_3\}$ emerges.

We observe that any two points in I_k cannot cancel each other, while since i_γ is finite, only finite number of points in I_k could possibly cancel some existing self-intersection points of γ. But note that the cardinality of I_k tends to $+\infty$ as $k \to +\infty$, we conclude that $i_{\gamma_k} \to$

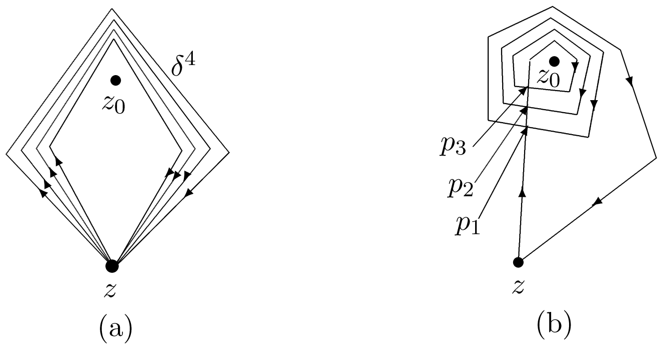

FIG. 6

$+\infty$ as $k \to +\infty$. Since $f \circ t^k_{\partial\Delta} \in \mathscr{F}(S)$, from the argument of Theorem 1.1 of [22], we obtain $\lambda(f \circ t^k_{\partial\Delta}) \to +\infty$ as $k \to +\infty$. Since $f \sim f \circ t^k_{\partial\Delta}$ and $f \circ t^k_{\partial\Delta}(\Delta) = \Delta_0$ for any k, we are done. The proof of (2) is the same as that of Theorem 1.3 in [23]. □

7. Examples

In [16] we constructed an example demonstrating that for any twice punctured disk Δ that encloses z and z_0, there are parabolic elements $h \in G$ such that $(\partial\Delta, h^*(\partial\Delta))$ fills S. In the example below, we present a simple hyperbolic element $h \in G$ such that $(\partial\Delta, h^*(\partial\Delta))$ fills S and $h^*(\partial\Delta) \in \mathscr{T}(\Delta)$.

Note that Δ is a twice punctured disk on S enclosing z; its deformation retract Λ is a path connecting z and another puncture z_1, say. The following constructions are suggested by the referee's comments on [18]. The surface S can be thought of as a surface with p handles $H_1, \ldots, H_p$ and $n+1$ punctures $z, z_1, \ldots, z_n$, where each handle is a copy of the handle H drawn in Fig. 7.

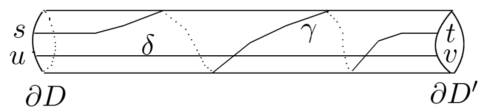

FIG. 7

H has two boundary components $\{\partial D, \partial D'\}$. Let γ, δ be two curves on H that are not homotopic to each other and fill H. Let $\{s, t\}$ and $\{u, v\}$ are endpoints of γ and δ, respectively.

We remove from the sphere $\mathbf{S}^2$ p pairs of small disks (D_i, D'_i) and $z, z_1, \ldots, z_n$. Then the surface S can be restored from attaching p handles along the boundary components $\partial D_i \cong \partial D$ and $\partial D'_i \cong \partial D'$ for $i = 1, 2, \ldots, p$.

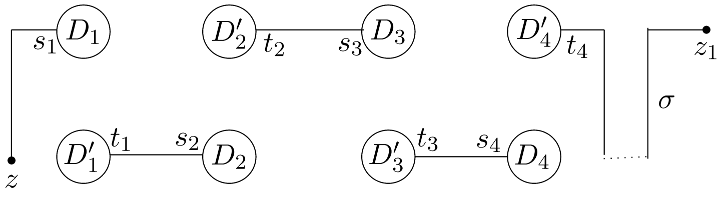

FIG. 8

Without loss of generality, we let Λ (the deformation retract of Δ) be the path described as follows. Connect z and s_1, followed by γ on H_1, then connect t_1 and s_2, and followed by

γ on H_2, and so forth. After p steps, we connect t_p and z_1 by a path σ that is away from all other punctures. Fig. 8 shows a path Λ in the case of $p = 4$.

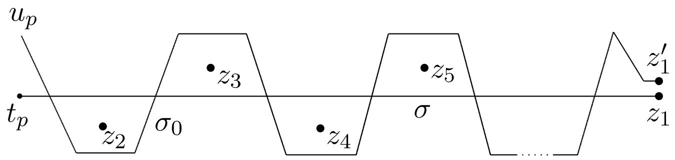

FIG. 9

Now we proceed to acquire a simple closed geodesic C as follows. Choose a point z' that is near to z, connect z' and v_1, followed by the inverse δ^{-1} of δ on H_1, then connect u_1 and v_2, followed by δ^{-1} on H_2, then connect u_2 and v_3, and so forth (see Fig. 10). After p steps, we draw a path σ_0 connecting u_p to a point z_1' that is near to the puncture z_1 in such a way that $\mathbf{S}^2\backslash\{\sigma, \sigma_0\}$ are $n-1$ once punctured disks each of which contains only one puncture in $\{z_2, z_3, \ldots, z_n\}$. See Fig. 9.

Finally, we connect z_1' and z' (the point we begin with) by a path away from all holes D_i, D_i' and all punctures $z, z_1, \ldots, z_n$. Fig. 10 shows an example for such a simple closed curve C in a surface of genus $p = 4$. We thus obtain a simple closed curve C on S so that the graph $C \cup \Lambda$ fills S, i.e., $S\backslash C \cup \Lambda$ consists of polygons and possibly once punctured polygons.

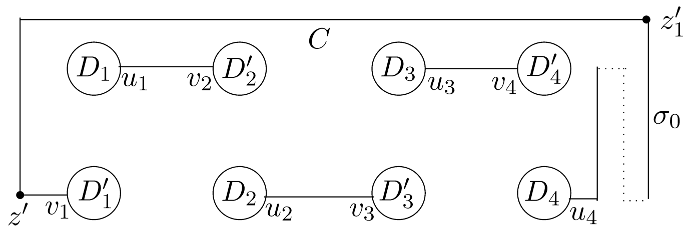

FIG. 10

Let $C_0 \subset S$ be another simple closed curve so that $\{C, C_0\}$ are boundary components of a punctured cylinder $\mathscr{P}$ with puncture z. Clearly, $\{C, C_0\} \cup \Lambda$ fills S. There exists a simple hyperbolic element $h \in G$ so that $h^* = t_{C_0} \circ t_C^{-1}$. Note also that Δ can be restored from Λ by a fattening process. We see that $\partial\mathscr{P} \cup \partial\Delta$ fills S. Let $\Delta_0 = h^*(\Delta)$.

PROPOSITION 7.1. *The pair $(\partial\Delta, \partial\Delta_0)$ fills S.*

PROOF. Assume that $(\partial\Delta, \partial\Delta_0)$ does not fill S. There is a geodesic $u \subset S$ such that $t_{\partial\Delta_0} \circ t_{\partial\Delta}^{-1}(u) = u$. Let $T \in G$ be the parabolic element corresponding to $t_{\partial\Delta}$. If $\tilde{u}$ is trivial, then the commutator $[h, T] = hTh^{-1}T^{-1}$ fixes a parabolic fixed point of G, so $[h, T] = hTh^{-1}T^{-1}$ is parabolic (otherwise G would not be discrete). This contradicts that $[h, T]$ is hyperbolic.

If $\tilde{u}$ is non-trivial, by Lemma 2.3, $[h, T]$ sends every maximal element $U \in \mathscr{U}_{\hat{u}}$ to a maximal element. On the other hand, we know from the hypothesis that u is disjoint from $\partial\Delta$. By Lemma 2.2, the fixed point x of T must lie in $\Omega_{\hat{u}} \cap \mathbf{S}^1$. Since $\{C, C_0\} \cup \Lambda$ fills S and u is disjoint from $\partial\Delta$, u must intersect $\{C, C_0\}$. By Lemma 2.1, axis(h) crosses a maximal element $U \in \mathscr{U}_{\hat{u}}$. Let $U' \in \mathscr{U}_{\hat{u}}$ be the other maximal element intersecting axis(h) (by Lemma 2.1 of [21]).

We are thus in the situation of Fig. 2 (with axis(g) being replaced by axis(h)). Since $x \notin \left(U \cup U'\right) \cap \mathbf{S}^1$, $x \in (CE) \cup (FD)$. Let us assume that $x \in (CE)$. By examining the action of the commutator $[h, T]$ on U', we find that $[h, T](U') \subset U'$, which says that $[h, T](U')$ is not a maximal element of $\mathscr{U}_{\hat{u}}$. This is a contradiction. $\square$

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