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Transformations Which Preserve Cauchy Distributions and Their Ergodic Properties

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Abstract. This paper is concerned with invariant densities for transformations on $\mathbf{R}$ which are the boundary restrictions of inner functions of the upper half plane. G. Letac [9] proved that if the corresponding inner function has a fixed point z_0 in $\mathbf{C}\backslash\mathbf{R}$ or a periodic point z_0 in $\mathbf{C}\backslash\mathbf{R}$ with period 2, then a Cauchy distribution $(1/\pi)\mathrm{Im}\,(1/(x-z_0))$ is an invariant probability density for the transformation. Using Cauchy's integral formula, we give an easier proof of Letac's result. An easy sufficient condition for such transformations to be isomorphic to piecewise expanding transformations on an finite interval is given by the explicit form of the density. Transformations of the forms $\alpha x+\beta-\sum_{k=1}^{n} b_k/(x-a_k)$, $\alpha x-\sum_{k=1}^{\infty}\{b_k/(x-a_k)+b_k/(x+a_k)\}$ and $\alpha x+\beta\tan x$ are studied as examples.

1. Introduction and Results

A various kind of 1-dimensional transformations have been found to have absolutely continuous invariant measures ([2], [3], [4], [8], [10]). However, there are not many transformations whose densities are explicitly known. The first aim of this article is to show that a transformation R on the real line $\mathbf{R}$ has an invariant probability density $(1/\pi)\mathrm{Im}\,(1/(x-z_0))$, if it is a boundary restriction of an inner function of the upper half plane and if there exists $z_0=x_0+\mathrm{i}y_0\in\mathbf{C}\backslash\mathbf{R}$ with $R(z_0)=z_0$ or with $R(z_0)=\overline{z_0}$. This explicit form of the invariant density allows us to obtain the ergodic properties of the transformation by using known results for transformations on finite intervals.

Precisely, we have the following Theorem 1, which was already proved by G. Letac ([9]). We give an easier proof by applying Cauchy's integral formula to the defining function in the theory of Sato's hyperfunction. The proofs and the examples will be found in Section 2 and Section 3, respectively. By $\mathbf{C}_+$ we denote the upper half plane $\{z\in\mathbf{C};\ \mathrm{Im}(z)>0\}$ and $\mathbf{C}_-=\{z\in\mathbf{C};\ \mathrm{Im}(z)<0\}$.

THEOREM 1. *Let $R(z)$ be a function defined on $\mathbf{C}_+\cup(\mathbf{R}\setminus E)$ and $R(\mathbf{R}\setminus E)\subset\mathbf{R}$, where E is a coutable subset of $\mathbf{R}$. Suppose further the following*:

(1) *$R(z)$ is holomorphic in $\mathbf{C}_+$ and satisfies that $R(\mathbf{C}_+)\subset\mathbf{C}_+$ or $R(\mathbf{C}_+)\subset\mathbf{C}_-$.*

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(2) *$R(z)$ is continuous on* $\mathbf{R} \setminus E$.

Then, for all $z_0 = x_0 + \mathrm{i}y_0 \in \mathbf{C}_+$, *the equation*

$$\int_{-\infty}^{\infty} f(R(x)) \left(\mathrm{Im} \frac{1}{x - z_0} \right) dx = \int_{-\infty}^{\infty} f(x) \left| \mathrm{Im} \frac{1}{x - R(z_0)} \right| dx$$

holds for all essentially bounded function $f(x)$.

From this Theorem 1 we can easily derive the following corollaries. If there exists a fixed point z_0 in $\mathbf{C} \setminus \mathbf{R}$, we can immediately get that a Cauchy distribution $(1/\pi)\mathrm{Im}\,(1/(x - z_0))\,dx$ is an invariant probability of the transformation R on $\mathbf{R}$.

COROLLARY 1. *Let $R(z)$ be a function which satisfies the assumptions in Theorem* 1. *Suppose also that R has a fixed point $z_0 = x_0 + \mathrm{i}y_0 \in \mathbf{C}_+$.*

Then we have

$$\int_{-\infty}^{\infty} f(R(x)) \left(\mathrm{Im} \frac{1}{x - z_0} \right) dx = \int_{-\infty}^{\infty} f(x) \left(\mathrm{Im} \frac{1}{x - z_0} \right) dx$$

for all bounded function $f(x)$, that is, a Cauchy distribution $(1/\pi)\mathrm{Im}(1/(x - z_0))dx$ is an invariant probability of the transformation R on $\mathbf{R}$.

In case there is a point $z_0 \in \mathbf{C}_+$ with $R(z_0) = \overline{z_0}$, the same result can be derived from the fact that $\mathrm{Im}\,(1/(x - z_0)) = -\mathrm{Im}\,(1/(x - \overline{z_0}))$.

COROLLARY 2. *Let $R(z)$ be a function which satisfies the assumptions in Theorem* 1. *Suppose also that R has a point $z_0 = x_0 + \mathrm{i}y_0 \in \mathbf{C}_+$ with $R(z_0) = \overline{z_0}$.*

Then we have

$$\int_{-\infty}^{\infty} f(R(x)) \left(\mathrm{Im} \frac{1}{x - z_0} \right) dx = \int_{-\infty}^{\infty} f(x) \left(\mathrm{Im} \frac{1}{x - z_0} \right) dx$$

for all bounded function $f(x)$, that is, a Cauchy distribution $(1/\pi)\mathrm{Im}(1/(x - z_0))dx$ is an invariant probability of the transformation R on $\mathbf{R}$.

In the article [7] the same results for a class of rational transformations

$$R(x) = \alpha x + \beta - \sum_{k=1}^{n} \frac{b_k}{x - a_k}$$

are given by using the factor theorem, and their applications are discussed. We can use the same idea in order to get the central limit theorem for more general transformations. Here we repeat the outline.

Suppose that R has a point $z_0 = x_0 + \mathrm{i}y_0 \in \mathbf{C}_+$ with $R(z_0) = z_0$ or $R(z_0) = \overline{z_0}$. We can also use our result to study the ergodic properties of (R, μ) on $\mathbf{R}$, where μ is an absolutely continuous probability measure with a density $(1/\pi)\mathrm{Im}\,(1/(x - z_0))$. Note that we clearly have

$$\mathrm{Im}\frac{1}{x-z_0}=\frac{y_0}{(x-x_0)^2+{y_0}^2}=\frac{d}{dx}\arctan\left(\frac{x-x_0}{y_0}\right)$$

for $z_0=x_0+\mathrm{i}y_0\in\mathbf{C}_+$. Denote

$$\varphi(x)=\arctan\left(\frac{x-x_0}{y_0}\right).$$

Then we can prove that the transformation $T(t):=\varphi(R(\varphi^{-1}(t)))$ on $(-\pi/2,\pi/2)$ preserves the normalized Lebesgue measure λ and that (T,λ) is isomorphic to (R,μ). Hence, the above results enable us to get the ergodic properties of the transformation R on $\mathbf{R}$ from those of T on $(-\pi/2,\pi/2)$.

If the transformation $R(x)$ is piecewise monotonic, it is clear that so is T. The piecewise monotonic transformations on finite intervals have been investigated by many authors. In particular, if piecewise monotonic transformations on finite intervals are uniformly expansive, then it has been shown that they have good ergodic properties ([2], [3], [4], [5], [6], [8]). As in [7] the relation

$$(T^n)'(t)=\frac{|x-z_0|^2}{|R^n(x)-z_0|^2}(R^n)'(x)$$

holds for almost all $t\in(-\pi/2,\pi/2)$, where $t=\varphi(x)$. This gives an easy sufficient condition for T to be piecewise expanding, while N. F. G. Martin gave another sufficient condition ([11]).

Consequently, combining this relation with the known results, we can easily prove the following Theorem 2, where $N(0,\sigma^2)(y)$ $(\sigma^2>0)$ stands for the distribution function of Gaussian measure with mean 0 and variance σ^2 and $N(0,0)(y)$ stands for that of Dirac measure. We give only a sketch of the proof in Section 2, because it is the same to the proof of Theorem 4 in the article [7]. Examples that satisfy the assumptions of Theorem 2 will be found in Section 3.

THEOREM 2. *Suppose that the function $R(z)$ on $\mathbf{C}_+$ is not* 1 *to* 1 *and satisfies the assumptions in Corollary* 1 *or Corollary* 2. *Suppose also the following*:

(i) *$\mathbf{R}\setminus E$ is a union of at most countable intervals I_j.*

(ii) *The restriction of $R(x)$ to each interval I_j is monotonic and of C^2-class.*

(iii) *The inequality*

$$\inf_{x\in\mathbf{R}\setminus E}\left|\frac{|x-z_0|^2}{|R^n(x)-z_0|^2}(R^n)'(x)\right|>1 \tag{1}$$

holds for some positive integer n.

(iv) *The set $\{R(I_j);\ j\}$ consists of a finite number of intervals.*

Suppose further that $f(x)$ is a function of bounded variation on $\mathbf{R}$ and that ν is a probability measure on $\mathbf{R}$ with a density $d\nu/d\mu$ with respect to the invariant probability measure μ.

Then the central limit theorem holds for the transformation R : the limit

$$\lim_{n\to\infty} \frac{1}{n} \int \left\{ \sum_{k=0}^{n-1} (f(R^k x) - \mu(f)) \right\}^2 d\mu =: \sigma^2 \tag{2}$$

exists and

$$\lim_{n\to\infty} \nu\left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(R^k x) - \mu(f)) \le y \right\} = N(0, \sigma^2)(y) \tag{3}$$

holds for all continuity points of $N(0,\sigma^2)(y)$. If we suppose further that $\sigma^2 > 0$ and that $(1 + x^2)(d\nu/dx)$ is of bounded variation, then there exists a constant $C > 0$ such that

$$\sup_{y\in\mathbf{R}} \left| \nu\left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(R^k x) - \mu(f)) \le y \right\} - N(0, \sigma^2)(y) \right| \le \frac{C}{\sqrt{n}} . \tag{4}$$

holds for all $n \in \mathbf{N}$.

Corollaries 1 and 2 can be also applied to get invariant densities for a class of transformations on finite intervals. Let us consider a transformation $T(t) := \{f(t)\}_\pi$ on the finite interval $[-\pi/2, \pi/2)$, where $f(t)$ is a real valued function on $[-\pi/2, \pi/2)$ and $\{a\}_\pi := a - k\pi$ for $-(\pi/2) + k\pi \le a < (\pi/2) + k\pi$. Remark that $R(x) := \tan T(\arctan x)$ is a transformation of $\mathbf{R}$ and $\tan\{f(\arctan x)\}_\pi = \tan f(\arctan x)$. Then Corollaries 1 and 2 can be applied to the transformation $R(x) := \tan T(\arctan x)$, and we can easily get the following.

PROPOSITION 1. *Suppose that the real valued function $f(\arctan x)$ on $\mathbf{R}$ can be extended to a function $g(z)$ which satisfies the assumptions in Theorem 1. Suppose also that there exists $z_0 = x_0 + iy_0 \in \mathbf{C}_+$ with $\tan g(z_0) = z_0$ or $\tan g(z_0) = \overline{z_0}$. Then the transformation $T(t) := \{f(t)\}_\pi$ on $[-\pi/2, \pi/2)$ has the invariant probability density*

$$\frac{1}{\pi}\left(\frac{y_0(1 + \tan^2 t)}{(\tan t - x_0)^2 + y_0{}^2} \right).$$

As examples, transformations $\{\alpha \tan t\}_\pi$ $(\alpha > 1)$ and $\{-\alpha \cot t\}_\pi$ $(\alpha > 0)$ on $[-\pi/2, \pi/2)$ are studied in Section 3.

2. Proofs

In this section we first give the proof of Theorem 1 and secondly the sketch of the proof of Theorem 2.

2.1. Proof of Theorem 1. Though G. Letac ([9]) has already proved a version of our Theorem 1, we give another proof by using Cauchy's integral formula.

Let us define

$$F(z) := \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{1}{x-z} f(x) dx \tag{5}$$

for a continuous function $f(x)$ on $\mathbf{R}$ with a compact support. Note that $F(z)$ is known as a defining function in the theory of Sato's hyperfunction ([13]). It is clear that $F(z)$ has the following properties:

LEMMA 1. *Let $f(x)$ be a continuous function on $\mathbf{R}$ with a compact support. Then $F(z)$, defined by* (5), *has the following properties*:

(1) *$F(z)$ is holomorphic in $\mathbf{C}_+ \cup \mathbf{C}_-$.*

(2) *$\mathrm{Re}F(z)$ is bounded on $\mathbf{C}_+ \cup \mathbf{C}_-$. Precisely, we have the estimation*

$$|\mathrm{Re}F(z)| \leq \frac{1}{2} \|f\|_\infty . \tag{6}$$

(3) *$\mathrm{Re}F(z)$ converges to $(1/2)f(x)$ as $z \in \mathbf{C}_+$ tends to $x \in \mathbf{R}$: that is,*

$$\lim_{z \to x, z \in \mathbf{C}_+} \mathrm{Re}F(z) = \frac{1}{2} f(x) . \tag{7}$$

PROOF. It seems that the above results are clear and well known. Hence we repeat the sketch of the proof of the equation (7) only.

First, we prove that $\mathrm{Re}F(x+\mathrm{i}y)$ uniformly converges to $(1/2)f(x)$ as $y \to +0$. Putting $t = (s-x)/y$, we get that

$$\begin{aligned} \left| \mathrm{Re}F(x+\mathrm{i}y) - \frac{1}{2} f(x) \right| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Im}\left(\frac{1}{s-(x+\mathrm{i}y)} \right) f(s) ds - \frac{1}{2} f(x) \right| \\ &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y}{(s-x)^2+y^2} f(s) ds - \frac{1}{2} f(x) \right| \\ &= \frac{1}{2} \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1} \left(f(x+yt) - f(x) \right) dt \right| . \end{aligned}$$

Since $f(x)$ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|x_1 - x_2| < \delta(\varepsilon)$, then $|f(x_1) - f(x_2)| < \varepsilon$. Note also that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1} dt = 1 .$$

Then for sufficiently large $M(\varepsilon)$ we clearly get

$$\frac{1}{\pi} \int_{|t| \geq M(\varepsilon)} \frac{1}{t^2+1} dt < \frac{\varepsilon}{2\|f\|_\infty} . \tag{8}$$

Hence, if $0 < y < \delta(\varepsilon)/M(\varepsilon)$, then we easily have the estimation

$$\begin{aligned}\left|\mathrm{Re}F(x+\mathrm{i}y)-\frac{1}{2}f(x)\right| &= \frac{1}{2}\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1}\,(f(x+yt)-f(x))\,dt\right| \\ &\le \frac{1}{2}\left\{\left|\frac{1}{\pi}\int_{|t|\ge M(\varepsilon)}\frac{2\|f\|_\infty}{t^2+1}dt\right|+\left|\frac{1}{\pi}\int_{-M(\varepsilon)}^{M(\varepsilon)}\frac{\varepsilon}{t^2+1}dt\right|\right\} \\ &< \varepsilon \end{aligned} \tag{9}$$

for any $x \in \mathbf{R}$.

This uniform convergence of $\mathrm{Re}F(x+\mathrm{i}y)$ and the uniform continuity of $f(x)$ immediately show the equation (7). □

Remarking the above properties of a defining function $F(z)$, we can get the following key lemma.

LEMMA 2. *Suppose that $R(z)$ satisfies the conditions in Theorem 1 and that f is a continuous function with a compact support. Suppose also $R(\mathbf{C}_+) \subset \mathbf{C}_+$. Then the equation*

$$\mathrm{Re}F(R(z_0)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\frac{1}{x-z_0}f(R(x))dx\,. \tag{10}$$

holds for all $z_0 \in \mathbf{C}_+$.

PROOF. Let us define

$$\phi_{z_0}(w) := \frac{\overline{z_0}w+z_0}{w+1}\,. \tag{11}$$

Then we have $\phi_{z_0}(U) = \mathbf{C}_+$ and $\phi_{z_0}(S^1) = \mathbf{R}$, where U stands for the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$ and S^1 for the unit circle $\{z \in \mathbf{C} : |z| = 1\}$. Lemma 1 shows that $F(R(\phi_{z_0}(w)))$ is holomorphic in U. By γ_ε we denote the positively oriented circle with center 0 and radius $1-\varepsilon$ $(0<\varepsilon<1)$. Cauchy's integral formula shows that

$$\begin{aligned}F(R(\phi_{z_0}(0))) &= \frac{1}{2\pi\mathrm{i}}\int_{\gamma_\varepsilon}\frac{F(R(\phi_{z_0}(w)))}{w}dw \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}F(R(\phi_{z_0}((1-\varepsilon)\mathrm{e}^{\mathrm{i}\theta})))d\theta\,.\end{aligned}$$

Therefore we get

$$\mathrm{Re}F(R(z_0)) = \mathrm{Re}F(R(\phi_{z_0}(0))) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Re}F(R(\phi_{z_0}((1-\varepsilon)\mathrm{e}^{\mathrm{i}\theta})))d\theta\,. \tag{12}$$

Since the inequality

$$|\mathrm{Re}F(z)| \le \frac{1}{2}\|f\|_\infty$$

and the equation

$$\lim_{\varepsilon\downarrow 0} \mathrm{Re}F(R(\phi_{z_0}((1-\varepsilon)\mathrm{e}^{\mathrm{i}\theta}))) = \frac{1}{2} f(R(\phi_{z_0}(\mathrm{e}^{\mathrm{i}\theta}))) \quad \text{a.e.}$$

hold from Lemma 1, the dominated convergence theorem shows that

$$\mathrm{Re}F(R(z_0)) = \frac{1}{4\pi}\int_{-\pi}^{\pi} f(R(\phi_{z_0}(\mathrm{e}^{\mathrm{i}\theta})))d\theta\,. \tag{13}$$

Note that $\phi_{z_0}^{-1}(z) = \dfrac{-z+z_0}{z-\overline{z_0}}$ and $(\phi_{z_0}^{-1})'(z) = \dfrac{\overline{z_0}-z_0}{(z-\overline{z_0})^2}$ hold. Then, putting $\mathrm{e}^{\mathrm{i}\theta} = \phi_{z_0}^{-1}(x)$, we can rewrite the right hand of (13) as

$$\begin{aligned}
\frac{1}{4\pi}\int_{-\pi}^{\pi} f(R(\phi_{z_0}(\mathrm{e}^{\mathrm{i}\theta})))d\theta &= \frac{1}{4\pi\mathrm{i}}\int_{-\infty}^{\infty} f(R(x))\frac{(\phi_{z_0}^{-1})'(x)}{\phi_{z_0}^{-1}(x)}dx \\
&= \frac{1}{4\pi\mathrm{i}}\int_{-\infty}^{\infty} f(R(x))\left(\frac{1}{(x-z_0)} - \frac{1}{(x-\overline{z_0})}\right)dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(R(x))\mathrm{Im}\left(\frac{1}{x-z_0}\right)dx\,.
\end{aligned}$$

This completes the proof. □

This Lemma 2 immediately shows Theorem 1. In fact, from Lemma 2

$$\mathrm{Re}F(R(z_0)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(R(x))\mathrm{Im}\left(\frac{1}{x-z_0}\right)dx\,.$$

holds for a continuous function f with compact support and $z_0 \in \mathbf{C}_+$. On the other hand, the definition (5) of $F(z)$ implies

$$\mathrm{Re}F(R(z_0)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\mathrm{Im}\left(\frac{1}{x-R(z_0)}\right)dx\,.$$

Hence, if we suppose the assumptions of Theorem 1 and $R(\mathbf{C}_+) \subseteq \mathbf{C}_+$, then the result of Theorem 1 for a continuous function f with compact support

$$\int_{-\infty}^{\infty} f(R(x))\mathrm{Im}\left(\frac{1}{x-z_0}\right)dx = \int_{-\infty}^{\infty} f(x)\mathrm{Im}\left(\frac{1}{x-R(z_0)}\right)dx \tag{14}$$

clearly follows. Remark that $\mathrm{Im}\dfrac{1}{x-z_0}$ and $\mathrm{Im}\dfrac{1}{x-R(z_0)}$ are integrable on $(-\infty,\infty)$. Then we can insist that the result (14) holds for all essentially bounded function $f(x)$. This proves Theorem 1 in case $R(\mathbf{C}_+) \subseteq \mathbf{C}_+$.

If $R(\mathbf{C}_+) \subseteq \mathbf{C}_-$, put $\widetilde{R}(z) := 2x_0 - R(z)$. Then we clearly get that $\widetilde{R}(z)$ satisfies the assumptions in Theorem 1 and $\widetilde{R}(\mathbf{C}_+) \subseteq \mathbf{C}_+$. Thus we have, putting $y = 2x_0 - x$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(R(x))\mathrm{Im}\left(\frac{1}{x - z_0}\right)dx &= \int_{-\infty}^{\infty} f(2x_0 - \widetilde{R}(x))\mathrm{Im}\left(\frac{1}{x - z_0}\right)dx \\ &= \int_{-\infty}^{\infty} f(2x_0 - x)\mathrm{Im}\left(\frac{1}{x - \widetilde{R}(z_0)}\right)dx \\ &= \int_{-\infty}^{\infty} f(y)\mathrm{Im}\left(\frac{1}{2x_0 - y - \widetilde{R}(z_0)}\right)dy \\ &= \int_{-\infty}^{\infty} f(y)\mathrm{Im}\left(\frac{1}{-y + R(z_0)}\right)dy \\ &= \int_{-\infty}^{\infty} f(y)\left|\mathrm{Im}\left(\frac{1}{y - R(z_0)}\right)\right|dy\,.\end{aligned}$$

This proves the result in the case $R(\mathbf{C}_+) \subseteq \mathbf{C}_-$. Thus the proof of Theorem 1 is completed.

2.2. Proof of Theorem 2. Let $z_0 = x_0 + \mathrm{i}y_0 \in \mathbf{C}_+$ satisfy the relation $R(z_0) = z_0$ or $R(z_0) = \overline{z_0}$. Then Corollary 1 or Corollary 2 shows that $d\mu := (1/\pi)\varphi'(x)dx$ is an invariant probability for the transformation R where $\varphi(x) := \arctan\{(x - x_0)/y_0\}$. Define the transformation T on the interval $(-\pi/2, \pi/2)$ by $T(t) := \varphi(R(\varphi^{-1}(t)))$. Then we can get the following Lemma, which is a key to the proof of Theorem 2.

LEMMA 3. *Suppose that the conditions on R in Theorem 2 are satisfied. Then (R, μ) is measure theoretically isomorphic to (T, λ), where λ denotes the normalized Lebesgue measure on the interval $(-\pi/2, \pi/2)$ and is invariant under T. Furthermore, T has the following properties*:

(1) *The restrictions $T|_{\varphi(I_j)}$ of T to the intervals $\varphi(I_j)$ are monotonic and of C^2-class.*
(2) *The set $\{T(\varphi(I_j));\ j\}$ consists of a finite number of intervals.*
(3) *The equation*

$$(T^n)'(t) = \frac{|x - z_0|^2}{|R^n(x) - z_0|^2}(R^n)'(x) \tag{15}$$

holds for all $t \notin \{\varphi(E)\}$ where $x = \varphi^{-1}(t)$, and hence the inequality

$$\inf_{t\notin\varphi(E)} \left|(T^n)'(t)\right| > 1 \tag{16}$$

holds for some positive integer n.

PROOF. Recall that we have

$$\mathrm{Im}\frac{1}{x - z_0} = \frac{y_0}{(x - x_0)^2 + y_0{}^2} = \frac{d}{dx}\arctan\left(\frac{x - x_0}{y_0}\right) = \varphi'(x)\,.$$

This is followed by

$$\lambda(A) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I_A(t)dt = \frac{1}{\pi}\int_{-\infty}^{\infty} I_A(\varphi(x))\varphi'(x)dx = \mu(\varphi^{-1}(A))\,.$$

Hence, we have that $\lambda(A) = \mu(\varphi^{-1}(A))$ and (R, μ) is measure theoretically isomorphic to (T, λ). This immediately shows that T preserves the normalized Lebesgue measure λ, since R preserves μ.

Because $R|_{I_j}$ and $\varphi$are monotonic and of C^2-class, $T|_{\varphi(I_j)}$ are also monotonic and of C^2-class. Similarly, it is also clear that the set $\{T(\varphi(I_j));\, j\}$ consists of a finite number of intervals.

Recall that $\varphi(x) = \arctan\{(x - x_0)/y_0\}$ and hence $\varphi^{-1}(t) = x_0 + y_0 \tan t$. Then we easily have

$$\begin{aligned}
(T^n)'(t) &= \varphi'(R^n(\varphi^{-1}(t)))\ (R^n)'(\varphi^{-1}(t))\ (\varphi^{-1})'(t)\\
&= \frac{y_0}{(R^n(\varphi^{-1}(t)) - x_0)^2 + {y_0}^2}\ (R^n)'(\varphi^{-1}(t))y_0(1 + \tan^2 t)\\
&= \frac{(x - x_0)^2 + {y_0}^2}{(R^n(x) - x_0)^2 + {y_0}^2}\ (R^n)'(x)\\
&= \frac{|x - z_0|^2}{|R^n(x) - z_0|^2}\ (R^n)'(x)\,,
\end{aligned}$$

where $x = \varphi^{-1}(t)$. This completes the proof. □

Lemma 3 shows that the dynamical system (R, μ) on the real line $\mathbf{R}$ is isomorphic to (T, λ) on the finite interval $(-\pi/2, \pi/2)$ and that (T, λ) is piecewise smooth and piecewise monotonic. The relation (16) implies that the transformation T is piecewise expanding and smooth enough if the assumptions in Theorem 2 are satisfied.

On the other hand it is already known that such T has a finite number of absolutely continuous ergodic invariant measures $\lambda_1, \lambda_2, \ldots, \lambda_M$ and the other absolutely continuous invariant measures are convex combinations of them (cf. [3], [4], [5] and [10]). Birkhoff's ergodic theorem shows that if $\tilde{f} \in L^1(\lambda_i)$ then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \tilde{f}(T^k t) = \int \tilde{f} d\lambda_i \quad (\lambda_i \text{ a.e.})$$

holds. Note that the supports of ergodic measures are mutually disjoint and that the normalized Lebesgue measure λ is also a convex combination of $\lambda_1, \lambda_2, \ldots, \lambda_M$. Hence if $\tilde{f}$ is a λ-integrable function, then $\tilde{f}$ is λ_i-integrable for all $i = 1, 2, \ldots, M$. This observation shows that for a λ-integrable function $\tilde{f}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \tilde{f}(T^k t) = \tilde{f}^*(t) \quad (\lambda \text{ a.e.}) \tag{17}$$

holds and $\tilde{f}^*(t) = \int \tilde{f} d\lambda_i$ for λ-a.e.$t \in \operatorname{supp}\{\lambda_i\}$ $(i = 1, 2, \ldots, M)$.

In order to prove our result, we apply Theorem 1 in [6] to the transformation in question (see also [5] and [12]). Hence, if $\tilde{f}$ is a function of bounded variation defined on the interval $(-\pi/2, \pi/2)$ and if $\tilde{\nu}$ is an absolutely continuous probability measure, then there exist nonnegative constants $c_1, c_2, \ldots, c_M$ with $\sum_{i=1}^{M} c_i = 1$ and ${\sigma_i}^2 \geq 0$ $(i = 1, 2, \ldots, M)$ for which

$$\lim_{n\to\infty} \tilde{\nu}\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}(\tilde{f}(T^k t) - \tilde{f}^*(t)) \leq y\right\} = \sum_{i=1}^{M} c_i N(0, {\sigma_i}^2)(y) \tag{18}$$

holds for all continuity points of the right hand side. If we assume further that ${\sigma_i}^2 > 0$ for all $i = 1, 2, \ldots, M$, and that $d\tilde{\nu}/d\lambda$ is of bounded variation, then

$$\sup_{y\in\mathbf{R}}\left|\tilde{\nu}\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}(\tilde{f}(T^k t) - \tilde{f}^*(t)) \leq y\right\} - \sum_{i=1}^{M} c_i N(0, {\sigma_i}^2)(y)\right| \leq \frac{C}{\sqrt{n}} \tag{19}$$

for some $C > 0$.

J. Aaronson ([1]) proved that (R, μ) is exact if $R(z)$ is not 1 to 1. Therefore the number M of absolutely continuous ergodic measures for T is equal to 1. It follows that λ is the unique invariant probability and $\tilde{f}^*(t) = \lambda(\tilde{f})$. Let $f(x)$ be a function of bounded variation on $\mathbf{R}$. Then $\tilde{f}(t) := (f \circ \varphi^{-1})(t)$ is also a function of bounded variation, because $\varphi^{-1}(t)$ is strictly increasing. Suppose that ν is a probability measure on $\mathbf{R}$ which is absolutely continuous with respect to μ. Then it is clear that the probability measure $\tilde{\nu}(A) := \nu(\varphi^{-1}A)$ is absolutely continuous with respect to λ.
Note that we have

$$\begin{aligned}
&\tilde{\nu}\left\{t \in (-\pi/2, \pi/2)\,;\ \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\big(\tilde{f}(T^k t) - \lambda(\tilde{f})\big) \leq y\right\} \\
&\quad = (\nu \circ \varphi^{-1})\left\{t \in (-\pi/2, \pi/2)\,;\ \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\big(\tilde{f}(T^k t) - \lambda(\tilde{f})\big) \leq y\right\} \\
&\quad = \nu\left\{x \in \mathbf{R}\,;\ \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\big((f \circ \varphi^{-1})\big(T^k \varphi(x)\big) - \lambda(f \circ \varphi^{-1})\big) \leq y\right\} \\
&\quad = \nu\left\{x \in \mathbf{R}\,;\ \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\big(f(R^k x) - \mu(f)\big) \leq y\right\}.
\end{aligned} \tag{20}$$

Therefore we get the relation (3), combining (18), (20) and the fact that $M = 1$.

On the other hand we have

$$\begin{aligned}\tilde{\nu}(A) &= \int_{\varphi^{-1}A} \frac{d\nu}{dx}(x)dx \\ &= \int_A \frac{d\nu}{dx}(\varphi^{-1}(t))(\varphi^{-1})'(t)dt \\ &= \int_A \frac{d\nu}{dx}(\varphi^{-1}(t))y_0(1+\tan^2 t)dt \\ &= \int_A \frac{d\nu}{dx}(\varphi^{-1}(t))y_0\left(1+\left(\frac{\varphi^{-1}(t)-x_0}{y_0}\right)^2\right)dt\,.\end{aligned}$$

This shows

$$\frac{d\tilde{\nu}}{d\lambda} = \frac{d\nu}{dx}(\varphi^{-1}(t))y_0\left(1+\left(\frac{\varphi^{-1}(t)-x_0}{y_0}\right)^2\right).$$

Therefore, because φ^{-1} is strictly increasing, the total variation of $d\tilde{\nu}/d\lambda$ on the interval $(-\pi/2, \pi/2)$ is equal to the one of

$$y_0\left(1+\left(\frac{x-x_0}{y_0}\right)^2\right)\frac{d\nu}{dx}(x)$$

on the real line $\mathbf{R}$. Thus, if $(x^2+1)(d\nu/dx)$ is of bounded variation, so is $d\tilde{\nu}/d\lambda$. This and (19) show the inequality (4) of Theorem 2.

3. Examples

In this section we consider examples which satisfy the assumptions in Theorems 1 and 2.

First let us note the following remark.

REMARK 1. Let us define $\widetilde{R}(z) := -R(z) + 2x_0$. Then we have the following:
(1) $\widetilde{R}(\mathbf{C}_+) \subseteq \mathbf{C}_-$ holds, if and only if $R(\mathbf{C}_+) \subseteq \mathbf{C}_+$.
(2) $\widetilde{R}(x_0 + \mathrm{i}y_0) = x_0 - \mathrm{i}y_0$, if and only if $R(x_0 + \mathrm{i}y_0) = x_0 + \mathrm{i}y_0$.

Therefore, it is enough to consider only the examples which satisfy $R(\mathbf{C}_+) \subseteq \mathbf{C}_+$.

3.1. Rational transformations $R(x) = \alpha x + \beta - \sum_{k=1}^n b_k/(x-a_k)$. Let us consider the rational transformations of the form

$$R(z) = \alpha z + \beta - \sum_{k=1}^{n} \frac{b_k}{z-a_k}\,, \tag{21}$$

where $0 \leq \alpha < 1, b_k > 0\ (k = 1, \ldots, n),\ a_1 < a_2 < \cdots < a_n$.

In the article [7], these transformations were proved, by using the different method, to have the same result. A sufficient condition for the existence of a fixed point $z_0 \in \mathbf{C} \setminus \mathbf{R}$ is given in Proposition 3.1 of [7].

PROPOSITION 2. *Let us define*

$$R(x) = \alpha x + \beta - \sum_{k=1}^{n} \frac{b_k}{x - a_k}$$

for $0 \le \alpha < 1$, $b_k > 0\,(k = 1, \ldots, n)$, $a_1 < a_2 < \cdots < a_n$. *Assume further that* $a_1 \le (\beta/(1-\alpha))$, $a_n \ge (\beta/(1-\alpha))$ *and*

$$a_{i+1} - a_i < \sqrt{\frac{\{{b_i}^{1/3} + {b_{i+1}}^{1/3}\}^3}{1-\alpha}} \tag{22}$$

for $i = 1, 2, \ldots, n-1$. *Then there exists* $z_0 \in \mathbf{C}_+ \setminus \mathbf{R}$ *with* $R(z_0) = z_0$.

Therefore, the transformations of the form (21) have the result of Corollary 1, if the assumptions of Proposition 2 are satisfied. The ergodic properties of these transformations are discussed in [7].

Let us consider the transformation $R(x) = \alpha x - x^{-1}$ with $0 < \alpha < 1$. We can easily get that there exists a unique fixed point $z_0 = \mathrm{i}y_0 = \mathrm{i}\sqrt{1/(1-\alpha)}$ of R in $\mathbf{C}_+$ in this case. Corollary 1 shows that $d\mu = \pi^{-1}\mathrm{Im}\,(1/(x - \mathrm{i}y_0))\,dx$ is an invariant probability for the transformation R.

Let us consider the transformation $T(t) := \varphi(R(\varphi^{-1}(t)))$, where $\varphi(x) := \arctan(x/y_0)$. Using Lemma 3 we have

$$\begin{aligned} T'(t) &= \frac{|x - z_0|^2}{|R(x) - z_0|^2} R'(x) \\ &= \frac{\alpha x^2 + 1}{\alpha^2 x^2 + 1 - \alpha}\,. \end{aligned} \tag{23}$$

Hence, the transformation T on $(-\pi/2, \pi/2)$ is uniformly expansive. Precisely, we have

$$T'(t) \ge \min\left(\frac{1}{\alpha}, \frac{1}{1-\alpha}\right) > 1$$

for all $t \neq 0$ ([7]). It is also clear that $R(z)$ is not 1 to 1. Therefore, $R(x) = \alpha x - x^{-1}$ $(0 < \alpha < 1)$ satisfies the assumptions of Theorem 2, so that we have the ergodic theorem and the ordinary central limit theorem for (R, μ).

If the number n of poles is more than 2, it is generally not easy to get the desired estimation of $|T'(t)|$. However, there are some examples that satisfy the assumption of Theorem 2.

Let us consider the transformation

$$R(x) = \alpha x - \frac{1}{x-1} - \frac{1}{x+1}$$

with $0 \leq \alpha < 1$. We can easily get that $R(\mathrm{i}y_0) = \mathrm{i}y_0$, where $y_0 = \sqrt{(1+\alpha)/(1-\alpha)}$. We can obtain

$$T'(t) = \frac{\alpha(x^2-1)^2 + 2x^2 + 2}{\alpha^2(x^2-1)^2 + (1-\alpha)^2x^2 + (1-\alpha)(1+\alpha)} \,. \tag{24}$$

Using the equation (24), for $0 < \alpha < 1$ we also get the inequality

$$T'(t) \geq \min\left(\frac{1}{\alpha}, \frac{1}{1-\alpha}\right) > 1$$

for all $t \notin \{-\pi/2, \varphi(-1), \varphi(1), \pi/2\}$ ([7]). If $\alpha = 0$, then it is clear that the right hand side of (24) is equal to 2 and

$$T(t) = \begin{cases} 2t + \pi & (-\pi/2 < t < -\pi/4)\,, \\ 2t & (-\pi/4 < t < \pi/4)\,, \\ 2t - \pi & (\pi/4 < t < \pi/2)\,. \end{cases}$$

Consequently, Theorem 2 can be also applied to these transformations.

3.2. Transformations $R(x) = \alpha x - \sum_{k=1}^{\infty}\{b_k/(x-a_k) + b_k/(x+a_k)\}$. Define

$$R(z) = \alpha z - \sum_{k=1}^{\infty}\left\{\frac{b_k}{z-a_k} + \frac{b_k}{z+a_k}\right\} \tag{25}$$

for $0 \leq \alpha < 1$, $0 < a_k$, and $0 < b_k (k = 1, 2, \ldots)$. Then it is clear that $R(\mathbf{C}_+) \subseteq \mathbf{C}_+$ and the assumptions of Theorem 1 are satisfied. In order to check the existence of the fixed point we have the following sufficient condition.

PROPOSITION 3. *Assume that*

$$\sum_{k=1}^{\infty} b_k < \infty \quad \textit{and} \quad 1 - \alpha < \sum_{n=1}^{\infty} \frac{2b_n}{a_n^2} < \infty\,. \tag{26}$$

Then there exists $y_0 > 0$ such that $R(y_0\mathrm{i}) = y_0\mathrm{i}$ holds.

PROOF. From the definition (25) of R we have the equation

$$\begin{aligned} R(y\mathrm{i}) &= \alpha y\mathrm{i} - \sum_{n=1}^{\infty}\left\{\frac{b_n}{y\mathrm{i}-a_n} + \frac{b_n}{y\mathrm{i}+a_n}\right\} \\ &= \left(\alpha + \sum_{n=1}^{\infty}\left\{\frac{2b_n}{y^2+a_n^2}\right\}\right)y\mathrm{i}\,. \end{aligned}$$

On the other hand the assumption (26) shows that there exists $y_0 > 0$ which satisfies the relation

$$\alpha + \sum_{n=1}^{\infty} \frac{2b_n}{y_0^2 + a_n^2} = 1 .$$

This completes the proof. □

Thus we have the result of Theorem 1 for transformations in question.

3.3. Transformations $R(x) = \alpha x + \beta \tan x$. Let us consider the transformation

$$R(z) = \alpha z + \beta \tan z \tag{27}$$

for $0 \leq \alpha$ and $0 < \beta$. Then $R(z)$ is holomorphic in $\mathbf{C}_+ \cup \mathbf{C}_-$. Remark that we have

$$\mathrm{Re}\,(\tan(x + \mathrm{i}y)) = \frac{2\sin 2x}{(e^{2y} + e^{-2y}) + 2\cos 2x} \tag{28}$$

and

$$\mathrm{Im}\,(\tan(x + \mathrm{i}y)) = \frac{e^{2y} - e^{-2y}}{(e^{2y} + e^{-2y}) + 2\cos 2x} . \tag{29}$$

Hence, we immediately get the relations $R(\mathbf{C}_+) \subset \mathbf{C}_+$ and $R(\mathbf{C}_-) \subset \mathbf{C}_-$ from the equation (29).

In order to apply Corollary 1 we have the following proposition which gives a sufficient condition for the existence of the fixed point $z_0 \in \mathbf{C}_+$.

PROPOSITION 4. *If $0 \leq \alpha < 1 < \alpha + \beta$, then there exists a unique positive number y_0 which satisfies $R(\mathrm{i}y_0) = \mathrm{i}y_0$.*

PROOF. The well known equation $\tan \mathrm{i}z = \mathrm{i} \tanh z$ shows that

$$R(\mathrm{i}y) = \mathrm{i}\{\alpha y + \beta \tanh y\} = \mathrm{i}\left\{\alpha y + \beta \frac{e^y - e^{-y}}{e^y + e^{-y}}\right\} .$$

Clearly $\tanh y$ has the following properties;

$$\tanh 0 = 0 , \quad \lim_{y\to\infty} \tanh y = 1 , \quad \tanh' 0 = 1 , \quad \tanh' y > 0 , \quad \tanh'' y < 0 \ \ (y > 0) .$$

These properties and the assumption $0 \leq \alpha < 1 < \alpha + \beta$ enable us to have a unique positive number y_0 which satisfies the equation $\alpha y_0 + \beta \tanh(y_0) = y_0$. This proves our assertion. □

From this proposition and Corollary 1, it follows that a transformation $R(x) = \alpha x + \beta \tan x$ has an invariant probability density $y_0/\pi(x^2 + y_0{}^2)$ if $0 \leq \alpha < 1 < \alpha + \beta$.

In order to check the condition (1) we have the following estimation.

PROPOSITION 5. *Suppose that* $0 \le \alpha < 1 < \alpha + \beta$. *Then for all* $n \ge 2$ *the estimation*

$$\text{ess. inf} \left| \frac{|x - z_0|^2}{|R^n(x) - z_0|^2} (R^n)'(x) \right| > \min \left\{ 1, \frac{\beta y_0{}^2}{(\alpha + \beta)} \right\} (\alpha + \beta)^{n-2} \tag{30}$$

holds, where $z_0 = iy_0$ *stands for the fixed point of* R.

PROOF. First, remark that the chain rule ensures us to have the inequality

$$\begin{aligned} &\frac{|x - z_0|^2}{|R^n(x) - z_0|^2} (R^n)'(x) \\ &\quad = \left(\frac{x^2 + y_0{}^2}{\left(\alpha x + \beta \tan R^{n-1}(x)\right)^2 + y_0{}^2} \right) \prod_{k=1}^{n} \left\{ R'(R^{k-1}(x)) \right\} \\ &\quad = \left(\frac{x^2 + y_0{}^2}{\left(\alpha x + \beta \tan R^{n-1}(x)\right)^2 + y_0{}^2} \right) \prod_{k=1}^{n} \left\{ \alpha + \beta + \beta \tan^2(R^{k-1}(x)) \right\} \\ &\quad \ge \left(\frac{\left(x^2 + y_0{}^2\right) \left(\alpha + \beta + \beta \tan^2(R^{n-1}(x))\right)}{\left(\alpha x + \beta \tan R^{n-1}(x)\right)^2 + y_0{}^2} \right) (\alpha + \beta)^{n-1} \end{aligned}$$

for all $x \in \mathbf{R} \setminus E$. If $|x| \le \left|\tan R^{n-1}(x)\right|$, then we clearly have the estimation

$$\begin{aligned} \frac{\left(x^2 + y_0{}^2\right) \left(\alpha + \beta + \beta \tan^2(R^{n-1}(x))\right)}{\left(\alpha x + \beta \tan R^{n-1}(x)\right)^2 + y_0{}^2} &\ge \frac{y_0{}^2 \left(\alpha + \beta + \beta \tan^2(R^{n-1}(x))\right)}{(\alpha + \beta)^2 \tan^2 R^{n-1}(x) + y_0{}^2} \\ &\ge \min \left\{ \frac{\beta y_0{}^2}{(\alpha + \beta)^2}, (\alpha + \beta) \right\}. \end{aligned}$$

If $|x| \ge \left|\tan R^{n-1}(x)\right|$, then it is also clear that the inequality

$$\frac{\left(x^2 + y_0{}^2\right) \left(\alpha + \beta + \beta \tan^2(R^{n-1}(x))\right)}{\left(\alpha x + \beta \tan R^{n-1}(x)\right)^2 + y_0{}^2} \ge \frac{\left(x^2 + y_0{}^2\right) (\alpha + \beta)}{(\alpha + \beta)^2 x^2 + y_0{}^2} \ge \frac{1}{\alpha + \beta}$$

holds. Thus we get the inequality (30). □

From Propositions 4 and 5 it follows that Theorem 2 can be applied to the transformation $R(x) = \alpha x + \beta \tan x$ if $0 \le \alpha < 1 < \alpha + \beta$. J. Aaronson remarked that transformations (R, μ) in question are exact if it is not 1 to 1 ([1]). Therefore we have the following.

THEOREM 3. *Let* $R(x) = \alpha x + \beta \tan x$ *and* $0 \le \alpha < 1 < \alpha + \beta$. *Suppose further that* $f(x)$ *is a function of bounded variation on* $\mathbf{R}$ *and that* ν *is a probability measure on* $\mathbf{R}$ *with a density* $d\nu/d\mu$. *Then the limit*

$$\lim_{n\to\infty} \frac{1}{n} \int \left\{ \sum_{k=0}^{n-1} (f(R^k x) - \mu(f)) \right\}^2 d\mu =: \sigma^2$$

exists and

$$\lim_{n\to\infty} \nu\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}(f(R^k x) - \mu(f)) \le y\right\} = N(0,\sigma^2)(y)$$

holds for all continuity points of $N(0,\sigma^2)(y)$.

If we suppose further that $\sigma^2 > 0$ *and that* $(1+x^2)(d\nu/dx)$ *is of bounded variation, then there exists a constant* $C > 0$ *such that*

$$\sup_{y\in\mathbf{R}}\left|\nu\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}(f(R^k x) - \mu(f)) \le y\right\} - N(0,\sigma^2)(y)\right| \le \frac{C}{\sqrt{n}}$$

holds for all $n \in \mathbf{N}$.

3.4. Transformations on a finite interval. We discuss some examples which satisfy the assumptions in Proposition 1.

Let us consider $f(t) := \alpha \tan t$ on $[-\pi/2, \pi/2)$. Then $g(x) = f(\arctan x)$ on $\mathbf{R}$ can be extended to a function $g(z) = \alpha z$ on $\mathbf{C}$, which satisfies the assumptions of Theorem 1. As in the above subsection, it is clear that for $\alpha > 1$ there exists $z_0 = \mathrm{i}y_0 \in \mathbf{C}_+$ such that $\tan(\alpha \mathrm{i} y_0) = \mathrm{i}y_0$. Hence, the transformation $T(t) := \{\alpha \tan t\}_\pi$ on $[-\pi/2, \pi/2)$ has the invariant probability density

$$\frac{1}{\pi}\left(\frac{y_0(1+\tan^2 t)}{\tan^2 t + {y_0}^2}\right).$$

Consider a function $f(t) := -\alpha \cot t$ on $[-\pi/2, \pi/2)$. Then $g(x) = f(\arctan x)$ on $\mathbf{R}$ can be extended to a function $g(z) = (-\alpha/z)$ on $\mathbf{C}$, which also satisfies the assumptions of Theorem 1. Using the equation $\tan(-\alpha/\mathrm{i}y_0) = \mathrm{i}\tanh(\alpha/y_0)$, we can prove that for $\alpha > 0$ there exists $z_0 = \mathrm{i}y_0 \in \mathbf{C}_+$ such that $\tan(-\alpha/\mathrm{i}y_0) = \mathrm{i}y_0$. Hence, the transformation $T(t) := \{-\alpha \cot t\}_\pi$ on $[-\pi/2, \pi/2)$ has the invariant probability density

$$\frac{1}{\pi}\left(\frac{y_0(1+\tan^2 t)}{\tan^2 t + {y_0}^2}\right).$$

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