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Curvature Pinching for Complete Kaehler Submanifolds of a Complex Projective Space

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Abstract. A classification of complete Kaehler submanifolds M_n in $P_{n+p}(C)$ with scalar curvature $\rho > n^2$ is given, resolving a conjecture of K. Ogiue.

1. Introduction

Let $P_{n+p}(C)$ be an $(n+p)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. There are a number of conjectures for Kaehler submanifolds in $P_{n+p}(C)$ suggested by K. Ogiue ([9]); some have been resolved under a suitable topological restriction (e.g. M_n is complete) (cf. [1], [6], [8], [9], [11], [12], [13], [14], [15], [17] and [18]). In this direction, one of the open problems is as follows:

CONJECTURE (K. Ogiue). Let M_n be an n-dimensional complete submanifold immersed in $P_{n+p}(C)$. If $\rho > n^2$, is M totally geodesic in $P_{n+p}(C)$?

In the case where M_n is a complete Kaehler submanifold immersed in $P_{n+p}(C)$ which has the Ricci curvature $S > \frac{n}{2}$, it was proved in [9] that such a submanifold M_n is totally geodesic in $P_{n+p}(C)$. Recently, in the case of M_n having $S \geq \frac{n}{2}$, Suh and Yang ([14]) proved that it is parallel, i.e., either totally geodesic or congruent to one of Q_n or $P_1(C) \times P_1(C)$. Also, the case where the scalar curvature $\rho > n(n+1) - \frac{n+2}{3}$ was studied by Tanno [16], and he proved that M is totally geodesic in $P_{n+p}(C)$. In [6] we prove that if M_n is compact, then $\rho \geq n^2$ if and only if M is either totally geodesic in $P_{n+p}(C)$ or $\rho = n^2$. In the latter case M^n is an imbedded submanifold congruent to the standard imbedding of one of the following submanifolds: $P_1(C) \times P_1(C)$ and the complex quadric Q_n, $n \geq 3$. Hence, we obtain that M_n is an n-dimensional compact Kaehler submanifold immersed in $P_{n+p}(C)$. If $\rho > n^2$, then M is totally geodesic in $P_{n+p}(C)$, so that the above conjecture is resolved partially.

In the present paper we would like to consider the case where M_n is complete and $\rho > n^2$, so that the above conjecture is resolved. The main result is the following:

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THEOREM. *Let M_n be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $\rho > n^2$ then M is totally geodesic in $P_{n+p}(C)$.*

2. Preliminaries

Let M_n be a Kaehler submanifold of complex dimension n, immersed in the complex projective space $P_{n+p}(C)$ endowed with the Fubini-Study metric of constant holomorphic sectional curvature 1. We denote by UM the unit tangent bundle over M and by UM_x its fibre over $x \in M$ and by J and $\langle\ ,\ \rangle$ the complex structure and the Fubini-Study metric. Let ∇ and h be the Riemannian connection and the second fundamental form of the immersion, respectively. A and $\nabla^{\perp}$ are the Weingarten endomorphism and the normal connection. The first and the second covariant derivatives of the normal valued tensor h are given by

$$(\nabla h)(X, Y, Z) = \nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

and

$$\begin{aligned}(\nabla^2 h)(X, Y, Z, W) = \nabla_X^{\perp}((\nabla h)(Y, Z, W)) &- (\nabla h)(\nabla_X Y, Z, W)\\ -(\nabla h)(Y, \nabla_X Z, W) &- (\nabla h)(Y, Z, \nabla_X W)\,,\end{aligned}$$

respectively, for any vector fields X, Y, Z and W tangent to M.

Let R and $R^{\perp}$ denote the curvature tensor associated with ∇ and $\nabla^{\perp}$, respectively. Then h and ∇h are symmetric and for $\nabla^2 h$ we have the Ricci-identity

$$\begin{aligned}&(\nabla^2 h)(X, Y, Z, W) - (\nabla^2 h)(Y, X, Z, W)\\ &= R^{\perp}(X, Y)h(Z, W) - h(R(X, Y)Z, W) - h(Z, R(X, Y)W)\,.\end{aligned}$$

We also consider the relations

$$h(JX, Y) = Jh(X, Y) \text{ and } A_{J\xi} = JA_{\xi} = -A_{\xi}J\,,$$

where ξ is a normal vector to M.

If S and ρ are the Ricci tensor of M and the scalar curvature of M, respectively, and M is a Kaehler submanifold in $P_{n+p}(C)$, then from the Gauss equation we have

$$S(v, w) = \frac{n+1}{2}\langle v, w\rangle - \sum_{i=1}^{2n}\langle A_{h(v,e_i)}e_i, w\rangle\,, \tag{1}$$

$$\rho = n(n+1) - |h|^2\,. \tag{2}$$

Now, let $v \in UM_x, x \in M$. If $e_2, \dots, e_{2n}$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2, \dots, e_{2n}\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1, e_2, \dots, e_{2n}\}$ is an orthonormal basis of $T_x M$. We denote the Laplacian of $UM_x \cong S^{2n-1}$ by Δ.

Define a function f_1 on UM_x, $x \in M$, by

$$f_1(v) = \sum_{i,j=1}^{2n} \langle A_{h(e_i,e_j)}e_j, A_{h(v,v)}e_i \rangle .$$

Noting that $\nabla_{e_k} v = -e_k$, $\nabla_{e_k} e_\ell = \delta_{k\ell} v$, $k, \ell = 2, \dots, 2n$, we have

$$\begin{aligned}
(\Delta f_1)(v) &= \sum_{k=2}^{2n} (\nabla f_1)(v, e_k, e_k) \\
&= -2 \sum_{k=2}^{2n} \nabla_{e_k} \Big(\sum_{i,j=1}^{2n} \langle A_{h(e_i,e_j)}e_j, A_{h(e_k,v)}e_i \rangle \Big) \\
&= -2 \sum_{k=2}^{2n} f_1(v) + 2 \sum_{k=2}^{2n} f_1(e_k).
\end{aligned}$$

Using the minimality of M, we can obtain that

$$\begin{aligned}
(\Delta f_1)(v) &= -2(2n-1) f_1(v) + 2 \sum_{k=2}^{2n} \langle A_{h(e_i,e_j)}e_j, A_{h(e_k,e_k)}e_i \rangle \\
&= -4n f_1(v) .
\end{aligned} \tag{3}$$

For more details on this, see [7], [11]. Similarly, define f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10} and f_{11} by

$$\begin{aligned}
f_2(v) &= \sum \langle A_{h(v,v)}v, A_{h(v,e_i)}e_i \rangle , \\
f_3(v) &= \sum \langle A_{h(e_i,e_j)}e_j, A_{h(v,e_i)}v \rangle , \\
f_4(v) &= \sum \langle A_{h(v,e_i)}e_i, A_{h(v,e_j)}e_j \rangle , \\
f_5(v) &= \sum \langle A_{h(v,v)}e_i, A_{h(v,v)}e_i \rangle , \\
f_6(v) &= \sum \langle A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i \rangle , \\
f_7(v) &= |h(v,v)|^2 , \\
f_8(v) &= \sum \langle A_{h(v,e_i)}e_i, v \rangle |h(v,v)|^2 , \\
f_9(v) &= \Big(\sum \langle A_{h(v,e_i)}e_i, v \rangle \Big)^2 , \\
f_{10}(v) &= \sum \langle A_{h(v,e_i)}e_i, v \rangle , \\
f_{11}(v) &= |h|^2 |h(v,v)|^2 ,
\end{aligned}$$

respectively. Then we obtain

$$(\Delta f_2)(v) = -4(2n+2)f_2(v) + 4f_3(v) + 4f_4(v) + 2f_1(v)\,, \tag{4}$$

$$(\Delta f_3)(v) = -4nf_3(v) + 2\sum\langle A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k\rangle\,, \tag{5}$$

$$(\Delta f_4)(v) = -4nf_4(v) + 2\sum\langle A_{h(e_j,e_i)}e_j, A_{h(e_k,e_i)}e_k\rangle\,, \tag{6}$$

$$(\Delta f_5)(v) = -4(2n+2)f_5(v) + 8\sum\langle A_{h(e_j,v)}e_i, A_{h(e_j,v)}e_i\rangle\,, \tag{7}$$

$$(\Delta f_6)(v) = -4nf_6(v) + 2\sum\langle A_{h(e_j,e_k)}e_i, A_{h(e_j,e_k)}e_i\rangle\,, \tag{8}$$

$$(\Delta f_7)(v) = -4(2n+2)f_7(v) + 8\sum\langle A_{h(v,e_i)}e_i, v\rangle\,, \tag{9}$$

$$(\Delta f_8)(v) = -6(2n+4)f_8(v) + 16f_2(v) + 2f_{11}(v) + 8f_9(v)\,, \tag{10}$$

$$(\Delta f_9)(v) = -4(2n+2)f_9(v) + 8f_4(v) + 4|h|^2\sum\langle A_{h(v,e_i)}e_i, v\rangle\,, \tag{11}$$

$$(\Delta f_{10})(v) = -4nf_{10}(v) + 2|h|^2\,, \tag{12}$$

$$(\Delta f_{11})(v) = -4(2n+2)f_{11}(v) + 8|h|^2\sum\langle A_{h(v,e_i)}e_i, v\rangle\,. \tag{13}$$

Since

$$\begin{aligned}\frac{1}{2}\sum(\nabla^2 f_7)(e_i, e_i, v) &= \sum\langle(\nabla^2 h)(e_i, e_i, v, v), h(v, v)\rangle \\ &= \sum\langle(\nabla h)(e_i, v, v), (\nabla h)(e_i, v, v)\rangle\,,\end{aligned}$$

we have the following (see [2], [3], [4], [5] and [7]):

LEMMA. *Let M be an n-dimensional complex Kaehler submanifold of $P_{n+p}(C)$. Then for $v \in UM_x$ we have*

$$\begin{aligned}\frac{1}{2}\sum(\nabla^2 f_7)(e_i, e_i, v) = \sum&|(\nabla h)(e_i, v, v)|^2 + \frac{n+2}{2}|h(v, v)|^2 \\ &+2\sum\langle A_{h(v,v)}e_i, A_{h(e_i,v)}v\rangle \\ &-2\sum\langle A_{h(v,e_i)}e_i, A_{h(v,v)}v\rangle \\ &-\sum\langle A_{h(v,v)}e_i, A_{h(v,v)}e_i\rangle\,.\end{aligned} \tag{14}$$

The following generalized maximum principle due to Omori [10] and Yau [19] will be used in order to prove our theorem.

GENERALIZED MAXIMUM PRINCIPLE ([10] and [19]). *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in C^2(M)$ a function bounded from above on M^n. Then, for any $\varepsilon > 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \sup f - \varepsilon\,, \quad \|grad\ f\| < \varepsilon\,, \quad \Delta f(p) < \varepsilon\,.$$

3. Proof of Theorem

From (2) we have

$$\rho = n(n+1) - |h|^2 .$$

Thus we have only to prove the Theorem under the assumption

$$|h|^2 < n. \tag{15}$$

We see that the following equation holds for $v \in UM_x$, $x \in M$,

$$\sum \langle A_{h(Jv,Jv)}e_i, A_{h(e_i,Jv)}Jv\rangle = -\sum \langle A_{h(v,v)}e_i, A_{h(e_i,v)}v\rangle . \tag{16}$$

From (14) and (16) we have

$$\begin{aligned}
&\frac{1}{4}\sum (\nabla^2 f_7)(e_i, e_i, v) + \frac{1}{4}\sum (\nabla^2 f_7)(e_i, e_i, Jv) \\
&\quad = \sum |(\nabla h)(e_i, v, v)|^2 + \frac{n+2}{2}|h(v,v)|^2 \\
&\qquad -2\sum \langle A_{h(v,e_i)}e_i, A_{h(v,v)}v\rangle - \sum \langle A_{h(v,v)}e_i, A_{h(v,v)}e_i\rangle .
\end{aligned} \tag{17}$$

Now, we choose an orthonormal basis $\{v = e_1, e_2, \dots, e_n\}$ such that the matrix $\sum_{\alpha=1}^{2p} A_{\xi_\alpha}^2$ is diagonalized, where $\{\xi_1, \xi_2, \dots, \xi_{2p}\}$ is any orthonormal normal basis and $1 \le \alpha \le 2p$. Then we have

$$f_2(v) = f_8(v) . \tag{18}$$

Now, we set

$$\begin{aligned}
g(v) = \frac{1}{2}\Bigg(&\frac{1}{6n(2n+2)}\Bigg(2f_2(v) + \frac{2}{n}f_3(v) - \frac{2}{n}f_4(v) + \frac{1}{n}f_1(v) \\
&-(2n+2)f_8(v) - 2f_9(v) + f_{11}(v)\Bigg) \\
&+\frac{n}{2}\left(\frac{1}{4(2n+2)}f_7(v) + \frac{2}{4n(2n+2)}f_{10}(v)\right) \\
&-\frac{1}{4(2n+2)}f_5(v) - \frac{2}{4n(2n+2)}f_6(v)\Bigg) .
\end{aligned}$$

In terms of (4), (5), (6), (7), (8), (9), (10), (11), (12) , (13) ,(17) and (18) we have

$$\begin{aligned}
&\frac{1}{4}\sum (\nabla^2 f_7)(e_i, e_i, v) + \frac{1}{4}\sum (\nabla^2 f_7)(e_i, e_i, Jv) \\
&\quad + (\Delta g)(v) + (\Delta g)(Jv)
\end{aligned} \tag{19}$$

$$\begin{aligned}
&= \sum |(\nabla h)(e_i, v, v)|^2 + \frac{n+2}{2}|h(v,v)|^2 \\
&\quad - \frac{1}{n} f_{11}(v) - f_5(v) \\
&\quad - \frac{n}{2}|h(v,v)|^2 + \frac{n}{2n(2n+2)}|h|^2 \\
&\quad + f_5(v) - \frac{2}{2n(2n+2)} \sum \langle A_{h(e_j,e_k)} e_i, A_{h(e_j,e_k)} e_i \rangle \\
&= \sum |(\nabla h)(e_i, v, v)|^2 + |h(v,v)|^2 - \frac{1}{n} f_{11}(v) \\
&\quad + \frac{n}{2n(2n+2)}|h|^2 - \frac{2}{2n(2n+2)} \sum_{\alpha,\beta=1}^{2p} (\mathrm{trace} A_{\xi_\alpha} A_{\xi_\beta})^2 \\
&\geq \sum |(\nabla h)(e_i, v, v)|^2 + |h(v,v)|^2 - \frac{1}{n} f_{11}(v) \\
&\quad + \frac{n}{2n(2n+2)}|h|^2 - \frac{1}{2n(2n+2)}|h|^4 ,
\end{aligned}$$

where we used $\sum(\mathrm{trace} A_{\xi_\alpha} A_{\xi_\beta})^2 \leq \frac{1}{2}|h|^4$ (see [9], p. 88) for any orthonormal basis $\{\xi_1, \xi_2, \dots, \xi_{2p}\}$, $1 \leq \alpha, \beta \leq 2p$, as above.

On the other hand, from the assumption of (15) we see that the Ricci curvature is bounded from below. Note that

$$\begin{aligned}
2g(v) = &\frac{1}{6n(2n+2)} \bigg(2f_2(v) + \frac{2}{n} f_3(v) - \frac{2}{n} f_4(v) + \frac{1}{n} f_1(v) \\
&-(2n+2) f_8(v) - 2f_9(v) + f_{11}(v) \bigg) \\
&+\frac{n}{2}\left(\frac{1}{4(2n+2)} f_7(v) + \frac{2}{4n(2n+2)} f_{10}(v) \right) \\
&-\frac{1}{4(2n+2)} f_5(v) - \frac{2}{4n(2n+2)} f_6(v) \\
\leq &\frac{1}{6n(2n+2)} \left(2f_8(v) + \frac{2}{n} f_3(v) + \frac{1}{n} f_1(v) + f_{11}(v) \right) \\
&+\frac{n}{2}\left(\frac{1}{4(2n+2)} f_7(v) + \frac{2}{4n(2n+2)} f_{10}(v) \right) .
\end{aligned}$$

Combining the scalar curvature $\rho\rangle n^2$ with the above equation, we know that there exist real numbers c_1, c_2 such that

$$R_{ii} > c_1 , \quad g(v) \leq c_2 ,$$

where R_{ii} is the Ricci curvature of M_n. Since we can also regard $\sum(\nabla^2 f_7)(e_i, e_i, v)$ as the Laplacian of $f_7(v)$, we obtain that M_n is totally geodesic. This proves the Theorem (see [8], pp. 662–663).

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