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Small-time Existence of a Strong Solution of Primitive Equations for the Ocean

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Abstract. Primitive equations derived originally by Richardson in 1920's have been considered as the model equations describing the motion of atmosphere, ocean and coupled atmosphere and ocean. In this paper, we discuss the free boundary problem of the primitive equations for the ocean in three-dimensional strip with surface tension. Using the so-called p-coordinates and a coordinate transformation similar to that in [2] in order to fix the time-dependent domain, we prove temporally local existence of the unique strong solution to the transformed problem in Sobolev-Slobodetskiĭ spaces.

1. Introduction

The idea of weather forecast was conceived by Bjerknes in 1904, and then numerical weather forecast was executed by Richardson in 1920's ([20]). He derived a system of equations describing the motion of atmosphere, which was similar to the Navier-Stokes equations. His attempt unfortunately failed because of mainly the lack of stability of the calculations, however many attempts have carried on him. Since around 1940–50's, digital computers made possible automatic calculations, so that the weather forecast with numerical calculation became practical; the first success was done by Charney, Fjörtoft and von Neumann (see [19] in detail). Until the present time many simplified models such as geostrophic and quasi-geostrophic models have been proposed in order to lessen the amount of numerical calculations. Nowadays, even the primitive equations can be solved numerically since the power of the computers intensively increases.

In 1969 Bryan [3] formulated the model of the ocean circulation similar to the Richardson's model of the atmosphere by applying the hydrostatic approximation. In it eddy or turbulent viscosity terms were introduced, which was anisotropic in the horizontal and the vertical directions. Now his model equations are called primitive equations for the ocean [30]. Based upon his formulation Semtner [21] proposed the general circulation model and studied it in detail numerically. In his model Boussinesq approximation and rigid lid hypothesis, which means that the ocean surface is fixed and flat, were used. On the other hand, Crowly [5] studied the free surface case, not the rigid lid, of the ocean numerically.

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Mathematical arguments of primitive equations were begun in 1990's. One of the main feature of primitive equations is the fact that the vertical velocity is determined by the horizontal velocities via the continuity equation, since the vertical velocity disappears in the vertical component of equations of motion due to the hydrostatic approximation. In [11], Lions, Temam and Wang formulated the evolution problem of primitive equations for the ocean and showed the existence of a weak solution in $L_2(0, T; H^1(\Omega)) \bigcap L_\infty(0, T; L_2(\Omega))$ by the Galerkin method, where $\Omega = \bigcup_s M_s$, each M_s is a connected domain with both horizontally and vertically flat boundaries. In [12], [13], [15]–[17], they also studied the evolutionary 3D coupled atmosphere and ocean model with rigid lid. In [17] they showed the well-posedness of the model formulated in [15] in the same function spaces as above. In [14] they derived quasi-geostrophic equations from primitive equations and showed the existence and uniqueness of a global weak solution in the similar function spaces as above.

Guillen-Gonzalez, Masmoudi and Rodriguez-Bellido in [7], [8] discussed the initial boundary value problem for the primitive equations for the ocean in the domain surrounded by the rigid lid, the vertically flat lateral boundary and the bottom. They showed the existence of a global strong solution with small data and a local strong solution with any data in $L^\infty(0, T; H^1(\Omega)) \bigcap L_2(0, T; H^2(\Omega)) \bigcap W_2^1(0, T; L^2(\Omega))$. In the similar situation as that in [8], Temam and Ziane [29] verified the existence and uniqueness of a strong local in time solution of primitive equations for the ocean in $C(0, T; H^1(\Omega)) \bigcap L_2(0, T; H^2(\Omega))$. Cao and Titi [4] showed the existence and uniqueness of a global solution in $C(0, T; H^1(\Omega)) \bigcap L_2(0, T; H^2(\Omega)) \bigcap W_1^1(0, T; L_2(\Omega))$.

While the depth of the ocean is finite in these papers, Azerad and Guillen-Gonzalez [1] showed the existence of the weak solution of the Navier-Stokes equations with anisotropic viscosity terms and its convergence to a weak solution of primitive equations as the aspect ratio of depth to width of the domain tends to zero.

Almost all the results cited above were obtained under the rigid lid hypothesis, and the turbulent viscosity terms were added as an empirical claim. In addition to the turbulent viscosity we take into account the effect of the surface movement following Crowly [5], and adopt f-plane approximation, *i.e.*, Coriolis parameter is a constant, where f is the deformation angle from the sphere over the plane. Furthermore, it is to be noted that in the papers cited above, ocean and atmosphere models are described in Cartesian and p-coordinates, respectively. Lions *et al.* [12], [13], [15]–[17] used these coordinates for the coupled ocean and atmosphere model in each layer, and made a physically unrealistic assumption that the height of the pressure isobar coincides with that of the ocean surface.

We use p-coordinates for the ocean model in this paper, which is the first study on the primitive equations for the ocean. Here, we enumerate the major features of this paper:

1. The surface of the ocean is free, not the rigid lid;
2. The boundary conditions on the free surface are described by the stress tensor and the effect of vapour;
3. The original equations are transformed by the p-coordinates system;

4. The existence of the strong solution is proved in the Sobolev-Slobodetskiĭ spaces.

The paper is organized as follows. In Section 2 we describe the mathematical formulation of our problem. Then the problem is rewritten in p-coordinates, and transformed into the one in the fixed time independent domain. In Section 3 we introduce function spaces, and describe the main theorem. In Section 4 some auxiliary lemmas are prepared, which are used in the proof of the main theorem. In Section 5 we solve the non-homogeneous linear problem in the transformed domain. In Section 6 we investigate the nonlinear problem, making use of the iteration method for a small time interval.

2. Formulation of the problem

As stated in Section 1, we are concerned with the free boundary problem of primitive equations for the ocean. By adopting f-plane approximation, our problem can be formulated in the strip-like region. By $x = (x_1, x_2, x_3)$, we denote orthogonal Cartesian coordinate system with x_3 being the vertical direction. Let the surface and the bottom of the ocean be described by $x_3 = d(x', t)$ and $x_3 = b(x')$ $(x' = (x_1, x_2))$, respectively, where $b(x')$ is a function satisfying $d(x', 0) = d_0(x') > b(x')$. Then the domain $\Omega(t)$ of the ocean at time t is represented as $\{(x', x_3) | x' \in \mathbf{R}^2, b(x') < x_3 < d(x', t)\}$. The equations that we consider in this paper are as follows:

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + w \dfrac{\partial \mathbf{v}}{\partial x_3} - \left[\mu_1 \Delta \mathbf{v} + \mu_2 \dfrac{\partial^2 \mathbf{v}}{\partial x_3^2} \right] + f \mathbf{A} \mathbf{v} = -\dfrac{1}{\varrho} \nabla p + \mathbf{F}_1' \,, \\ \dfrac{\partial p}{\partial x_3} = (F_{13} - \varrho g) =: \tilde{F}_{13} \,, \\ \nabla \cdot \mathbf{v} + \dfrac{\partial w}{\partial x_3} = 0 \,, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla)\theta + w \dfrac{\partial \theta}{\partial x_3} - \left[\mu_3 \Delta \theta + \mu_4 \dfrac{\partial^2 \theta}{\partial x_3^2} \right] = F_2 \,, \\ \dfrac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla)S + w \dfrac{\partial S}{\partial x_3} - \left[\mu_5 \Delta S + \mu_6 \dfrac{\partial^2 S}{\partial x_3^2} \right] = F_3 \,, \quad x \in \Omega(t) \,, \quad t > 0 \,. \end{cases} \tag{2.1}$$

Here, $f \mathbf{A} \mathbf{v}$ is a Coriolis force with $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Coriolis parameter f (a positive constant); ∇ and Δ are 2 dimensional gradient and Laplacian, respectively; $\mathbf{F}_1'$ and $\tilde{F}_{13}$ are the horizontal and vertical components of external forces given in $\mathbf{R}^3 \times [0, \infty)$. The horizontal component of the velocity is represented by $\mathbf{v}$ and the vertical component w; p is the pressure, ϱ is the density (a positive constant), θ is the temperature, S is the salinity; F_2 and F_3 are the sources of heat and salinity, respectively; μ_1 and μ_2 are the coefficients of turbulent viscosity;

(μ_3, μ_4) and (μ_5, μ_6) are, respectively, given by scaling sum of turbulent and molecular diffusivities of heat and salinity.

The conditions on the free surface $\Gamma_s(t) = \{x \in \mathbf{R}^3 | x_3 = d(x', t), t > 0\}$ are as follows:

$$\begin{cases} \mathbf{T}(\mathbf{v})\mathbf{n} - (\mathbf{T}(\mathbf{v})\mathbf{n} \cdot \mathbf{n}')\mathbf{n}' = |\mathbf{v}|^\alpha \mathbf{v} \,, \\ -\left(\mu_3 \nabla\theta \cdot \mathbf{n}' + \mu_4 \dfrac{\partial \theta}{\partial x_3} n_3\right) = -la(\theta_e)\mathcal{V} + g_1 |\mathbf{v}|^\alpha \theta + \sigma L H \,, \\ (\theta, S, p) = (\theta_e, S_e, p_0) \,, \end{cases} \tag{2.2}$$

where

$$\mathbf{T}(\mathbf{v}) = \begin{pmatrix} \mu_1 \dfrac{\partial v_1}{\partial x_1} & \mu_1 \dfrac{\partial v_1}{\partial x_2} & \mu_2 \dfrac{\partial v_1}{\partial x_3} \\ \mu_1 \dfrac{\partial v_2}{\partial x_1} & \mu_1 \dfrac{\partial v_2}{\partial x_2} & \mu_2 \dfrac{\partial v_2}{\partial x_3} \end{pmatrix} \tag{2.3}$$

is a part of the stress tensor, $\mathcal{V}$ is the normal velocity of the free surface, $la(\theta_e)$ is a latent heat with saturation temperature θ_e, σ is the surface tension coefficient, H is twice mean curvature, $\mathbf{n} = (n_1, n_2, n_3) = (\mathbf{n}', n_3)$ is the unit normal vector to $\Gamma_s(t)$ at time t pointing to the atmospheric region, L is the heat capacity, g_1 is a given function representing the turbulent transition on the free surface including the albedo of the earth, and p_0 is atmospheric pressure at the ocean surface (constant). The conditions of the form (2.2) are called bulk formulae ([6], [11], [18], [30]).

Since $\mathcal{V} = \frac{\partial d}{\partial t} / \sqrt{1 + |\nabla d|^2}$, the condition $(2.2)_2$ can be written in the explicit form

$$\frac{\partial d}{\partial t} = L_{4,d} d + G_{6,d}(\mathbf{v}, w, \theta) \,, \quad x' \in \mathbf{R}^2 \,, \quad t > 0 \,, \tag{2.4}$$

where

$$L_{4,d} d := \frac{\sigma L}{la(\theta_e)(1 + |\nabla d|^2)} \left\{ \left(1 + \left(\frac{\partial d}{\partial x_2}\right)^2\right) \frac{\partial^2 d}{\partial x_1^2} - 2 \frac{\partial d}{\partial x_1} \frac{\partial d}{\partial x_2} \frac{\partial^2 d}{\partial x_1 \partial x_2} + \left(1 + \left(\frac{\partial d}{\partial x_1}\right)^2\right) \frac{\partial^2 d}{\partial x_2^2} \right\} ,$$

$$G_{6,d}(\mathbf{v}, w, \theta) := \frac{1}{la(\theta_e)} \left[\left(-\mu_3 \nabla\theta \cdot \nabla d + \mu_4 \frac{\partial \theta}{\partial x_3} \right) + g_1 \sqrt{1 + |\nabla d|^2} |\mathbf{v}|^\alpha \theta \right] .$$

The conditions on the bottom $\Gamma_b = \{(x', b(x')) | x' \in \mathbf{R}^2\}$ are

$$(\mathbf{v}, w, \theta, S)(x', b(x'), t) = (\mathbf{0}, 0, \theta_b, S_b)(x', t) \,, \quad x' \in \mathbf{R}^2 \,, \quad t > 0 \,. \tag{2.5}$$

Initial conditions are

$$(\mathbf{v}, \theta, S)(x, 0) = (\mathbf{v}_0, \theta_0, S_0)(x), \quad x \in \Omega := \Omega(0), \tag{2.6}$$

$$d(x', 0) = d_0(x'), \quad x' \in \mathbf{R}^2. \tag{2.7}$$

Let us introduce the "p-coordinate system". From $(2.1)_2$ and $(2.2)_3$, p can be represented as

$$p = p_0 + \int_d^{x_3} \frac{\partial p}{\partial x_3}\, \mathrm{d}x_3 = p_0 + \int_d^{x_3} \tilde{F}_{13}\, \mathrm{d}x_3. \tag{2.8}$$

We assume that $|F_{13}| < \varrho g$ in $\mathbf{R}^3 \times [0, \infty)$, which means the gravity force is dominant in the vertical directon. Now we denote the pressure at the bottom of the ocean by $h(x', t) := p_0 + \int_d^b \tilde{F}_{13}\, \mathrm{d}x_3$, and $h_0 := p_0 + \int_{d_0}^b \tilde{F}_{13}|_{t=0}\, \mathrm{d}x_3$. We assume $d_0(x') > b(x')$ for any $x' \in \mathbf{R}^2$. Since $\partial p/\partial x_3 = \tilde{F}_{13} < 0$, we can take p as an independent variable in place of $x_3 = X_3(x', p, t; h)$ with X_3 being the inverse function of (2.8), and d can be represented as an implicit function, $d = \Psi(x', t; h)$. From (2.8), we get

$$\nabla p = -\tilde{F}_{13}(x', d, t)\nabla d + \int_d^{x_3} \nabla \tilde{F}_{13}\, \mathrm{d}x_3 =: \mathbf{F}_5(x', x_3, t), \tag{2.9}$$

$$\frac{\partial p}{\partial t} = -\tilde{F}_{13}(x', d, t)\frac{\partial d}{\partial t} + \int_d^{x_3} \frac{\partial \tilde{F}_{13}}{\partial t}\, \mathrm{d}x_3 =: F_6(x', x_3, t). \tag{2.10}$$

Note that after introducing p-coordinates, the ocean surface becomes flat and is represented by the equation $p = p_0$. Hereafter, we denote a function $f(x', x_3, t)$ after this coordinate transform by $f^{(h)}(x', p, t) = f(x', X_3(x', p, t; h), t)$. Moreover, we introduce another mapping:

$$y' = x', \quad y_3 = (p_0 - h_0)\frac{p - h(x', t)}{p_0 - h(x', t)} + h_0. \tag{2.11}$$

For simplicity, instead of $\tilde{F}_{13} w$ we use $\bar{w}$. By composing these transformations, it is clear that the regions $\bigcup_{0 \le t \le T}(\Omega(t) \times \{t\})$, $\bigcup_{0 \le t \le T}(\Gamma_b \times \{t\})$, $\bigcup_{0 \le t \le T}(\Gamma_s(t) \times \{t\})$ are transformed onto the regions $\tilde{\Omega}_T := \tilde{\Omega} \times [0, T]$, $\tilde{\Gamma}_{bT} := \tilde{\Gamma}_b \times [0, T]$, $\tilde{\Gamma}_{sT} := \tilde{\Gamma}_s \times [0, T]$, respectively, where $\tilde{\Omega} = \{(y', y_3) | y' \in \mathbf{R}^2,\ p_0 < y_3 < h_0(y')\}$, $\tilde{\Gamma}_b = \{(y', y_3) | y' \in \mathbf{R}^2,\ y_3 = h_0(y')\}$ and $\tilde{\Gamma}_s = \{(y', y_3) | y' \in \mathbf{R}^2,\ y_3 = p_0\}$. We denote the inverse of transposed matrix of the Jacobian matrix by $J[(x', p)/(y', y_3)]^{-\mathrm{T}} = (a^{ij}) = (a^{ij}(h))$ $(i, j = 1, 2, 3)$. Then one can easily derive

$$\begin{aligned}\mathbf{a}^3 := \left(a^{31},\ a^{32}\right) &= \frac{(p_0 - h_0)(p(y, t) - p_0)}{(p_0 - h)^2}\nabla h + \frac{p_0 - p(y, t)}{p_0 - h}\nabla h_0 \\ &=: A_1(y, t)\nabla h + \mathbf{B}_1(y, t),\end{aligned}$$

$$p(y,t) = \frac{(p_0 - h)(y_3 - h_0)}{p_0 - h_0} + h\,,$$

$$a^{33} = \frac{p_0 - h_0}{p_0 - h}\,, \quad a^{ij} = \delta_{ij} \quad (i = 1, 2,\ j = 1, 2, 3)\,.$$

In the following, we use the notation

$$\begin{pmatrix} \nabla_h \\ \nabla_{h,3} \end{pmatrix} = J\left[\frac{(x', p)}{(y', y_3)}\right]^{-\mathrm{T}} \begin{pmatrix} \nabla \\ \dfrac{\partial}{\partial p} \end{pmatrix};$$

$$X_3(x', p, t; h)\big|_{x'=y',\ p=p(y,t)} = \tilde{X}_3(y', y_3, t; h)\,;$$
$$f^{(h)}(x', p, t)\big|_{x'=y',\ p=p(y,t)} = f^{(h)*}(y', y_3, t)\,.$$

Now let us derive the explicit representation of $\mathbf{F}_5^{(h)*}$ and $F_6^{(h)*}$. Representing the integral term in (2.8) by p-coordinate system, we have

$$\mathbf{F}_5^{(h)}(x', p, t) = -\tilde{F}_{13}\big|_{x_3=\Psi}\nabla\Psi + \int_{p_0}^{p} \frac{1}{\tilde{F}_{13}^{(h)}}\left(\nabla\tilde{F}_{13}^{(h)} + \mathbf{F}_5^{(h)}\frac{\partial\tilde{F}_{13}^{(h)}}{\partial p}\right)\mathrm{d}p\,.$$

From this integral equation with the condition $\mathbf{F}_5^{(h)}|_{p=p_0} = -\tilde{F}_{13}|_{x_3=\Psi}\nabla\Psi$, we derive the explicit representation

$$\mathbf{F}_5^{(h)}(x', p, t) = \tilde{F}_{13}^{(h)}\left(-\nabla\Psi + \int_{p_0}^{p}\frac{\nabla\tilde{F}_{13}^{(h)}}{\tilde{F}_{13}^{(h)2}}\,\mathrm{d}p\right),$$

and hence

$$\begin{aligned}\mathbf{F}_5^{(h)*}(y', y_3, t) &= \tilde{F}_{13}^{(h)*}\left\{-\nabla\Psi + \int_{p_0}^{y_3}\frac{1}{\tilde{F}_{13}^{(h)*2}}\left(\nabla\tilde{F}_{13}^{(h)*} + \mathbf{a}^3(h)\frac{\partial\tilde{F}_{13}^{(h)*}}{\partial y_3}\right)\mathrm{d}y_3\right\}\\ &=: -\tilde{F}_{13}^{(h)*}\nabla\Psi + \mathbf{C}_1(y,t)\,.\end{aligned} \tag{2.12}$$

Similarly, we obtain

$$F_6^{(h)}(x', p, t) = \tilde{F}_{13}^{(h)}\left(-\frac{\partial\Psi}{\partial t} + \int_{p_0}^{p}\frac{1}{\tilde{F}_{13}^{(h)2}}\frac{\partial\tilde{F}_{13}^{(h)}}{\partial t}\,\mathrm{d}p\right),$$

$$\begin{aligned}F_6^{(h)*}(y', y_3, t) &= \tilde{F}_{13}^{(h)*}\left\{-\frac{\partial\Psi}{\partial t} + \int_{p_0}^{y_3}\frac{1}{\tilde{F}_{13}^{(h)*2}}\left(\frac{\partial\tilde{F}_{13}^{(h)*}}{\partial t} + \frac{\partial y_3}{\partial t}\frac{\partial\tilde{F}_{13}^{(h)*}}{\partial y_3}\right)\mathrm{d}y_3\right\}\\ &=: -\tilde{F}_{13}^{(h)*}\frac{\partial\Psi}{\partial t} + \tilde{C}_1(y,t),\end{aligned} \tag{2.13}$$

where $\dfrac{\partial y_3}{\partial t} = A_1(y,t)\dfrac{\partial h}{\partial t}$.

Differentiating the relation $h(x',t) = p_0 + \int_d^b \tilde{F}_{13}\, \mathrm{d}x_3$ with respect to x' and t leads us to the following equalities, respectively:

$$\nabla\Psi = \frac{1}{\tilde{F}_{13}|_{x_3=\Psi}}\left\{-\nabla h + \tilde{F}_{13}|_{x_3=b}\nabla b + \int_d^b \nabla\tilde{F}_{13}\, \mathrm{d}x_3\right\} =: -\frac{\nabla h}{\tilde{F}_{13}|_{x_3=\Psi}} + \mathbf{D}_1\,, \tag{2.14}$$

where $\mathbf{D}_1 = (D_1,\ D_2)$,

$$\begin{aligned}\frac{\partial^2\Psi}{\partial y_i\partial y_j} &= -\frac{1}{\tilde{F}_{13}|_{x_3=\Psi}}\frac{\partial^2 h}{\partial y_i\partial y_j} - \frac{\partial}{\partial y_j}\left(\frac{1}{\tilde{F}_{13}|_{x_3=\Psi}}\right)\frac{\partial h}{\partial y_i} + \frac{\partial D_i}{\partial y_j}\\ &=: -\frac{1}{\tilde{F}_{13}|_{x_3=\Psi}}\frac{\partial^2 h}{\partial y_i\partial y_j} + H_{ij} \quad (i,\,j = 1,2),\end{aligned} \tag{2.15}$$

$$\frac{\partial\Psi}{\partial t} = \frac{1}{\tilde{F}_{13}|_{x_3=\Psi}}\left\{-\frac{\partial h}{\partial t} + \int_d^b \frac{\partial\tilde{F}_{13}}{\partial t}\, \mathrm{d}x_3\right\} =: -\frac{1}{\tilde{F}_{13}|_{x_3=\Psi}}\frac{\partial h}{\partial t} + E_1. \tag{2.16}$$

Now, rewriting the problem (2.1)–(2.7) in y-coordinates and denoting $(\mathbf{v}^*, \bar{w}^*, \theta^*, S^*, h)$ by $(\mathbf{u}, u_3, \tilde{\theta}, \tilde{S}, h)$ for brevity, we have

$$\begin{cases}\dfrac{\partial\mathbf{u}}{\partial t} = L_{1,h}\mathbf{u} + \tilde{\mathbf{G}}_{1,h}(\mathbf{u}, u_3)\,,\\[2mm] \nabla_{h,3}u_3 - (\nabla_{h,3}\tilde{F}_{13}^{(h)*})\dfrac{u_3}{\tilde{F}_{13}^{(h)*}} = \tilde{G}_{3,h}(\mathbf{u})\,,\\[2mm] \dfrac{\partial\tilde{\theta}}{\partial t} = L_{2,h}\tilde{\theta} + \tilde{G}_{4,h}(\mathbf{u}, u_3, \tilde{\theta})\,,\\[2mm] \dfrac{\partial\tilde{S}}{\partial t} = L_{3,h}\tilde{S} + \tilde{G}_{5,h}(\mathbf{u}, u_3, \tilde{S}) \quad \text{in } \tilde{\Omega}_T\,,\\[2mm] \dfrac{\partial h}{\partial t} = L_{4,h}h + \tilde{G}_{6,h}(\mathbf{u}, u_3, \tilde{\theta}) \quad \text{in } \mathbf{R}^2_T\,,\end{cases} \tag{2.17}$$

$$\begin{cases}B_h\mathbf{u} = \tilde{\mathbf{G}}_2(\mathbf{u})\,,\\[2mm] (\tilde{\theta}, \tilde{S}) = (\theta_e, S_e)|_{x_3=\Psi} \quad \text{on } \tilde{\Gamma}_{sT}\,,\\[2mm] (\mathbf{u}, u_3, \tilde{\theta}, \tilde{S}) = (\mathbf{0}, 0, \theta_b, S_b)|_{x_3=b} \quad \text{on } \tilde{\Gamma}_{bT}\,,\\[2mm] (\mathbf{u}, \tilde{\theta}, \tilde{S})(y,0) = (\mathbf{v}_0^{(h_0)*}, \theta_0^{(h_0)*}, S_0^{(h_0)*})(y) \quad \text{on } \tilde{\Omega}\,,\\[2mm] h(y',0) = h_0(y') \quad \text{on } \mathbf{R}^2\,,\end{cases} \tag{2.18}$$

where

$$L_{1,h}\mathbf{u} := \mu_1 L_{11,h}\mathbf{u} + \mu_2 L_{12,h}\mathbf{u}\,,$$

$$L_{11,h}\mathbf{u} := \left[l_{11,h} + 2l_{12,h} + |\mathbf{F}_5^{(h)*}|^2 (a^{33})^2 \frac{\partial^2}{\partial y_3^2} \right]\mathbf{u}\,,$$

$$L_{12,h}\mathbf{u} := \tilde{F}_{13}^{(h)*2} (a^{33})^2 \frac{\partial^2 \mathbf{u}}{\partial y_3^2}\,,$$

$$l_{11,h} := \nabla^2 + 2\mathbf{a}^3 \cdot \nabla \frac{\partial}{\partial y_3} + |\mathbf{a}^3|^2 \frac{\partial^2}{\partial y_3^2}\,, \qquad l_{12,h} := a^{33}\mathbf{F}_5^{(h)*} \cdot \nabla_h \frac{\partial}{\partial y_3}\,,$$

$$\begin{aligned}
\tilde{\mathbf{G}}_{1,h}(\mathbf{u}, u_3) := \mu_1 \Bigg[& (\nabla_h^2 - l_{11,h}) + \nabla_h \cdot (a^{33}\mathbf{F}_5^{(h)*}) \frac{\partial}{\partial y_3} \\
& + a^{33}\mathbf{F}_5^{(h)*} \cdot \left(\frac{\partial \mathbf{a}^3}{\partial y_3} + \frac{\partial}{\partial y_3}(a^{33}\mathbf{F}_5^{(h)*}) \right) \frac{\partial}{\partial y_3} \Bigg]\mathbf{u} \\
& + \mu_2 \left[\left(a^{33}\tilde{F}_{13}^{(h)*} \frac{\partial}{\partial y_3} \right)^2 - (a^{33}\tilde{F}_{13}^{(h)*})^2 \frac{\partial^2}{\partial y_3^2} \right]\mathbf{u} \\
& - \left[(\mathbf{u}\cdot\nabla_h) + \left((\mathbf{F}_5^{(h)*}\cdot\mathbf{u})a^{33} + u_3 a^{33} + F_6^{(h)*}a^{33} + A_1(y,t)\frac{\partial h}{\partial t} \right) \frac{\partial}{\partial y_3} \right]\mathbf{u} \\
& + f\mathbf{A}\mathbf{u} - \frac{1}{\varrho}\mathbf{F}_5^{(h)*} + \mathbf{F}_1^{\prime(h)*} \\
=: {} & \mu_1 \tilde{G}_{11,h}\mathbf{u} + \mu_2 \tilde{G}_{12,h}\mathbf{u} - \tilde{G}_{13,h}(\mathbf{u}, u_3)\mathbf{u} + f\mathbf{A}\mathbf{u} - \frac{1}{\varrho}\mathbf{F}_5^{(h)*} + \mathbf{F}_1^{\prime(h)*}\,,
\end{aligned}$$

$$\tilde{G}_{3,h}(\mathbf{u}) := -\nabla_h \cdot \mathbf{u} - a^{33}\mathbf{F}_5^{(h)*} \cdot \frac{\partial \mathbf{u}}{\partial y_3}\,,$$

$$L_{2,h}\tilde{\theta} := \mu_3 L_{11,h}\tilde{\theta} + \mu_4 L_{12,h}\tilde{\theta}\,,$$

$$\tilde{G}_{4,h}(\mathbf{u}, u_3, \tilde{\theta}) =: \mu_3 \tilde{G}_{11,h}\tilde{\theta} + \mu_4 \tilde{G}_{12,h}\tilde{\theta} - \tilde{G}_{13,h}(\mathbf{u}, u_3)\tilde{\theta} + F_2^{(h)*}\,,$$

$$L_{3,h}\tilde{S} := \mu_5 L_{11,h}\tilde{S} + \mu_6 L_{12,h}\tilde{S}\,,$$

$$\tilde{G}_{5,h}(\mathbf{u}, u_3, \tilde{S}) =: \mu_5 \tilde{G}_{11,h}\tilde{S} + \mu_6 \tilde{G}_{12,h}\tilde{S} - \tilde{G}_{13,h}(\mathbf{u}, u_3)\tilde{S} + F_3^{(h)*}\,,$$

$$B_h\mathbf{u} := \left\{\mu_1\left[(\mathbf{n}'\cdot\nabla_h)\mathbf{u} + (\mathbf{F}_5^{(h)*}\cdot\mathbf{n}')a^{33}\frac{\partial\mathbf{u}}{\partial y_3}\right] + \mu_2\tilde{F}_{13}^{(h)*}a^{33}\frac{\partial\mathbf{u}}{\partial y_3}n_3\right\}$$

$$-\left\{\mu_1\left[(\mathbf{n}'\cdot\nabla_h)\mathbf{u}\cdot\mathbf{n}' + (\mathbf{F}_5^{(h)*}\cdot\mathbf{n}')a^{33}\frac{\partial\mathbf{u}}{\partial y_3}\cdot\mathbf{n}'\right] + \mu_2\tilde{F}_{13}^{(h)*}\left(a^{33}\frac{\partial\mathbf{u}}{\partial y_3}\cdot\mathbf{n}'\right)n_3\right\}\mathbf{n}',$$

$$\tilde{\mathbf{G}}_2(\mathbf{u}) := |\mathbf{u}|^\alpha\mathbf{u},$$

$$L_{4,h}h := \frac{\sigma L}{la(\theta_e)\left(1+|\nabla\Psi|^2\right)}\sum_{i,j=1}^{2}c_{ij}\frac{\partial^2 h}{\partial y_i\partial y_j},$$

$$c_{11} := 1 + \left(D_2 - \frac{1}{\tilde{F}_{13}|_{x_3=b}}\frac{\partial h}{\partial y_2}\right)^2,\quad c_{22} := 1 + \left(D_1 - \frac{1}{\tilde{F}_{13}|_{x_3=b}}\frac{\partial h}{\partial y_1}\right)^2,$$

$$c_{12} = c_{21} := -\left(\frac{1}{\tilde{F}_{13}|_{x_3=b}}\frac{\partial h}{\partial y_1} - D_1\right)\left(\frac{1}{\tilde{F}_{13}|_{x_3=b}}\frac{\partial h}{\partial y_2} - D_2\right),$$

$$\tilde{G}_{6,h}(\mathbf{u},\tilde{\theta}) := \tilde{F}_{13}|_{x_3=\Psi}\Bigg[E_1 - \frac{\sigma L}{la(\theta_e)\left(1+|\nabla\Psi|^2\right)}\sum_{i,j=1}^{2}c_{ij}H_{ij}$$

$$-\frac{1}{la(\theta_e)}\Bigg\{-\mu_3\left(\nabla_h + a^{33}(h)\mathbf{F}_5^{(h)*}\frac{\partial}{\partial y_3}\right)\tilde{\theta}\cdot\nabla\Psi$$

$$+\mu_4\tilde{F}_{13}^{(h)*}\big|_{x_3=\Psi}a^{33}(h)\frac{\partial\tilde{\theta}}{\partial y_3} + g_1^{(h)*}(1+|\nabla\Psi|^2)^{\frac{1}{2}}|\mathbf{u}|^\alpha\tilde{\theta}\Bigg\}\Bigg],$$

$$\mathbf{n} = (\mathbf{n}', n_3)^{\mathrm{T}} = \frac{\mathbf{a}}{|\mathbf{a}|},\quad \mathbf{a} = \left(a^{31}(h)F_{51}^{(h)*},\ a^{32}(h)F_{52}^{(h)*},\ a^{33}(h)\tilde{F}_{13}^{(h)*}\right)^{\mathrm{T}}.$$

It is to be noted that we can extend $(\mathbf{v}_0^{(h_0)*}, \theta_0^{(h_0)*}, S_0^{(h_0)*})(y) = (\mathbf{v}_0, \theta_0, S_0)(x)$ and d_0 into the half space $t > 0$ preserving the regularity, which is denoted by $(\bar{\mathbf{u}}_0, \bar{\tilde{\theta}}_0, \bar{\tilde{S}}_0, \bar{d}_0)$. We also define the extension of h_0 by $\bar{h}_0 = p_0 + \int_{\bar{d}_0}^{b}\tilde{F}_{13}\,\mathrm{d}x_3$. For the detail, see Section 4.

Then the problem (2.17), (2.18) can be rewritten as the following one for

$(\mathbf{u}', u_3', \tilde{\theta}', \tilde{S}', h') := (\mathbf{u} - \bar{\mathbf{u}}_0, u_3, \tilde{\theta} - \bar{\tilde{\theta}}_0, \tilde{S} - \bar{\tilde{S}}_0, h - \bar{h}_0)$:

$$\begin{cases} \dfrac{\partial \mathbf{u}'}{\partial t} = L_{1,h}\mathbf{u}' + L_{1,h}\bar{\mathbf{u}}_0 - \dfrac{\partial \bar{\mathbf{u}}_0}{\partial t} + \tilde{\mathbf{G}}_{1,h}(\mathbf{u}, u_3)\,, \\ \nabla_{h,3}u_3' - (\nabla_{h,3}\tilde{F}_{13}^{(h)*})\dfrac{u_3'}{\tilde{F}_{13}^{(h)*}} = \tilde{G}_{3,h}(\mathbf{u})\,, \\ \dfrac{\partial \tilde{\theta}'}{\partial t} = L_{2,h}\tilde{\theta}' + L_{2,h}\bar{\tilde{\theta}}_0 - \dfrac{\partial \bar{\tilde{\theta}}_0}{\partial t} + \tilde{G}_{4,h}(\mathbf{u}, u_3, \tilde{\theta})\,, \\ \dfrac{\partial \tilde{S}'}{\partial t} = L_{3,h}\tilde{S}' + L_{3,h}\bar{\tilde{S}}_0 - \dfrac{\partial \bar{\tilde{S}}_0}{\partial t} + \tilde{G}_{5,h}(\mathbf{u}, u_3, \tilde{S}) \quad \text{in } \tilde{\Omega}_T\,, \\ \dfrac{\partial h'}{\partial t} = L_{4,h}h' + L_{4,h}h_0 - \dfrac{\partial \bar{h}_0}{\partial t} + \tilde{G}_{6,h}(\mathbf{u}, \tilde{\theta}) \quad \text{in } \mathbf{R}^2_T\,, \end{cases} \tag{2.19}$$

$$\begin{cases} B_h\mathbf{u}' = -B_h\bar{\mathbf{u}}_0 + \tilde{\mathbf{G}}_2(\mathbf{u}), \\ (\tilde{\theta}', \tilde{S}') = (\theta_e|_{x_3=\Psi} - \underline{a}r\tilde{\theta}_0, S_e|_{x_3=\Psi} - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{sT}\,, \\ (\mathbf{u}', u_3', \tilde{\theta}', \tilde{S}') = (-\bar{\mathbf{u}}_0, 0, \theta_b|_{x_3=b} - \bar{\tilde{\theta}}_0, S_b|_{x_3=b} - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{bT}\,, \\ (\mathbf{u}', \tilde{\theta}', \tilde{S}')|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega}\,, \\ h'|_{t=0} = 0 \quad \text{on } \mathbf{R}^2\,, \end{cases} \tag{2.20}$$

where $(\mathbf{u}, \tilde{\theta}, \tilde{S}, h)$ is replaced by $(\mathbf{u}' + \bar{\mathbf{u}}_0, \tilde{\theta}' + \bar{\tilde{\theta}}_0, \tilde{S}' + \bar{\tilde{S}}_0, h' + \bar{h}_0)$ in the right-hand sides.

3. Main Theorem

Before stating the main theorem, we introduce function spaces (see, for instance, [23], [24]).

Let G be a domain in $\mathbf{R}^n$ and l a non-negative number. By $W_2^l(G)$ we mean a space of functions $u(x)$, $x \in G$, equipped with the norm $\|u\|^2_{W_2^l(G)} = \sum_{|\alpha|<l} \|D^\alpha u\|^2_{L_2(G)} + \|u\|^2_{\dot{W}_2^l(G)}$, where

$$\begin{cases} \|u\|^2_{\dot{W}_2^l(G)} = \displaystyle\sum_{|\alpha|=l} \|D^\alpha u\|^2_{L_2(G)} = \sum_{|\alpha|=l} \int_G |D^\alpha u(x)|^2 \mathrm{d}x \qquad \text{if } l \text{ is an integer}, \\ \|u\|^2_{\dot{W}_2^l(G)} = \displaystyle\sum_{|\alpha|=[l]} \int_G \int_G \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}} \mathrm{d}x\mathrm{d}y \\ \qquad\qquad\qquad \text{if } l \text{ is a non-integer}, \ l = [l] + \{l\}, \ 0 < \{l\} < 1. \end{cases}$$

We also define the following function spaces with $m > 1$:

$$\overline{W}_2^m(G) = \left\{ f(x) \middle| \|f\|_{\overline{W}_2^m(G_T)} := \sup_{x\in G} |f|^2 + \|f\|^2_{\dot{W}_2^{m-[m]}(G)} + \|Df\|^2_{W_2^{m-1}(G)} < \infty \right\}.$$

Next we introduce anisotropic Sobolev-Slobodetskiĭ spaces $W_2^{l,\frac{l}{2}}(G_T) := W_2^{l,0}(G_T) \cap W_2^{0,\frac{l}{2}}(G_T)$ $(G_T := G \times [0,T])$, whose norms are defined by

$$\begin{aligned} \|u\|^2_{W_2^{l,\frac{l}{2}}(G_T)} &= \int_0^T \|u(\cdot,t)\|^2_{W_2^l(G)} \mathrm{d}t + \int_G \|u(x,\cdot)\|^2_{W_2^{\frac{l}{2}}(0,T)} \mathrm{d}x \\ &=: \|u\|^2_{W_2^{l,0}(G_T)} + \|u\|^2_{W_2^{0,\frac{l}{2}}(G_T)}. \end{aligned}$$

We also define function spaces

$$\widetilde{W}_2^{l,\frac{l}{2}}(G_T) = \left\{ f \in W_2^{l,\frac{l}{2}}(G_T) \middle| \frac{\partial f}{\partial x_3} \in W_2^{l,\frac{l}{2}}(G_T) \right\}$$

with the norm

$$\|f\|^2_{\widetilde{W}_2^{l,\frac{l}{2}}(G_T)} = \|f\|^2_{W_2^{l,\frac{l}{2}}(G_T)} + \left\| \frac{\partial f}{\partial x_3} \right\|^2_{W_2^{l,\frac{l}{2}}(G_T)},$$

and for $m > 2$,

$$\begin{aligned} \overline{W}_2^{m,\frac{m}{2}}(G_T) = \Bigg\{ f(x,t) \Bigg| \|f\|_{\overline{W}_2^{m,\frac{m}{2}}(G_T)} &:= \sup_{G_T} |f|^2 + \sup_{x\in G} \|f\|^2_{\dot{W}_2^{\frac{m-[m]}{2}}(0,T)} \\ &+ \sup_{t\in(0,T)} \|f\|^2_{\dot{W}_2^{m-[m]}(G)} + \|D_x f\|^2_{W_2^{m-1,\frac{m-1}{2}}(G_T)} + \|D_t f\|^2_{W_2^{m-2,\frac{m-2}{2}}(G_T)} < \infty \Bigg\}, \end{aligned}$$

where D_x and D_t represent the differential operators with respect to x and t, respectively. The norms of the vector spaces and the product spaces are defined by the standard vector norm and the sum of the norms of each space, respectively.

Now the following is our main result,

THEOREM 3.1. *Let $l \in (1/2, 1)$, and T be an arbitrary positive number. Assume that*

(i) $\alpha = 2$ *or* $\alpha > 2l + 1$;

(ii) $la(\cdot) : \mathbf{R} \to \mathbf{R}^+$ *satisfies* $la(x) > 0$, *and is bounded and Lipschitz continuous together with its derivatives such that*

$$|||la||| := \sum_{i=0}^{2} \left[\sup_{x\in\mathbf{R}} \left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i la(x) \right| + \left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i la \right|^{(L)} \right] < \infty,$$

where $|la|^{(L)}$ is Lipschitz coefficient of la;

(iii) $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, θ_0, $S_0 \in \overline{W}_2^{1+l}(\mathbf{R}^3)$, $d_0 \in W_2^{\frac{3}{2}+l}(\mathbf{R}^2)$, $0 < \underline{\theta}_0 \le \theta_0(x) < \infty$ *and* $0 < \underline{S}_0 \le S_0(x) < \infty$ *with positive constants* $\underline{\theta}_0$ *and* $\underline{S}_0$, *respectively*;

(iv) $0 < \underline{\theta}_0 \le \theta_e(x), \theta_b(x) < \infty$, $0 < \underline{S}_0 \le S_e(x), S_b(x) < \infty$, θ_e, θ_b, S_e, $S_b \in \overline{W}_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$. $\frac{\partial\theta_e}{\partial x_3}, \frac{\partial\theta_b}{\partial x_3}, \frac{\partial S_e}{\partial x_3}, \frac{\partial S_b}{\partial x_3} \in W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$, *and* $\theta_e - \theta_0$, $\theta_b - \theta_0$, $S_e - S_0$, $S_b - S_0 \in \widetilde{W}_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$;

(v) $b \in \overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)$ *and* $d_0(x') - b(x') > c_0$ *on* $x' \in \mathbf{R}^2$ *with a positive constant* c_0;

(vi) $\mathbf{F}_1'$, F_2 *and* $F_3 \in \widetilde{W}_2^{l,\frac{l}{2}}(\mathbf{R}_T^3)$, *and their derivatives with respect to* x_3 *satisfy Hölder condition with exponent* $\beta > l/2$ *with respect to* x_3 *(we call this property as condition* (A)*). For the function with this property*, *we introduce the notation*

$$|||f|||_{0,T}^2 := \|f\|^2_{\widetilde{W}_2^{l,\frac{l}{2}}(\mathbf{R}_T^3)} + \left(\left| \frac{\partial f}{\partial x_3} \right|_{x_3}^{(\beta)} \right)^2,$$

where $|f|_{x_3}^{(\beta)}$ *stands for the Hölder coefficient of* f *in* x_3 *with exponent* β;

(vii) $g_1 \in \widetilde{W}_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)$;

(viii) $F_{13} \in \widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)$, *and* $|F_{13}| < \varrho g$ *in* $\mathbf{R}_T^3$.

Moreover, *the following compatibility conditions are satisfied*:

$$\mathbf{u}_0'(y,0) = \mathbf{0}\,, \quad y \in \tilde{\Gamma}_b\,,$$

$$B_{h_0}\bar{\mathbf{u}}_0(y,0) = \tilde{\mathbf{G}}_2(\bar{\mathbf{u}}_0\big|_{t=0})\,, \quad y \in \tilde{\Gamma}_s\,,$$

$$\theta_e(y', \Psi(y',0),0) = \theta_0^{(h_0)*}(y)\,, \quad S_e(y', \Psi(y',0),0) = S_0^{(h_0)*}(y)\,, \quad y \in \tilde{\Gamma}_s\,,$$

$$\theta_b(y', b(y'),0) = \theta_0^{(h_0)*}(y)\,, \quad S_b(y', b(y'),0) = S_0^{(h_0)*}(y), \quad y \in \tilde{\Gamma}_b\,.$$

Then, *there exists* $T^* \in (0,T]$ *such that the problem* (2.19)–(2.20) *has a unique solution* $(\mathbf{u}', u_3, \tilde{\theta}', \tilde{S}', h') \in Z(T^*) := W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times \widetilde{W}_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_{T^*}) \times W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_{T^*}^2)$ *satisfying* $0 < \tilde{\theta} = \tilde{\theta}' + \bar{\tilde{\theta}} < \infty$ *and* $0 < \tilde{S} = \tilde{S}' + \bar{\tilde{S}}$ *on* $\tilde{\Omega}_{T^*}$.

4. Auxiliary Lemmas

In this section, we prepare some lemmas used in the proof of the main theorem in Section 6. It is to be noted that $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, θ_0, $S_0 \in \overline{W}_2^{1+l}(\mathbf{R}^3)$, $d_0 \in W_2^{\frac{3}{2}+l}(\mathbf{R}^2)$ $(1/2 < l < 1)$ imply $\mathbf{v}_0^{(h_0)*} \in W_2^{1+l}(\tilde{\Omega})$, $\theta_0^{(h_0)*}$, $S_0^{(h_0)*} \in \overline{W}_2^{1+l}(\tilde{\Omega})$. By trace theorem, they are extensible into the half space $t > 0$ so that the extended functions $(\bar{\mathbf{u}}_0, \bar{\tilde{\theta}}_0, \bar{\tilde{S}}_0, \bar{d}_0)$ satisfy for some constant C (see, for instance, [31])

$$\begin{cases} \|\bar{\mathbf{u}}_0\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\|\mathbf{v}_0^{(h_0)*}\|_{W_2^{1+l}(\tilde{\Omega})}\,, \\ \|\bar{\bar{\theta}}_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\|\theta_0^{(h_0)*}\|_{\overline{W}_2^{1+l}(\tilde{\Omega})}\,, \\ \|\bar{\bar{S}}_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le C\|S_0^{(h_0)*}\|_{\overline{W}_2^{1+l}(\tilde{\Omega})}\,, \\ \|\bar{d}_0\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \le C\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}\,. \end{cases} \tag{4.1}$$

Now, making use of the well known inequalities

$$\begin{cases} ||uv||_{W_2^{1+l}(\tilde{\Omega})} \le c_1||u||_{W_2^{1+l}(\tilde{\Omega})}||v||_{W_2^{1+l}(\tilde{\Omega})} \quad \text{for } \forall u,\ v \in W_2^{1+l}(\tilde{\Omega})\,, \\ ||uv||_{W_2^{l}(\tilde{\Omega})} \le c_1||u||_{W_2^{1+l}(\tilde{\Omega})}||v||_{W_2^{l}(\tilde{\Omega})} \quad \text{for } \forall u \in W_2^{1+l}(\tilde{\Omega}),\ \forall v \in W_2^{l}(\tilde{\Omega}) \end{cases} \tag{4.2}$$

with a positive constant c_1 (see, for instance, [25]-[28]), we prepare some lemmas concerning the estimates used later. In the followings, C stands for a constant depending on $\|b\|_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)}$, $\|F_{13}\|_{\widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)}$, $\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}$, $\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}$, $\|\theta_0\|_{\overline{W}_2^{1+l}(\mathbf{R}^3)}$, $\|S_0\|_{\overline{W}_2^{1+l}(\mathbf{R}^3)}$, and P a polynomial of its arguments with coefficients having the same dependency as C.

LEMMA 4.1. *Let* $h', h_1', h_2' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ *and* $h = h' + \bar{h}_0$, $h_i = h_i' + \bar{h}_0$ $(i = 1, 2)$. *The following estimates hold*:

$$\|\Psi(\cdot;h)\|^2_{W_2^{i-\frac{1}{2}+l,\frac{i+l}{2}-\frac{1}{4}}(\mathbf{R}_T^2)} \le P\left(\|h'\|_{W_2^{i-\frac{1}{2}+l,\frac{i+l}{2}-\frac{1}{4}}(\mathbf{R}_T^2)}\right) \quad (i = 0, 1, 2, 3)\,, \tag{4.3}$$

$$\begin{aligned} &\|\Psi(\cdot;h_1) - \Psi(\cdot;h_2)\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)} \\ &\quad \le P\left(\sum_{j=1}^{2}\|h_j'\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\|h_1' - h_2'\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)} \quad (i = 0, 1, 2)\,. \end{aligned} \tag{4.4}$$

l in the norm $\|h_1' - h_2'\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}_T^2)}$ *when $i = 2$ can be replaced by any l', $1/2 < l' < l$.*

PROOF. Making use of the relation (2.16), we have

$$\left\|\frac{\partial\Psi}{\partial t}\right\|^2_{L_2(\mathbf{R}^2)} \le C\left(\left\|\frac{\partial h}{\partial t}\right\|^2_{L_2(\mathbf{R}^2)} + \|E_1\|^2_{L_2(\mathbf{R}^2)}\right). \tag{4.5}$$

This and

$$\|\Psi\|^2_{L_2(\mathbf{R}_T^2)} \le C\left(\|d_0\|^2_{L_2(\mathbf{R}^2)} + t\left\|\frac{\partial\Psi}{\partial t}\right\|^2_{L_2(\mathbf{R}_T^2)}\right) \tag{4.6}$$

yield the estimate of $\|\Psi\|^2_{L_2(\mathbf{R}^2_T)}$.

The fractional norm $\|\Psi\|^2_{W_2^{l-\frac{1}{2},0}(\mathbf{R}^2_T)}$ is easily estimated from the inequality

$$\begin{aligned}
&|\Psi(y^{1\prime},t;h)-\Psi(y^{2\prime},t;h)|^2\\
&\quad\le C\left(\frac{1}{\inf_{x,t}|\tilde F_{13}|}\right)^2\Bigg[|h(y^{1\prime},t)-h(y^{2\prime},t)|^2+|b(y^{1\prime})-b(y^{2\prime})|^2\sup_{x_3}|\tilde F_{13}(y^{1\prime},\cdot,t)|^2\\
&\qquad+\sup_{x_3}|\tilde F_{13}(y^{1\prime},x_3,t)-\tilde F_{13}(y^{2\prime},x_3,t)|^2(|b(y^{2\prime})|^2+|\Psi(y^{1\prime},t;h)|^2)\Bigg] \qquad (4.7)
\end{aligned}$$

with $h=p_0+\int_d^b\tilde F_{13}\,\mathrm{d}x_3$. Indeed, applying the Hölder inequality and the Sobolev embedding theorem, one can easily confirm that

$$\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\frac{|\Psi(y^{1\prime},t;h)|^2|F_{13}(y^{1\prime},x_3,t)-F_{13}(y^{2\prime},x_3,t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}}\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\\
&\quad\le\int_{\mathbf{R}^2}\frac{\|\Psi(y^{1\prime},t;h)\|^2_{L_p(\mathbf{R}^2)}\|F_{13}(y^{1\prime},x_3,t)-F_{13}(y^{1\prime}-z',x_3,t)\|^2_{L_q(\mathbf{R}^2)}}{|z'|^{1+2l}}\mathrm{d}z'\\
&\quad\le C\|\Psi(t)\|^2_{L_p(\mathbf{R}^2)}\left(\|\nabla F_{13}(\cdot,x_3,t)\|^2_{\dot W_2^{l-\frac{1}{2}}(\mathbf{R}^2)}+\|F_{13}(\cdot,x_3,t)\|^2_{\dot W_2^{l-\frac{1}{2}}(\mathbf{R}^2)}\right)
\end{aligned}$$

$(1/p+1/q=1/2)$ holds with a postive constant C. Other terms in (4.7) can be estimated in the same way.

For $\|\Psi\|_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)}$, using the relation $h=p_0+\int_d^b\tilde F_{13}\,\mathrm{d}x_3$ again, we have

$$\|\Psi\|^2_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)}\le C\left[\|h\|^2_{W_2^{0,\frac{l}{2}-\frac{1}{4}}(\mathbf{R}^2_T)}+\left(\|b\|^2_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)}+\sup_{\mathbf{R}^2_T}|\Psi|^2\right)\|F_{13}\|^2_{W_2^{l,\frac{l}{2}}(\mathbf{R}^3_T)}\right].$$

Making use of these facts, we arrive at (4.3) with $i=0$.

For $i=1$, we make use of (2.14). Since

$$\begin{aligned}
&|\nabla\Psi(y^{1\prime},t;h)-\nabla\Psi(y^{2\prime},t;h)|\\
&\quad\le C\Bigg[|F_{13}(y^{1\prime},\Psi(y^{1\prime},t;h),t)-F_{13}(y^{2\prime},\Psi(y^{2\prime},t;h),t)|\\
&\qquad\times\left|-\nabla h+\tilde F_{13}(y',b(y'),t)\nabla b+\int_b^d\nabla\tilde F_{13}\,\mathrm{d}x_3\right|+|\nabla h(y^{1\prime},t)-\nabla h(y^{2\prime},t)|\\
&\qquad+\left|\tilde F_{13}(y^{1\prime},b(y^{1\prime}),t)\nabla b(y^{1\prime})-\tilde F_{13}(y^{2\prime},b(y^{2\prime}),t)\nabla b(y^{2\prime})\right|
\end{aligned}$$

$$+ \sup_{\mathbf{R}^3_T} |\nabla \tilde{F}_{13}|(|\Psi(y^{1\prime}, t; h) - \Psi(y^{2\prime}, t; h)| + |b(y^{1\prime}) - b(y^{2\prime})|)$$

$$+ \left(\sup_{\mathbf{R}^2_T} |\Psi| + \sup_{\mathbf{R}^2} |b| \right) \sup_{x_3} |\nabla \tilde{F}_{13}(y^{1\prime}, x_3, t) - \nabla \tilde{F}_{13}(y^{2\prime}, x_3, t)| \Bigg],$$

the estimate of $\|\nabla\Psi(\cdot, t; h)\|_{W_2^{l-\frac{1}{2}}(\mathbf{R}^2)}$ follows by tracing the argument in the case of (4.3) with $i = 0$. Similarly $\|\Psi\|_{W_2^{0,\frac{l}{2}+\frac{1}{4}}(\mathbf{R}^2_T)}$ can be estimated as in the case $i = 0$, and hence we have the estimate of $\|\Psi\|_{W_2^{l+\frac{1}{2},\frac{l}{2}+\frac{1}{4}}(\mathbf{R}^2_T)}$.

For the higher order norms ($i = 2, 3$) of Ψ, one can easily estimate them from (2.14)–(2.16).

Estimates in (4.4) are obtained in exactly the same way as (4.3). □

LEMMA 4.2. *Let* $h' \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$ *and* $h = h' + \bar{h}_0$. *The following inequality holds*:

$$|\tilde{X}_3(y^{1\prime}, y_3^1, t; h) - \tilde{X}_3(y^{2\prime}, y_3^2, t; h)|^2$$
$$\le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\Big)\big(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2\big).$$

PROOF. Since $p = p_0 + \int_{\Psi}^{X_3(x',p,t)} \tilde{F}_{13}\, \mathrm{d}x_3$ by the definition of X_3,

$$p - \tilde{p} = \int_{X_3(x^{1\prime},p,t)}^{X_3(x^{2\prime},\tilde{p},t)} \tilde{F}_{13}(x^{1\prime}, x_3, t)\, \mathrm{d}x_3$$
$$+ \int_{\Psi(x^{1\prime},t;h)}^{X_3(x^{2\prime},\tilde{p},t)} \Big[\tilde{F}_{13}(x^{1\prime}, x_3, t) - \tilde{F}_{13}(x^{2\prime}, x_3, t)\Big] \mathrm{d}x_3$$
$$+ \int_{\Psi(x^{2\prime},t;h)}^{\Psi(x^{1\prime},t;h)} \tilde{F}_{13}(x^{2\prime}, x_3, t)\, \mathrm{d}x_3$$

holds. Each term in the right-hand side is estimated as follows:

$$\left| \int_{X_3(x^{1\prime},p,t)}^{X_3(x^{2\prime},\tilde{p},t)} \tilde{F}_{13}(x^{1\prime}, x_3, t)\, \mathrm{d}x_3 \right| \ge C|X_3(x^{1\prime}, p, t) - X_3(x^{2\prime}, \tilde{p}, t)|,$$

$$\left| \int_{\Psi(x^{1\prime},t;h)}^{X_3(x^{2\prime},\tilde{p},t)} \Big[\tilde{F}_{13}(x^{1\prime}, x_3, t) - \tilde{F}_{13}(x^{2\prime}, x_3, t)\Big] \mathrm{d}x_3 \right|$$
$$\le \Big(\sup_{\mathbf{R}^2} |b| + \sup_{\mathbf{R}^2_T} |\Psi| \Big) \sup_{x_3} |F_{13}(x^{1\prime}, x_3, t) - F_{13}(x^{2\prime}, x_3, t)|$$

$$\leq \Big(\sup_{\mathbf{R}^2} |b| + \sup_{\mathbf{R}^2_T} |\Psi| \Big) \sup_{\mathbf{R}^3_T} |\nabla F_{13}| |x^{1\prime} - x^{2\prime}| \,,$$

$$\left| \int_{\Psi(x^{2\prime},t;h)}^{\Psi(x^{1\prime},t;h)} \tilde{F}_{13}(x^{2\prime}, x_3, t)\, \mathrm{d}x_3 \right| \leq \Big(\sup_{\mathbf{R}^3_T} |F_{13}| + \varrho g \Big) \sup_{\mathbf{R}^2_T} |\nabla \Psi| |x^{1\prime} - x^{2\prime}| \,.$$

Thus we have

$$|X_3(x^{1\prime}, p, t; h) - X_3(x^{2\prime}, \tilde{p}, t; h)|^2$$
$$\leq C\bigg[\Big(\|b\|^2_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)} + \|\Psi\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_\mathbf{T})} \Big) \|F_{13}\|^2_{\widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}^3_T)}$$
$$+ \Big(1 + \|F_{13}\|^2_{\widetilde{W}_2^{3+l,\frac{3+l}{2}}(\mathbf{R}^3_T)} \Big) \|\Psi\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_\mathbf{T})} \bigg] |x^{1\prime} - x^{2\prime}|^2 + |p - \tilde{p}|^2 \,.$$

From this and (2.11) the assertion is derived easily. □

Now we turn to the estimates of the functions appearing in the conditions of Theorem 3.1.

LEMMA 4.3. *Let* $h', h'_1, h'_2 \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$, $h = h' + \bar{h}_0$, $h_i = h'_i + \bar{h}_0$ $(i = 1, 2)$ *and* $1/2 < l' < l$.

(1) *For a function* f *satisfying condition* (A), *the following estimates hold*:

$$\|f^{(h)*}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} \leq P\Big(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}^2_T)} \Big) |\!|\!| f |\!|\!|^2_{0,T} \,, \tag{4.8}$$

$$\|f^{(h_1)*} - f^{(h_2)*}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)}$$
$$\leq P\Big(\sum_{j=1}^2 \|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)} \Big) \|h'_1 - h'_2\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)} |\!|\!| f |\!|\!|^2_{0,T} \,. \tag{4.9}$$

(2) *For a function* $f \in \widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}^3_T)$, *the following estimates hold*:

$$\|f^{(h)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \leq P\Big(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}^2_T)} \Big) \|f\|^2_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}^3_T)} \,, \tag{4.10}$$

$$\|f^{(h_1)*} - f^{(h_2)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)}$$
$$\leq P\Big(\sum_{j=1}^2 \|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)} \Big) \|h'_1 - h'_2\|^2_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}^2_T)} \|f\|^2_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\mathbf{R}^3_T)} \,. \tag{4.11}$$

(3) *For a function* $f \in \widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)$, *the following estimates hold*:

$$\|f^{(h)*}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le P\left(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\right)\|f\|^2_{\widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)}, \tag{4.12}$$

$$\|f^{(h_1)*} - f^{(h_2)*}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)}$$

$$\le P\left(\sum_{j=1}^{2}\|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\|h'_1 - h'_2\|^2_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\|f\|^2_{\widetilde{W}_2^{2+l,1+\frac{l}{2}}(\mathbf{R}_T^3)}. \tag{4.13}$$

PROOF. To prove (4.8), (4.10), (4.12), we use the relation $f^{(h)*}(y', y_3, t) = f(y', \tilde{X}_3(y', y_3, t; h), t)$ and

$$\int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{|f^{(h)*}(y^{1\prime}, y_3^1, t) - f^{(h)*}(y^{2\prime}, y_3^2, t)|^2}{(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2)^{\frac{3+2l}{2}}}\, dy^{1\prime}dy^{2\prime}dy_3^1dy_3^2$$

$$= \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{|f(y^{1\prime}, \tilde{X}_3(y^{1\prime}, y_3^1, t; h), t) - f(y^{2\prime}, \tilde{X}_3(y^{2\prime}, y_3^2, t; h), t)|^2}{(|y^{1\prime} - y^{2\prime}|^2 + |\tilde{X}_3(y^{1\prime}, y_3^1, t; h) - \tilde{X}_3(y^{2\prime}, y_3^2, t; h)|^2)^{\frac{3+2l}{2}}}$$

$$\times \frac{(|y^{1\prime} - y^{2\prime}|^2 + |\tilde{X}_3(y^{1\prime}, y_3^1, t; h) - \tilde{X}_3(y^{2\prime}, y_3^2, t; h)|^2)^{\frac{3+2l}{2}}}{(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2)^{\frac{3+2l}{2}}}\, dy^{1\prime}dy^{2\prime}dy_3^1dy_3^2.$$

Applying Lemma 4.2, changing the variables $\tilde{X}_3(y^{1\prime}, y_3^1, t; h)$ and $\tilde{X}_3(y^{2\prime}, y_3^2, t; h)$ to $\tilde{y}_3^1$ and $\tilde{y}_3^2$, respectively in the integrand, and noting $\dfrac{\partial(y', y_3)}{\partial(y', \tilde{y}_3^i)} = \tilde{F}_{13}(y^{i\prime}, \tilde{y}_3^i, t)\dfrac{p_0 - h_0}{p_0 - h}$, we can get the estimate

$$\|f^{(h)*}(t)\|^2_{\dot{W}_2^l(\tilde{\Omega})} \le P\left(\|h'\|_{W_2^{\frac{5}{2}+l',\frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_T^2)}\right)\|f(t)\|^2_{\dot{W}_2^l(\mathbf{R}^3)}. \tag{4.14}$$

The estimate of $\|f^{(h)*}(y)\|^2_{\dot{W}_2^{\frac{l}{2}}(0,T)}$ can be obtained in a similar way, and hence we arrive at the estimate of $\|f^{(h)*}\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)}$.

Higher order derivatives of $f^{(h)*}$ can be estimated analogously.

To prove (4.9), (4.11), (4.13), we use the following expression of $\tilde{f} := f^{(h_1)*}(y', y_3, t) - f^{(h_2)*}(y', y_3, t)$ derived from the mean value theorem (see, for instance, [22]):

$$\tilde{f}(y', y_3, t) = \bar{X}_3(y', y_3, t)\int_0^1 \frac{\partial f}{\partial x_3}(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t)\, ds,$$

$$\bar{X}_3 = \tilde{X}_3(y', y_3, t; h_1) - \tilde{X}_3(y', y_3, t; h_2)\,.$$

First, we show

$$\|\tilde{f}\|_{W_2^{l,0}(\tilde{\Omega})} \leq P\left(\sum_{j=1}^{2} \|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\right)\|h'_1 - h'_2\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \|\!|f|\!\|_{0,T}\,.$$

Indeed, the above expression yields

$$\begin{aligned}
&|\tilde{f}(y^{1\prime}, y_3^1, t) - \tilde{f}(y^{2\prime}, y_3^2, t)|^2 \\
&\leq \bigg|\int_0^1 \frac{\partial f}{\partial x_3}\big(y^{1\prime}, s\tilde{X}_3(y^{1\prime}, y_3^1, t; h_1) + (1-s)\tilde{X}_3(y^{1\prime}, y_3^1, t; h_2), t\big) \\
&\quad - \frac{\partial f}{\partial x_3}\big(y^{2\prime}, s\tilde{X}_3(y^{2\prime}, y_3^2, t; h_1) + (1-s)\tilde{X}_3(y^{2\prime}, y_3^2, t; h_2), t\big)\,\mathrm{d}s\bigg|^2 \big|\bar{X}_3(y^{1\prime}, y_3^1, t)\big|^2 \\
&\quad + \bigg|\int_0^1 \frac{\partial f}{\partial x_3}\big(y^{2\prime}, s\tilde{X}_3(y^{2\prime}, y_3^2, t; h_1) + (1-s)\tilde{X}_3(y^{2\prime}, y_3^2, t; h_2), t\big)\,\mathrm{d}s\bigg|^2 \\
&\quad \times \big|\bar{X}_3(y^{1\prime}, y_3^1, t) - \bar{X}_3(y^{2\prime}, y_3^2, t)\big|^2 =: I_1 + I_2\,.
\end{aligned} \tag{4.15}$$

On the other hand, for $p(y,t;h) = \frac{(p_0-h)(y_3-h_0)}{p_0-h_0} + h$, we have from (2.8)

$$p(y,t;h_1) - p(y,t;h_2) = \int_{X_3(y',p(y,t;h_2),t;h_2)}^{X_3(y',p(y,t;h_1),t;h_1)} \tilde{F}_{13}\,\mathrm{d}x_3 + \int_{\Psi(y',t;h_1)}^{\Psi(y',t;h_2)} \tilde{F}_{13}\,\mathrm{d}x_3\,,$$

which easily leads to

$$\big|\bar{X}_3(y,t)\big|^2 \leq C\left(|p(y,t;h_1) - p(y,t;h_2)|^2 + \left|\int_{\Psi(y',t;h_1)}^{\Psi(y',t;h_2)} \tilde{F}_{13}\,\mathrm{d}x_3\right|^2\right).$$

Noting that

$$|p(y,t;h_1) - p(y,t;h_2)| = \left|\frac{(p_0 - y_3)(h'_1 - h'_2)}{p_0 - h_0}\right| \leq |h'_1 - h'_2|\,,$$

$$|\Psi(y',t;h_1) - \Psi(y',t;h_2)| \leq \left|\frac{h'_1 - h'_2}{\inf_{x_3} \tilde{F}_{13}(y', x_3, t)}\right|,$$

we obtain

$$\big|\bar{X}_3(y,t)\big|^2 \leq C\left(1 + \|F_{13}\|^2_{W_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)}\right)\big|(h'_1 - h'_2)(y',t)\big|^2\,. \tag{4.16}$$

In the similar manner, using a notation $\tilde{h} := h'_1 - h'_2$, we obtain

$$\big[p(y^{1\prime}, y_3^1, t; h_1) - p(y^{1\prime}, y_3^1, t; h_2)\big] - \big[p(y^{2\prime}, y_3^2, t; h_1) - p(y^{2\prime}, y_3^2, t; h_2)\big]$$

$$= \left[\frac{1}{p_0 - h_0(y^{1\prime})} - \frac{1}{p_0 - h_0(y^{2\prime})}\right](p_0 - y_3^1)\tilde{h}(y^{1\prime}, t)$$

$$+ \frac{(y_3^2 - y_3^1)\tilde{h}(y^{1\prime}, t)}{p_0 - h_0(y^{2\prime})} + \frac{p_0 - y_3^2}{p_0 - h_0(y^{2\prime})}[\tilde{h}(y^{1\prime}, t) - \tilde{h}(y^{2\prime}, t)],$$

$$\left|\int_{\Psi(y^{1\prime},t;h_1)}^{\Psi(y^{1\prime},t;h_2)} \tilde{F}_{13}(y^{1\prime}, x_3, t)\, \mathrm{d}x_3 - \int_{\Psi(y^{2\prime},t;h_1)}^{\Psi(y^{2\prime},t;h_2)} \tilde{F}_{13}(y^{2\prime}, x_3, t)\, \mathrm{d}x_3\right|$$

$$\leq \sup_{x_3}\left|\tilde{F}_{13}(y^{1\prime}, x_3, t)\right|\left|\Psi(y^{1\prime}, t; h_1) - \Psi(y^{1\prime}, t; h_2)\right|$$

$$+ \sup_{x_3}\left|\tilde{F}_{13}(y^{2\prime}, x_3, t)\right|\left|\Psi(y^{2\prime}, t; h_1) - \Psi(y^{2\prime}, t; h_2)\right|.$$

These yield

$$\left|\bar{X}_3(y^{1\prime}, y_3^1, t) - \bar{X}_3(y^{2\prime}, y_3^2, t)\right|^2$$

$$\leq C\bigg[\Big(1 + \sup_{\mathbf{R}^2}|h_0|\Big)^2\left|h_0(y^{1\prime}) - h_0(y^{2\prime})\right|^2|\tilde{h}(y^{1\prime}, t)|^2$$

$$+ \left|y_3^1 - y_3^2|^2\right|\tilde{h}(y^{1\prime}, t)|^2 + \Big(1 + \sup_{\mathbf{R}^2}|h_0|\Big)^2\left|\tilde{h}(y^{1\prime}, t) - \tilde{h}(y^{2\prime}, t)\right|^2$$

$$+ \Big(1 + \|F_{13}\|_{W_2^{3+l,\frac{3+l}{2}}(\mathbf{R}_T^3)}\Big)^2\big(|\tilde{h}(y^{1\prime}, t)|^2 + |\tilde{h}(y^{2\prime}, t)|^2\big). \tag{4.17}$$

Here we have used the estimates $|p_0 - y_3| < C(1 + \sup_{\mathbf{R}^2}|h_0|)$, and $\dfrac{1}{p_0 - h_0(y')} < C$ due to $h_0 = p_0 + \displaystyle\int_b^{d_0} \tilde{F}_{13}\, \mathrm{d}x_3$ with $d_0 - b > c_0$.

We apply (4.16) to the term I_1 in (4.15), and proceed to evaluate in the same way as (4.14), so that

$$\int_0^T \mathrm{d}t \int_{\tilde{\Omega}}\int_{\tilde{\Omega}} \frac{I_1}{(|y^{1\prime} - y^{2\prime}|^2 + |y_3^1 - y_3^2|^2)^{\frac{3+2l}{2}}}\, \mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}dy_3^1dy_3^2$$

$$\leq P\bigg(\sum_{i=1}^{2}\|h_i'\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\bigg)\left\|\frac{\partial f}{\partial x_3}\right\|^2_{W_2^{l,\frac{l}{2}}(\mathbf{R}_T^3)}\|\tilde{h}\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}. \tag{4.18}$$

As for I_2 in (4.15), we need to estimate the right-hand side of (4.17). For the terms except the second, we make use of the estimate

$$\|fg\|_{W_2^{l,0}(\tilde{\Omega}_T)} \leq C\Big(1 + \sup_{\mathbf{R}^2}|h_0|\Big)^\delta \|f\|^2_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)}\|g\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)} \tag{4.19}$$

for any $f \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$ and $g \in W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$ in general with some positive constant δ. Indeed, let $\gamma = 2-2l-\delta$ with $0<\delta<\min\{2l(2l-1)/(3-2l),\ 2-2l\}$. Then, it is easy to see that

$$\int_{\tilde{\Omega}} \frac{|g(y^{1\prime},t)-g(y^{2\prime},t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y_3^1$$

$$\le 2\sup_{\mathbf{R}^2}|g(t)|^{\gamma}\sup_{\mathbf{R}^2}|\nabla g(t)|^{2-\gamma}\int_{\tilde{\Omega}}\frac{|y^{1\prime}-y^{2\prime}|^{2-\gamma}}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y_3^1$$

$$\le 2\pi\sup_{\mathbf{R}^2}|g(t)|^{\gamma}\sup_{\mathbf{R}^2}|\nabla g(t)|^{2-\gamma}B\left(2-\frac{\gamma}{2},\ l-\frac{1}{2}+\frac{\gamma}{2}\right)\int_{p_0}^{\sup|h_0|}|y_3^1-y_3^2|^{1-\gamma-2l}\,\mathrm{d}y_3^1,$$

where $B(x,y)$ is the beta function. This leads to the estimate

$$\int_{\tilde{\Omega}}\int_{\tilde{\Omega}}\frac{|f(y^{2\prime},y_3^2,t)|^2|g(y^{1\prime},t)-g(y^{2\prime},t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2$$

$$\le C(1+\sup_{\mathbf{R}^2}h_0)^{\delta}\|g(t)\|^{\gamma}_{W_2^{\frac{1}{2}+l}(\mathbf{R}^2)}\|g(t)\|^{2-\gamma}_{W_2^2(\mathbf{R}^2)}\|f(t)\|^2_{L_2(\tilde{\Omega})}.$$

Applying the Hölder inequality yields the estimate

$$\int_0^T\|g(t)\|^{\gamma}_{W_2^{\frac{1}{2}+l}(\mathbf{R}^2)}\|g(t)\|^{2-\gamma}_{W_2^2(\mathbf{R}^2)}\|f(t)\|^2_{L_2(\tilde{\Omega})}\,\mathrm{d}t$$

$$\le \sup_t\|g(t)\|^{\gamma}_{W_2^{\frac{1}{2}+l}(\mathbf{R}^2)}\|g\|^{2-\gamma}_{L_{\frac{2-\gamma}{l}}(0,T;W_2^2(\mathbf{R}^2))}\|f\|^2_{L_{\frac{2}{1-l}}(0,T;L_2(\tilde{\Omega}))}.$$

The assumpition of δ and the Sobolev embedding theorem imply $W_2^{\frac{l}{2}}(0,T)\subset L_{\frac{2}{1-l}}(0,T)$ and $W_2^{\frac{1}{2}\left(l-\frac{1}{2}\right)}(0,T)\subset L_{\frac{2-\gamma}{l}}(0,T)$, so that (4.19) holds.

For the second term in the right-hand side of (4.17), it is sufficient to consider

$$\int\frac{|f(y^{2\prime},y_3^2,t)|^2|y_3^1-y_3^2|^2|\tilde{h}(y^{1\prime},t)|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}\mathrm{d}y_3^1\mathrm{d}y_3^2$$

in the region $|y_3^1-y_3^2|\neq 0$ with $f\in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$. Then we have

$$\sup_{\mathbf{R}_T^2}|\tilde{h}|^2\int_0^T\mathrm{d}t\int_{\tilde{\Omega}}\left(\int_{\tilde{\Omega}}\frac{|y_3^1-y_3^2|^2}{(|y^{1\prime}-y^{2\prime}|^2+|y_3^1-y_3^2|^2)^{\frac{3+2l}{2}}}\,\mathrm{d}y^{1\prime}\,\mathrm{d}y_3^1\right)|f(y^{2\prime},y_3^2,t)|^2\,\mathrm{d}y^{2\prime}\mathrm{d}y_3^2$$

$$\le \pi B\left(1,\ \frac{1}{2}+l\right)\|f\|^2_{L_2(\tilde{\Omega}_T)}\|\tilde{h}(t)\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2)}\left(\sup_{y_3^2}\int_{p_0}^{\sup h_0}|y_3^1-y_3^2|^{1-2l}\,\mathrm{d}y_3^1\right)$$

$$\leq C(p_0 + \sup_{\mathbf{R}^2} |h_0|)^{2-2l} \|f\|^2_{L_2(\tilde{\Omega}_T)} \|\tilde{h}\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}.$$

Therefore, we can estimate the second term of (4.17), and consequently $\|\tilde{f}\|_{W_2^{l,0}(\tilde{\Omega}_T)}$.

The estimate of $\|\tilde{f}\|_{L_2(\tilde{\Omega};\dot{W}_2^{\frac{l}{2}}(0,T))}$ is obtained in a similar manner under the assumption of the Hölder continuity of $\partial f/\partial x_3$. Actually, in this case, the right-hand side of (4.15) is replaced by

$$\begin{aligned}
&\bigg[\bigg|\int_0^1 \frac{\partial f}{\partial x_3}\big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t\big) \\
&\qquad - \frac{\partial f}{\partial x_3}\big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t-\tau\big)\, \mathrm{d}s\bigg|^2 \\
&+ \bigg|\int_0^1 \frac{\partial f}{\partial x_3}\big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t-\tau\big) \\
&\qquad - \frac{\partial f}{\partial x_3}\big(y', s\tilde{X}_3(y', y_3, t-\tau; h_1) + (1-s)\tilde{X}_3(y', y_3, t-\tau; h_2), t-\tau\big)\, \mathrm{d}s\bigg|^2\bigg] \\
&\times \big|\bar{X}_3(y', y_3, t)\big|^2 \\
&+ \bigg|\int_0^1 \frac{\partial f}{\partial x_3}\big(y', s\tilde{X}_3(y', y_3, t-\tau; h_1) + (1-s)\tilde{X}_3(y', y_3, t-\tau; h_2), t-\tau\big)\, \mathrm{d}s\bigg|^2 \\
&\times \big|\bar{X}_3(y', y_3, t) - \bar{X}_3(y', y_3, t-\tau)\big|^2 \\
&=: J_1 + J_2 + J_3\,. \qquad (4.20)
\end{aligned}$$

It is easily seen that J_1 and J_3 in (4.20) can be estimated in exactly the same way as in (4.15). For J_2, we have

$$\begin{aligned}
&\bigg|\int_0^1 \frac{\partial f}{\partial x_3}\big(y', s\tilde{X}_3(y', y_3, t; h_1) + (1-s)\tilde{X}_3(y', y_3, t; h_2), t-\tau\big) \\
&\qquad - \frac{\partial f}{\partial x_3}\big(y', s\tilde{X}_3(y', y_3, t-\tau; h_1) + (1-s)\tilde{X}_3(y', y_3, t-\tau; h_2), t-\tau\big)\, \mathrm{d}s\bigg|^2 \\
&\leq C\bigg(\bigg|\frac{\partial f}{\partial x_3}\bigg|^{(\beta)}_{x_3}\bigg)^2 \sum_{i=1}^{2} \big|\tilde{X}_3(y', y_3, t; h_i) - \tilde{X}_3(y', y_3, t-\tau; h_i)\big|^{2\beta}.
\end{aligned}$$

In the same way as the proof of Lemma 4.2, we have

$$|\tilde{X}_3(y', y_3, t; h) - \tilde{X}_3(y', y_3, t-\tau; h)|$$

$$\le C\Bigg[|p(y,t)-p(y,t-\tau)| + \Big(\sup_{\mathbf{R}^2}|b| + \sup_{\mathbf{R}^2_T}|\Psi|\Big)\sup_{x_3}|F_{13}(y',x_3,t)-F_{13}(y',x_3,t-\tau)| + \Big(1+\sup_{\mathbf{R}^3_T}|F_{13}|\Big)|\Psi(y',t;h)-\Psi(y',t-\tau;h)|\Bigg].$$

Thus we have

$$\int_{\tilde{\Omega}} \mathrm{d}y \int_0^T \int_0^t \frac{J_2}{\tau^{1+l}}\,\mathrm{d}\tau\,\mathrm{d}t \le C\bigg(\Big|\frac{\partial f}{\partial x_3}\Big|^{(\beta)}_{x_3}\bigg)^2 T^{2\beta-l}\sum_{i=1}^{2}\bigg[\|h_i'\|^{2\beta}_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)} + \Big(\|b\|^{2\beta}_{\overline{W}_2^{\frac{5}{2}+l}(\mathbf{R}^2)} + \|\Psi(\cdot;h_i)\|^{2\beta}_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\Big)\|F_{13}\|^{2\beta}_{W_2^{3+l,\frac{3+l}{2}}(\mathbf{R}^3_T)} + \Big(1+\|F_{13}\|^{2\beta}_{W_2^{3+l,\frac{3+l}{2}}(\mathbf{R}^3_T)}\Big)\|\Psi(\cdot;h_i)\|^{2\beta}_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\bigg]\|h_1'-h_2'\|^2_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}.$$

These completes the estimate of $\|\tilde{f}\|_{W_2^{0,\frac{l}{2}}(\tilde{\Omega})}$, and hence $\|\tilde{f}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega})}$.

Higher order norms can be estimated in a similar manner. □

LEMMA 4.4. *Let $h', h_1', h_2' \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)$ and $h = h' + \bar{h}_0$, $h_i = h_i' + \bar{h}_0$ $(i = 1,2)$. The following estimates hold*:

$$\|\mathbf{F}_5^{(h)*}\|^2_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\Big),$$

$$\|\mathbf{F}_5^{(h_1)*} - \mathbf{F}_5^{(h_2)*}\|^2_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\sum_{j=1}^{2}\|h_j'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\Big) \times \|h_1'-h_2'\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}^2_T)} \quad (i = 0,1,2).$$

l in the norm $\|h_1'-h_2'\|^2_{W_2^{\frac{1}{2}+i+l,\frac{1}{4}+\frac{i+l}{2}}(\mathbf{R}^2_T)}$ when $i = 2$ can be replaced by any l', $1/2 < l' < l$.

PROOF. According to the explicit form of $\mathbf{F}_5^{(h)*}$ given in (2.12), it is sufficient to estimate the second term $\mathbf{C}_1$. Since $\nabla_y F_{13}^{(h)*} \in W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)$ and

$$\|\mathbf{a}^3(h_{(m)})\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_T)}\Big), \tag{4.21}$$

we have

$$\|\mathbf{C}_1\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big)\|F_{13}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\mathbf{R}_T^3)}$$

with the aid of multiplicative inequalities. The latter assertion is proved in the same way. □

Similarly as Lemma 4.4, we can obtain the following lemma.

LEMMA 4.5. *Let* h', h'_1 *and* $h'_2 \in W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$, $h = h' + \bar{h}_0$, $h_i = h'_i + \bar{h}_0$ $(i = 1, 2)$.

The following estimates hold:

$$\|F_6^{(h)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\|h'\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big),$$

$$\|F_6^{(h_1)*} - F_6^{(h_2)*}\|^2_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_T)} \le P\Big(\sum_{j=1}^{2}\|h'_j\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}\Big) \times \|h'_1 - h'_2\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_T^2)}.$$

5. Linear Problems

Let us introduce the linear operators $L_{i,\bar{h}_0}$ $(i = 1, 2, 3, 4)$, which are obtained from $L_{i,h}$ $(i = 1, 2, 3, 4)$ with (h, Ψ) replaced by $(\bar{h}_0, \bar{d}_0)$. From the assumptions of Theorem 3.1, it is easily seen that the coefficients of $L_{i,\bar{h}_0}$ $(i = 1, 2, 3)$ belong to $W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$, and those of $L_{4,\bar{h}_0}$ to $\overline{W}_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_T^2)$. In this section we consider the following linear problems.

$$\begin{cases} \dfrac{\partial \mathbf{u}'}{\partial t} - L_{1,\bar{h}_0}\mathbf{u}' = \mathbf{l}_1, \\ \dfrac{\partial \tilde{\theta}'}{\partial t} - L_{2,\bar{h}_0}\tilde{\theta}' = l_4\,, \\ \dfrac{\partial \tilde{S}'}{\partial t} - L_{3,\bar{h}_0}\tilde{S}' = l_5 \quad \text{in } \tilde{\Omega}_T\,, \\ B_{\bar{h}_0}\mathbf{u}' = \mathbf{l}_2,\ (\tilde{\theta}', \tilde{S}') = (\bar{\theta}_e, \bar{S}_e) \quad \text{on } \tilde{\Gamma}_{sT}\,, \\ (\mathbf{u}', \tilde{\theta}', \tilde{S}') = (-\bar{\mathbf{u}}_0, \bar{\theta}_b, \bar{S}_b) \quad \text{on } \tilde{\Gamma}_{bT}\,, \\ (\mathbf{u}', \tilde{\theta}', \tilde{S}') = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega}\,; \end{cases} \tag{5.1}$$

$$\begin{cases} \dfrac{\partial h'}{\partial t} - L_{4,\bar{h}_0} h' = l_6 \quad \text{in } \mathbf{R}^2_T\,, \\ h' = 0 \quad \text{on } \mathbf{R}^2\,; \end{cases} \tag{5.2}$$

$$\begin{cases} \nabla_{\bar{h}_0,3} u'_3 - (\nabla_{\bar{h}_0,3} \tilde{F}^{(\bar{h}_0)*}_{13}) \dfrac{u'_3}{\tilde{F}^{(\bar{h}_0)*}_{13}} = l_3 \quad \text{in } \tilde{\Omega}_T\,, \\ u'_3 = 0 \quad \text{on } \tilde{\Gamma}_{bT}\,. \end{cases} \tag{5.3}$$

For problems (5.1), (5.2), we have

LEMMA 5.1. (i) *Let* $\mathbf{l}_1 \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$, $\mathbf{l}_2 \in W_2^{\frac12+l,\frac14+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $l_4,\ l_5 \in W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)$, $\bar{\mathbf{u}}_0 \in W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{bT})$ $\bar{\theta}_e,\ \bar{S}_e \in W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{sT})$, $\bar{\theta}_b,\ \bar{S}_b \in W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{bT})$, *and satisfy the compatibility conditions*

$$\begin{cases} \bar{\mathbf{u}}_0 = \mathbf{0}\,, \quad \bar{\theta}_b(y,0) = 0\,, \quad \bar{S}_b(y,0) = 0\,, \quad y \in \tilde{\Gamma}_b\,, \\ \mathbf{l}_2|_{t=0} = \mathbf{0}\,, \quad \bar{\theta}_e(y,0) = 0\,, \quad \bar{S}_e(y,0) = 0\,, \quad y \in \tilde{\Gamma}_s\,. \end{cases}$$

Then the problem (5.1) *has a unique solution* $(\mathbf{u}', \tilde{\theta}', \tilde{S}') \in Z'(T) := W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T) \times W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_T)$ *satisfying*

$$\begin{aligned} \|(\mathbf{u}', \tilde{\theta}', \tilde{S}')\|_{Z'(T)} \le C_1 \Bigg[&\|\mathbf{l}_1\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} + \|\mathbf{l}_2\|_{W_2^{\frac12+l,\frac14+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \sum_{i=4}^{5} \|l_i\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_T)} \\ &+ \|\bar{\theta}_e\|_{W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{sT})} + \|\bar{S}_e\|_{W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{sT})} \\ &+ \|\bar{\mathbf{u}}_0\|_{W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{bT})} + \|\bar{\theta}_b\|_{W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{bT})} + \|\bar{S}_b\|_{W_2^{\frac32+l,\frac34+\frac{l}{2}}(\tilde{\Gamma}_{bT})} \Bigg]. \end{aligned} \tag{5.4}$$

(ii) *For* $l_6 \in W_2^{\frac12+l,\frac14+\frac{l}{2}}(\mathbf{R}^2_T)$, *problem* (5.2) *has a unique solution* $h' \in W_2^{\frac52+l,\frac54+\frac{l}{2}}(\mathbf{R}^2_T)$ *satisfying*

$$\|h'\|_{W_2^{\frac52+l,\frac54+\frac{l}{2}}(\mathbf{R}^2_T)} \le C_2 \|l_6\|_{W_2^{\frac12+l,\frac14+\frac{l}{2}}(\mathbf{R}^2_T)}\,. \tag{5.5}$$

PROOF. Note that the operator $L_{i,\bar{h}_0}$ $(i = 1,2,3,4)$ is uniformly elliptic. Indeed, for $(\xi',\ \xi_3) := (\xi_1,\ \xi_2,\ \xi_3)^{\mathrm{T}} \in \mathbf{R}^3\backslash\{0\}$ and $\mathbf{a}^3_0 := \left(a^{31}_0,\ a^{32}_0\right)^{\mathrm{T}} = \left(a^{31}(\bar{h}_0),\ a^{32}(\bar{h}_0)\right)^{\mathrm{T}}$, $a^{33}_0 := a^{33}(\bar{h}_0)$, the charasteristic polynomial of $L_{1,\bar{h}_0}$ is

$$\begin{aligned} \mu_1 \left(|\xi'|^2 + 2\mathbf{a}^3_0 \cdot \xi'\xi_3 + |\mathbf{a}^3_0|^2 \xi_3^2\right) + 2\mu_1 a^{33}_0 \left(\mathbf{F}^{(\bar{h}_0)*}_5 \cdot \xi'\xi_3 + \mathbf{a}^3_0 \cdot \mathbf{F}^{(\bar{h}_0)*}_5 \xi_3^2\right) \\ + \mu_1 |\mathbf{F}^{(\bar{h}_0)*}_5|^2 (a^{33}_0)^2 \xi_3^2 + \mu_2 \tilde{F}^{(\bar{h}_0)*2}_{13} (a^{33}_0)^2 \xi_3^2 \end{aligned}$$

$$= \mu_1 |\xi' + \mathbf{a}^3 \xi_3 + a^{33} \mathbf{F}_5^{(\bar{h}_0)*} \xi_3|^2 + \mu_2 \tilde{F}_{13}^{(\bar{h}_0)*2} (a_0^{33})^2 \xi_3^2 > 0\,.$$

This means that $L_{1,\bar{h}_0}$ is uniformly elliptic. In exactly the same method, other operators $L_{i,\bar{h}_0}$ $(i = 2, 3, 4)$ are also uniformly elliptic. Then the general theory for linear partial differential equations of parabolic type [10] leads to the desired result. □

LEMMA 5.2. *For* $l_3 \in W_2^{1+l, \frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$, *the problem* (5.3) *has a unique solution* $u_3' \in \widetilde{W}_2^{1+l, \frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$ *such that*

$$\|u_3'\|_{\widetilde{W}_2^{1+l, \frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)} \le C_3 \|l_3\|_{W_2^{1+l, \frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)}\,.$$

PROOF. Problem (5.3) has an exact solution given by

$$u_3'(y', y_3, t) = \tilde{F}_{13}^{(\bar{h}_0)*}(y', y_3, t) \left[\int_{h_0}^{y_3} \frac{l_3}{a_0^{33} \tilde{F}_{13}^{(\bar{h}_0)*}} \, \mathrm{d}y_3 \right].$$

This directly implies $u_3' \in \widetilde{W}_2^{1+l, \frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_T)$. □

6. Nonlinear Problem (Proof of Theorem 3.1)

In this section, we prove Theorem 3.1 by an iteration method. Let $(\mathbf{u}'_{(0)}, u'_{3(0)}, \tilde{\theta}'_{(0)}, \tilde{S}'_{(0)}, h'_{(0)}) = (\mathbf{0}, 0, 0, 0, 0)$ and $(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}, h'_{(m+1)})$ $(m = 0, 1, 2, \ldots)$ be a solution of the following problem for a given $(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{S}'_{(m)}, h'_{(m)})$

$\in Z(T)$.

$$
\begin{cases}
\dfrac{\partial \mathbf{u}'_{(m+1)}}{\partial t} - L_{1,\bar{h}_0}\mathbf{u}'_{(m+1)} = \big[L_{1,h_{(m)}} - L_{1,\bar{h}_0}\big]\mathbf{u}'_{(m+1)} + L_{1,h_{(m)}}\bar{\mathbf{u}}_0 \\
\qquad\qquad\qquad\qquad -\dfrac{\partial \bar{\mathbf{u}}_0}{\partial t} + \tilde{G}_{1,h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}) =: \mathbf{l}_1^{(m+1)}, \\
\nabla_{\bar{h}_0,3} u'_{3(m+1)} - (\nabla_{\bar{h}_0,3}\tilde{F}_{13}^{(\bar{h}_0)*})\dfrac{u'_{3(m+1)}}{\tilde{F}_{13}^{(\bar{h}_0)*}} = -(\nabla_{\bar{h}_0,3} - \nabla_{h_{(m)},3})u'_{3(m+1)} \\
\qquad\qquad\qquad\qquad +\tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)}, \mathbf{u}_{(m)}) =: l_3^{(m+1)}, \\
\dfrac{\partial \tilde{\theta}'_{(m+1)}}{\partial t} - L_{2,\bar{h}_0}\tilde{\theta}'_{(m+1)} = \big[L_{2,h_{(m)}} - L_{2,\bar{h}_0}\big]\tilde{\theta}'_{(m+1)} + L_{2,h_{(m)}}\bar{\tilde{\theta}}_0 \\
\qquad\qquad\qquad\qquad -\dfrac{\partial \bar{\tilde{\theta}}_0}{\partial t} + \tilde{G}_{4,h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}, \tilde{\theta}_{(m)}) =: l_4^{(m+1)}, \\
\dfrac{\partial \tilde{S}'_{(m+1)}}{\partial t} - L_{3,\bar{h}_0}\tilde{S}'_{(m+1)} = \big[L_{3,h_{(m)}} - L_{3,\bar{h}_0})\big]\tilde{S}'_{(m+1)} + L_{3,h_{(m)}}\bar{\tilde{S}}_0 \\
\qquad\qquad\qquad\qquad -\dfrac{\partial \bar{\tilde{S}}_0}{\partial t} + \tilde{G}_{5,h_{(m)}}(\mathbf{u}_{(m)}, u_{3(m)}, \tilde{\theta}_{(m)}) =: l_5^{(m+1)} \quad \text{in } \tilde{\Omega}_T,
\end{cases}
\tag{6.1}
$$

$$
\begin{cases}
B_{\bar{h}_0}\mathbf{u}'_{(m+1)} = -B_{h_{(m)}}\bar{\mathbf{u}}_0 + (B_{\bar{h}_0}\mathbf{u}'_{(m+1)} - B_{h_{(m)}}\mathbf{u}'_{(m+1)}) + \tilde{\mathbf{G}}_2(\mathbf{u}_{(m)}) =: \mathbf{l}_2^{(m+1)}, \\
(\tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}) = (\theta_e - \bar{\tilde{\theta}}_0, S_e - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{sT}, \\
(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}) = (-\bar{\mathbf{u}}_0, 0, \theta_b - \bar{\tilde{\theta}}_0, S_b - \bar{\tilde{S}}_0) \quad \text{on } \tilde{\Gamma}_{bT}, \\
(\mathbf{u}'_{(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)})|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega},
\end{cases}
\tag{6.2}
$$

$$
\begin{cases}
\dfrac{\partial h'_{(m+1)}}{\partial t} - L_{4,\bar{h}_0}h'_{(m+1)} = (L_{4,h_{(m)}} - L_{4,\bar{h}_0})h'_{(m+1)} + L_{4,h_{(m)}}\bar{h}_0 \\
\qquad\qquad -\dfrac{\partial \bar{h}_0}{\partial t} + \tilde{G}_{6,h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)}) =: l_6^{(m+1)} \quad \text{in } \mathbf{R}^2_T, \\
h'_{(m+1)}|_{t=0} = 0 \quad \text{on } \mathbf{R}^2.
\end{cases}
\tag{6.3}
$$

Here $\mathbf{u}_{(m)} = \mathbf{u}'_{(m)} + \bar{\mathbf{u}}_0$, $\tilde{\theta}_{(m)} = \tilde{\theta}'_{(m)} + \bar{\bar{\tilde{\theta}}}_0$, $\tilde{S}_{(m)} = \tilde{S}'_{(m)} + \bar{\bar{\tilde{S}}}_0$, and

$$\tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)}, \mathbf{u}_{(m)}) = -\nabla_{h_{(m)}} \cdot \mathbf{u}_{(m+1)} - a^{33}(h_{(m)})\mathbf{F}_5^{(h_{(m)})*} \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_3},$$

$$\begin{aligned}
&\tilde{G}_{6,h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)}) \\
&\quad = \tilde{F}_{13}(y', \Psi(y', t; h_{(m)}), t)\Bigg[E_1 - \frac{\sigma L}{la(\theta_e)\left(1 + |\nabla\Psi(y', t; h_{(m)})|^2\right)} \sum_{i,j=1}^{2} c_{ij} H_{ij} \\
&\qquad - \frac{1}{la(\theta_e)}\Bigg\{-\mu_3\left(\nabla_{h_{(m)}} + a^{33}(h_{(m)})\mathbf{F}_5^{(h_{(m)})*}\frac{\partial}{\partial y_3}\right)\tilde{\theta}_{(m)} \cdot \nabla\Psi(y', t; h_{(m)}) \\
&\qquad + g_1^{(h_{(m)})*}\left(1 + \left|\nabla\Psi(y', t; h_{(m)})\right|^2\right)^{\frac{1}{2}} |\mathbf{u}_{(m)}|^\alpha \tilde{\theta}_{(m)}\Bigg\}\Bigg] \\
&\qquad - \frac{\mu_4}{la(\theta_e)}\Bigg[\left(F_{13}(y', \Psi(y', t; h_{(m)}), t)^2 - (\varrho g)^2\right)a^{33}(h_{(m)})\frac{\partial\tilde{\theta}_m}{\partial y_3} \\
&\qquad + (\varrho g)^2 a^{33}(h_{(m)})\frac{\partial\tilde{\theta}_{m+1}}{\partial y_3}\Bigg]
\end{aligned}$$

with $la(\theta_e) = la(\theta_e(y', \Psi(y', t; h_{(m)}), t))$.

The unique existence of $(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}, h'_{(m+1)})$ is guaranteed by Lemmas 5.1 and 5.2 and the fixed point arguments [10].

Now applying Lemmas 5.1 and 5.2 again, we shall estimate $(\mathbf{u}'_{(m+1)}, u'_{3(m+1)}, \tilde{\theta}'_{(m+1)}, \tilde{S}'_{(m+1)}, h'_{(m+1)})$.

By the interpolation and Young's inequalities, it is easy to confirm that

$$\begin{aligned}
\|u\|^2_{W_2^{k,0}(\tilde{\Omega}_t)} &\le \varepsilon\|u\|^2_{W_2^{m,0}(\tilde{\Omega}_t)} + C_\varepsilon \int_0^t \|u\|^2_{L_2(\tilde{\Omega})}\, \mathrm{d}t \\
&\le \varepsilon\|u\|^2_{W_2^{m,0}(\tilde{\Omega}_t)} + C_\varepsilon t^2 \left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(\tilde{\Omega}_t)} \qquad (6.4)
\end{aligned}$$

for any $\varepsilon > 0$ if $m > k$ and $u|_{t=0} = 0$. Using (4.2), (6.4) and Lemmas 4.1-4.5, we shall estimate the right-hand side of (6.1)-(6.3). In the followings, $P(\cdot)$ stands for the polynomial of its arguments, ε an arbitrary positive number, and $1/2 < l' < l$.

First, making use of (4.2), (4.21), (6.4), multiplicative inequalities and the fact that

$$\|a^{33}(h_{(m)})f\|_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_t)} \le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|f\|_{W_2^{i+l,\frac{i+l}{2}}(\tilde{\Omega}_t)} \qquad (6.5)$$

$(i=1,2)$ hold for any $f \in W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)$, we can show the inequalities

$$\begin{aligned}
&\left\|\tilde{G}_{11,h_{(m)}}\mathbf{u}_{(m)}\right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\quad\le \left(\left\|\sum_{i=1}^{2}\left[\frac{\partial a^{3i}(h_{(m)})}{\partial y_i}+a^{3i}(h_{(m)})\frac{\partial a^{3i}(h_{(m)})}{\partial y_3}\right]\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}\right. \\
&\qquad\left.+\left\|\nabla_{h_{(m)}}\cdot\left(a^{33}(h_{(m)})\mathbf{F}_5^{(h_{(m)})*}\right)\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}\right)\left\|\frac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right\|_{W_2^{1+l',\frac{1+l'}{2}}(\tilde{\Omega}_t)} \\
&\quad+\|a^{33}(h_{(m)})\mathbf{F}_5^{(h_{(m)})*}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\left(\left\|\frac{\partial \mathbf{a}^3(h_{(m)})}{\partial y_3}\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}\right. \\
&\qquad\left.+\left\|\frac{\partial}{\partial y_3}\left(a^{33}(h_{(m)})\mathbf{F}_5^{(h_{(m)})*}\right)\right\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}\right)\left\|\frac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right\|_{W_2^{1+l',\frac{1+l'}{2}}(\tilde{\Omega}_t)} \\
&\quad\le (\varepsilon+C_\varepsilon t)\,P(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)})\left\|\mathbf{u}_{(m)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left\|\tilde{G}_{12,h_{(m)}}\mathbf{u}_{(m)}\right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\quad\le (\varepsilon+C_\varepsilon t)\,P(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)})\left\|F_{13}^{(h_{(m)})*}\right\|^2_{W_2^{3+l,\frac{3+l}{2}}(\tilde{\Omega}_t)}\left\|\mathbf{u}_{(m)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}.
\end{aligned}$$

It is easy to estimate the lower order terms in $\mathbf{l}_1^{(m+1)}$, for example,

$$\begin{aligned}
&\left\|\left(\mathbf{u}_{(m)}\cdot\nabla_{h_{(m)}}\right)\mathbf{u}_{(m)}\right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\quad\le (\varepsilon+C_\varepsilon t)\|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\left\|\left(\nabla+\mathbf{a}^3(h_{(m)})\frac{\partial}{\partial y_3}\right)\mathbf{u}_{(m)}\right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)},
\end{aligned}$$

$$\begin{aligned}
&\left\|u_{3(m)}a^{33}(h_{(m)})\frac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} \\
&\quad\le (\varepsilon+C_\varepsilon t)P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\|u_{3(m)}\|_{W_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}\left\|\mathbf{u}_{(m)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)},
\end{aligned}$$

$$\left\|F_6^{(h_{(m)})*}a^{33}(h_{(m)})\frac{\partial \mathbf{u}_{(m)}}{\partial y_3}\right\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}$$

$$\le (\varepsilon + C_\varepsilon t) \left\| F_6^{(h_{(m)})*} \right\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)} \left\| a^{33}(h_{(m)}) \frac{\partial \mathbf{u}_{(m)}}{\partial y_3} \right\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)}$$

$$\le (\varepsilon + C_\varepsilon t) P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \left\| F_6^{(h_{(m)})*} \right\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)} \left\| \mathbf{u}_{(m)} \right\|_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)}$$

by virtue of (4.2) and (6.5). It is to be noted that the extended functions of the initial data have explicit representations

$$\frac{\partial}{\partial y_i} u_0^{(h_0)*}(y', y_3) = \frac{\partial u_0}{\partial x_i}(y', \tilde{X}_3(y', y_3, 0; h_0))(1 - \delta_{3i})$$
$$+ \frac{\partial u_0}{\partial x_3}(y', \tilde{X}_3(y', y_3, 0; h_0)) \frac{\partial \tilde{X}_3}{\partial y_i}(y', y_3, 0; h_0) \quad (i = 1, 2, 3)\,.$$

Then, we get

$$\left\| L_{1,h_{(m)}} \bar{\mathbf{u}}_0 - \frac{\partial \bar{\mathbf{u}}_0}{\partial t} \right\|^2_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)} \le \left[P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) + 1 \right] \|\bar{\mathbf{u}}_0\|^2_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)}$$
$$\le \left[P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) + 1 \right] \|\mathbf{v}_0^{(h_0)*}\|^2_{W_2^{1+l}(\tilde{\Omega})}$$
$$\le \left[P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) + 1 \right] \|\mathbf{v}_0\|^2_{W_2^{1+l}(\Omega)}\,.$$

Moreover, the first term in the right-hand side of $\mathbf{l}_1^{(m+1)}$ is estimated by noting the inequalities, for example,

$$\|(l_{11,h_{(m)}} - l_{11,\bar{h}_0}) \mathbf{u}'_{(m+1)}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)}$$
$$\le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l', \frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_t^2)} \right) \|\mathbf{u}'_{(m+1)}\|_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)}\,,$$

and in general

$$\|(f^{(h_{(m)})*} - f^{(\bar{h}_0)*}) \nabla^2 \mathbf{u}'_{(m+1)}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)}$$
$$\le P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l', \frac{5}{4}+\frac{l'}{2}}(\mathbf{R}_t^2)} \right) \|f\|_{W_2^{2+l, \frac{2+l}{2}}(\mathbf{R}_t^3)} \|\mathbf{u}'_{(m+1)}\|_{W_2^{2+l, \frac{2+l}{2}}(\tilde{\Omega}_t)}$$

for $f \in W_2^{2+l, \frac{2+l}{2}}(\mathbf{R}_t^3)$. Consequently, with the aid of Lemmas 4.3–4.5, we have

$$\|\mathbf{l}_1^{(m+1)}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)} \le \big(\varepsilon + C_\varepsilon t\big) P\left(\|u_{3(m)}\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right)$$

$$\times \left(\|\mathbf{u}_{(m)}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \right)$$

$$+ P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_t)} \right) \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)} .$$

The terms $l_4^{(m+1)}$, $l_5^{(m+1)}$ are estimated in exactly the same way as $\mathbf{l}_1^{(m+1)}$.

Now we proceed to estimate $\mathbf{l}_2^{(m+1)}$. First, note that

$$\| |\mathbf{u}_{(m)}|^{\alpha}\big|_{y_3=p_0}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}^2_t)} \leq C\|\mathbf{u}_{(m)}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \left(1 + \|\mathbf{u}_{(m)}\|^{\alpha-2}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right) \tag{6.6}$$

holds for $\alpha \geq 2$. Indeed, we first show

$$\| |\mathbf{u}_{(m)}|^{\alpha}\big|_{y_3=p_0}\|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}^2_t)} \leq C\|\mathbf{u}_{(m)}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \left(1 + \|\mathbf{u}_{(m)}\|^{\alpha-2}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right). \tag{6.7}$$

Since in the case $\alpha = 2$ (6.7) is obvious by (6.4), we consider the case $\alpha > 2$, *i.e.*, $\alpha = 2 + \delta$ with $\delta > 0$. Then we have

$$\begin{aligned}
&\left| \frac{\partial}{\partial y_i}\left(|\mathbf{u}_{(m)}|^{\alpha}\right)(y^{1\prime}, p_0, t) - \frac{\partial}{\partial y_i}\left(|\mathbf{u}_{(m)}|^{\alpha}\right)(y^{2\prime}, p_0, t) \right|^2 \\
&= \left| \alpha \left[|\mathbf{u}_{(m)}|^{\delta}\mathbf{u}_{(m)} \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_i}(y^{1\prime}, p_0, t) - |\mathbf{u}_{(m)}|^{\delta}\mathbf{u}_{(m)} \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_i}(y^{2\prime}, p_0, t) \right] \right|^2 \\
&\leq C \left[\left| \left|\mathbf{u}_{(m)}\right|^{\delta}\mathbf{u}_{(m)}(y^{1\prime}, p_0, t) - \left|\mathbf{u}_{(m)}\right|^{\delta}\mathbf{u}_{(m)}(y^{2\prime}, p_0, t) \right|^2 \left| \frac{\partial \mathbf{u}_{(m)}}{\partial y_i}(y^{1\prime}, p_0, t) \right|^2 \right. \\
&\qquad \left. + |\mathbf{u}_{(m)}(y^{2\prime}, p_0, t)|^{2\delta+2} \left| \frac{\partial \mathbf{u}_{(m)}}{\partial y_i}(y^{1\prime}, p_0, t) - \frac{\partial \mathbf{u}_{(m)}}{\partial y_i}(y^{2\prime}, p_0, t) \right|^2 \right].
\end{aligned}$$

The first term is estimated by using the mean value theorem, so that

$$\begin{aligned}
&\left| \left|\mathbf{u}_{(m)}\right|^{\delta}\mathbf{u}_{(m)}(y^{1\prime}, p_0, t) - \left|\mathbf{u}_{(m)}\right|^{\delta}\mathbf{u}_{(m)}(y^{2\prime}, p_0, t) \right|^2 \\
&\leq C\Big[|\mathbf{u}_{(m)}(y^{1\prime}, p_0, t)| + |\mathbf{u}_{(m)}(y^{2\prime}, p_0, t)| \Big]^{2\delta} \left| \mathbf{u}_{(m)}(y^{1\prime}, p_0, t) - \mathbf{u}_{(m)}(y^{2\prime}, p_0, t) \right|^2 .
\end{aligned}$$

This makes it possible to get the desired estimate for the first term. The second term and the lower order norm $\left\| |\mathbf{u}_{(m)}|^{\alpha}\big|_{y_3=p_0} \right\|_{L_2(0,t;W_2^1(\mathbf{R}^2))}$ are estimated easily, and finally we have

(6.7). For $\left\| |\mathbf{u}_{(m)}|^\alpha |_{y_3=p_0} \right\|_{W_2^{0,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$, we have

$$\left\| |\mathbf{u}_{(m)}|^\alpha |_{y_3=p_0} \right\|_{L_2(\mathbf{R}^2; W_2^{\frac{1}{4}+\frac{l}{2}}(0,t))} \leq C \left\| |\mathbf{u}_{(m)}|^\alpha |_{y_3=p_0} \right\|_{L_2(\mathbf{R}^2; W_2^1(0,t))}$$
$$\leq C \|\mathbf{u}_{(m)}\|^\alpha_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}.$$

Thus we have (6.6), and consequently

$$\left\| |\mathbf{u}_m|^\alpha \mathbf{u}_m |_{y_3=p_0} \right\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$$
$$\leq (\varepsilon + C_\varepsilon t)\, \|\mathbf{u}_{(m)}\|^3_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \left(1 + \|\mathbf{u}_{(m)}\|^{\alpha-2}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right).$$

Other terms in $\mathbf{l}_2^{(m+1)}$ are estimated in the same way as the corresponding terms in $\mathbf{l}_1^{(m+1)}$. Finally we have

$$\|\mathbf{l}_2^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{st})} \leq (\varepsilon + C_\varepsilon t) \left(\|\mathbf{u}_{(m)}\|^{\alpha+1}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|^3_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \right)$$
$$+ (\varepsilon + C_\varepsilon t)\, P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \left\|\mathbf{u}_{(m+1)}\right\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + C\|\mathbf{v}_0\|_{W_2^{1+l}(\tilde{\Omega})}.$$

Since

$$\|\tilde{G}_{3,h_{(m)}}(\mathbf{u}_{(m+1)}, \mathbf{u}_{(m)})\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_t)}$$
$$\leq \left\| \left(\nabla + \mathbf{a}^3(h_{(m)})\right) \cdot \frac{\partial \mathbf{u}_{(m+1)}}{\partial y_3} + a^{33}(h_{(m)}) \mathbf{F}_5^{(h_{(m)})*} \cdot \frac{\partial \mathbf{u}_{(m)}}{\partial y_3} \right\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Omega}_t)}$$
$$\leq P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \|\mathbf{u}_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}$$
$$+ (\varepsilon + C_\varepsilon t)\, P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \|\mathbf{F}_5^{(h_{(m)})*}\|_{W_2^{2+l',\frac{2+l'}{2}}(\tilde{\Omega}_t)} \|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}$$

by virtue of (6.4) and (6.5), we easily have

$$\|l_3^{(m+1)}\|_{W_2^{1+l,\frac{1}{2}+\frac{l}{2}}(\tilde{\Gamma}_{st})} \leq P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right) \|\mathbf{u}_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}$$
$$+ (\varepsilon + C_\varepsilon t)\, P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \right)$$

$$\times \left[\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \right].$$

For $l_6^{(m+1)}$, we begin with estimating the term $\tilde{G}_{6,h(m)}$. First, it is easy to get

$$\|la(\theta_e)\|^2_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} \leq C\|\!|la|\!\|^2 \left(1 + \|\Psi(\cdot;h_{(m)})\|^2_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2)}\right)^2 \times \left(1 + \|\theta_e\|^2_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)^2.$$

Second, the term containing $|\mathbf{u}_{(m)}|^\alpha$ is estimated in exactly the same way as $\tilde{\mathbf{G}}_2(\mathbf{u}_{(m)})$. Indeed, we have

$$\left\| |\mathbf{u}_m|^\alpha \tilde{\theta}_m \big|_{y_3=p_0} \right\|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2)} \leq (\varepsilon + C_\varepsilon t) \left(1 + \|\tilde{\theta}'_m\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right) \left(\|\mathbf{u}_m\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_m\|^\alpha_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)$$

by making use of (6.7), and we also have

$$\left\| |\mathbf{u}_m|^\alpha \tilde{\theta}_m \big|_{y_3=p_0} \right\|_{W_2^{0,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \leq (\varepsilon + C_\varepsilon t) \left(1 + \|\tilde{\theta}'_m\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right) \|\mathbf{u}_m\|^\alpha_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}.$$

Then we easily obtain

$$\|\tilde{G}_{6,h_{(m)}}(\mathbf{u}_{(m)}, \tilde{\theta}_{(m+1)}, \tilde{\theta}_{(m)})\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \leq (\varepsilon + C_\varepsilon t)\, P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)$$
$$\times \left[1 + \left(1 + \|\tilde{\theta}'_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\left(\|\mathbf{u}_{(m)}\|^\alpha_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{u}_{(m)}\|^2_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right) + \|\tilde{\theta}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right],$$

and finally

$$\|l_6^{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \leq \|(L_{4,h_{(m)}} - L_{4,\bar{h}_0})h'_{(m+1)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$$
$$+ \|L_{4,h_{(m)}}\bar{h}_0\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} + \left\|\frac{\partial \bar{h}_0}{\partial t}\right\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} + \|\tilde{G}_{6,h_{(m)}}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$$
$$\leq (\varepsilon + C_\varepsilon t)\, P\left(\|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right)\left(1 + \|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\tilde{\mathbf{R}}_t^2)}\right)$$

$$\times\left[1+\left(1+\|\tilde{\theta}'_{(m)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\left(\|\mathbf{u}_{(m)}\|^{\alpha}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}+\|\mathbf{u}_{(m)}\|^{2}_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)\right]$$

$$+C\left(1+\|\theta_e\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)}\right)\left(1+\|\tilde{\theta}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\right)+\|d_0\|_{W_2^{\frac{3}{2}+l}(\mathbf{R}^2)}.$$

Introduce the notation

$$E_m(t):=\|(\mathbf{u}'_{(m)},u'_{3(m)},\tilde{\theta}'_{(m)},\tilde{S}'_{(m)},h'_{(m)})\|_{Z(t)},\ \ E'_m(t):=\|(\mathbf{u}'_{(m)},\tilde{\theta}'_{(m)},\tilde{S}'_{(m)})\|_{Z'(t)}.$$

Making use of the above estimates, we arrive at the inequalities

$$E'_{m+1}(t)\le C_1\big[(\varepsilon+C_\varepsilon t)\big\{\phi_1(E_m(t))+\phi_2(E_m(t))E'_{m+1}(t)\big\}+1\big] \tag{6.8}$$

and

$$\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}+\|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_t)}$$

$$\le\tilde{C}_2\Bigg[\phi_3(E_m(t))\left(\|\mathbf{u}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}+\|\tilde{\theta}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\right)$$

$$+(\varepsilon+C_\varepsilon t)\Bigg\{\phi_1(E_m(t))+\phi_2(E_m(t))$$

$$\times\left(\|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)}+\|h'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}^2_t)}\right)\Bigg\}+1\Bigg] \tag{6.9}$$

for any $t\in(0,T]$, where C_1 is the constant in Lemma 5.1, $\tilde{C}_2=C_2+C_3$ with $C_2,\ C_3$ being the constants in Lemmas 5.1 and 5.2, respectively, and $\phi_i\ (i=1,2,3)$ are monotone increasing. Adding (6.8) and (6.9) multiplied by $1/(2\tilde{C}_2\phi_3(E_m(t))$, we get the inequality

$$E_{m+1}(t)\le C_4(t)\big[(\varepsilon+C_\varepsilon t)\big\{\phi_1(E_m(t))+\phi_2(E_m(t))E_{m+1}(t)\big\}+1\big]$$

with some constant $C_4(t)$ depending on t monotonically increasingly.

Let a positive constant M such that $C_4(T)<M$. Take ε first small enough so that

$$\varepsilon C_4(T)\phi_2(M)<1\,,\quad \varepsilon C_4(T)\big[\phi_1(M)+\phi_2(M)M\big]<M-C_4(T)$$

hold, and then $T_1\in(0,T]$ so that

$$C_4(T)C_\varepsilon\phi_2(M)T_1<1-C_4(T)\varepsilon\phi_2(M)\,,$$

$$C_4(T)C_\varepsilon T_1\{\phi_1(M)+\phi_2(M)M\}<M-C_4(T)-\varepsilon C_4(T)\{\phi_1(M)+\phi_2(M)M\}$$

hold. Consequently we obtain

$$E_{m+1}(T_1)<\frac{C_4(T)\big\{(\varepsilon+C_\varepsilon T_1)\phi_1(M)+1\big\}}{1-C_4(T)(\varepsilon+C_\varepsilon T_1)\phi_2(M)}<M$$

from the assumption $E_m(T_1) < M$. By induction the sequence $\{(\mathbf{u}'_{(m)}, u'_{3(m)}, \tilde{\theta}'_{(m)}, \tilde{S}'_{(m)}, h'_{(m)})\}_{m=0}^{\infty}$ is well defined in $Z(T_1)$ and $E_m(T_1) < M$ for $m = 0, 1, 2, \ldots$.

Now we prove its convergence. Subtract (6.1)-(6.3) with m replaced by $(m-1)$ from (6.1)–(6.3). Then

$$\begin{aligned}&(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{u}'_{3(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)}, \tilde{h}'_{(m+1)})\\ &:= (\mathbf{u}'_{(m+1)} - \mathbf{u}'_{(m)}, u'_{3(m+1)} - u'_{3(m)}, \tilde{\theta}'_{(m+1)} - \tilde{\theta}'_{(m)}, \tilde{S}'_{(m+1)} - \tilde{S}'_{(m)}, h'_{(m+1)} - h'_{(m)})\end{aligned}$$

satisfies the equations

$$\begin{cases} \dfrac{\partial \tilde{\mathbf{u}}'_{(m+1)}}{\partial t} - L_{1,\bar{h}_0}\tilde{\mathbf{u}}'_{(m+1)} = \mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}, \\ \nabla_{h_0,3}\tilde{u}'_{3(m+1)} - (\nabla_{h_0,3}\tilde{F}_{13}^{(h_0)*})\dfrac{\tilde{u}'_{3(m+1)}}{\tilde{F}_{13}^{(h_0)*}} = l_3^{(m+1)} - l_3^{(m)}, \\ \dfrac{\partial \tilde{\tilde{\theta}}'_{(m+1)}}{\partial t} - L_{2,\bar{h}_0}\tilde{\tilde{\theta}}'_{(m+1)} = l_4^{(m+1)} - l_4^{(m)}, \\ \dfrac{\partial \tilde{\tilde{S}}'_{(m+1)}}{\partial t} - L_{3,\bar{h}_0}\tilde{\tilde{S}}'_{(m+1)} = l_5^{(m+1)} - l_5^{(m)} \quad \text{in } \tilde{\Omega}_{T_1}, \end{cases}$$

$$\begin{cases} B_{\bar{h}_0}\tilde{\mathbf{u}}'_{(m+1)} = \mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}, \\ (\tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)}) = (\theta_e|_{x_3=\Psi(\cdot;h_{(m)})} - \theta_e|_{x_3=\Psi(\cdot;h_{(m-1)})}, \\ \qquad\qquad S_e|_{x_3=\Psi(\cdot;h_{(m)})} - S_e|_{x_3=\Psi(\cdot;h_{(m-1)})}) \quad \text{on } \tilde{\Gamma}_{sT_1}, \\ (\tilde{\mathbf{u}}'_{(m+1)}, \tilde{u}'_{3(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)}) = (\mathbf{0}, 0, 0, 0) \quad \text{on } \tilde{\Gamma}_{bT_1}, \\ (\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)})|_{t=0} = (\mathbf{0}, 0, 0) \quad \text{on } \tilde{\Omega}, \end{cases}$$

$$\begin{cases} \dfrac{\partial \tilde{h}'_{(m+1)}}{\partial t} - L_{4,\bar{h}_0}\tilde{h}'_{(m+1)} = l_6^{(m+1)} - l_6^{(m)} \quad \text{in } \mathbf{R}^2_{T_1}, \\ \tilde{h}'_{(m+1)}\big|_{t=0} = 0 \quad \text{on } \mathbf{R}^2. \end{cases}$$

Then Lemmas 5.1 and 5.2 yield for any $t \le T_1$ the estimates

$$\begin{aligned}&\|(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)})\|_{Z'(t)}\\ &\le C_1\Big[\|\mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{st})}\\ &\qquad + \|l_4^{(m+1)} - l_4^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)} + \|l_5^{(m+1)} - l_5^{(m)}\|_{W_2^{l,\frac{l}{2}}(\tilde{\Omega}_t)}\end{aligned}$$

$$+ \|\theta_e(y', \Psi(y', t; h_{(m)}), t) - \theta_e(y', \Psi(y', t; h_{(m-1)}), t)\|_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$$

$$+ \|S_e(y', \Psi(y', t; h_{(m)}), t) - S_e(y', \Psi(y', t; h_{(m-1)}), t)\|_{W_2^{\frac{3}{2}+l, \frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg],$$

$$\|\tilde{h}'_{(m+1)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} \leq C_2\|l_6^{(m+1)} - l_6^{(m)}\|_{W_2^{\frac{1}{2}+l, \frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)},$$

$$\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)} \leq C_3\|l_3^{(m+1)} - l_3^{(m)}\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)}.$$

Each term in the right-hand side of the above inequalities except for $\|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\|_{W_2^{\frac{1}{2}+l, \frac{1}{4}+\frac{l}{2}}(\tilde{\Gamma}_{st})}$ and $\|l_6^{(m+1)} - l_6^{(m)}\|_{W_2^{\frac{1}{2}+l, \frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$ can be estimated just as we have done for $\|\mathbf{l}_1^{(m+1)}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)}$, $\|l_i^{(m+1)}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)}$ $(i = 4, 5)$ and $\|l_3^{(m+1)}\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)}$. Then we have

$$\|\mathbf{l}_1^{(m+1)} - \mathbf{l}_1^{(m)}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)} + \sum_{i=4}^{5}\|l_i^{(m+1)} - l_i^{(m)}\|_{W_2^{l, \frac{l}{2}}(\tilde{\Omega}_t)}$$

$$\leq (\varepsilon + C_\varepsilon t)\Bigg[P\Bigg(\big\|\big(\mathbf{u}'_{(m+1)}, \mathbf{u}'_{(m)}, \mathbf{u}'_{(m-1)}\big)\big\|_{W_2^{2+l, 1+\frac{l}{2}}(\tilde{\Omega}_t)}, \|u'_{3(m-1)}\|_{\widetilde{W}_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)},$$

$$\big\|\big(\tilde{\theta}'_{(m+1)}, \tilde{\theta}'_{(m)}\big)\big\|_{W_2^{2+l, 1+\frac{l}{2}}(\tilde{\Omega}_t)}, \big\|\big(\tilde{S}'_{(m+1)}, \tilde{S}'_{(m)}\big)\big\|_{W_2^{2+l, 1+\frac{l}{2}}(\tilde{\Omega}_t)},$$

$$\big\|\big(h'_{(m)}, h'_{(m-1)}\big)\big\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg)\Big\|(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\tilde{\theta}}'_{(m)}, \tilde{\tilde{S}}'_{(m)}, \tilde{h}'_{(m)})\Big\|_{Z(t)},$$

$$+ P\Bigg(\|\tilde{h}'_{(m-1)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg)\Big\|(\tilde{\mathbf{u}}'_{(m+1)}, \tilde{\tilde{\theta}}'_{(m+1)}, \tilde{\tilde{S}}'_{(m+1)})\Big\|_{Z'(t)}\Bigg],$$

$$\|l_3^{(m+1)} - l_3^{(m)}\|_{W_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)}$$

$$\leq (\varepsilon + C_\varepsilon t)\Bigg[P\Bigg(\big\|\big(\mathbf{u}'_{(m+1)}, \mathbf{u}'_{(m)}\big)\big\|_{W_2^{2+l, 1+\frac{l}{2}}(\tilde{\Omega}_t)}, \|u'_{3(m+1)}\|_{\widetilde{W}_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)},$$

$$\big\|\big(h'_{(m)}, h'_{(m-1)}\big)\big\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg)$$

$$\times\Bigg(\|\tilde{\mathbf{u}}'_{(m)}\|_{W_2^{2+l, 1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l, \frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m)}\|_{W_2^{\frac{5}{2}+l, \frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Bigg)$$

$$+ P(\|\tilde{h}'_{(m-1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)})\|\tilde{\mathbf{u}}'_{(m+1)}\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_t)}\Bigg] .$$

In estimating $\left\|l_6^{(m+1)} - l_6^{(m)}\right\|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2)}$, the terms except the one containing $\left|\mathbf{u}_{(m)}\right|^\alpha \tilde{\theta}_{(m)} - \left|\mathbf{u}_{(m-1)}\right|^\alpha \tilde{\theta}_{(m-1)}$ are rather easy to do. Hence we show only its estimates. Let $f(\mathbf{u},\theta) := |\mathbf{u}|^\alpha\theta|_{y_3=p_0}$. Then the mean value theorem implies

$$\begin{aligned}
f(\mathbf{u}_{(m)},&\tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)},\tilde{\theta}_{(m-1)}) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} f\left(s(\mathbf{u}_{(m)},\tilde{\theta}_{(m)}) + (1-s)(\mathbf{u}_{(m-1)},\tilde{\theta}_{(m-1)})\right)\mathrm{d}s \\
&= \int_0^1 \left[\sum_{i=1}^2 \frac{\partial f}{\partial u_i}(\mathbf{u}_s,\theta_s)(u_{(m)i} - u_{(m-1)i}) + \frac{\partial f}{\partial \theta}(\mathbf{u}_s,\theta_s)(\tilde{\theta}_{(m)} - \tilde{\theta}_{(m-1)})\right]\mathrm{d}s \\
&= \int_0^1 \left[\alpha\,|\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s + |\mathbf{u}_s|^\alpha\,\tilde{\tilde{\theta}}'_{(m)}\right]\mathrm{d}s\,,
\end{aligned}$$

where $\mathbf{u}_s = s\mathbf{u}_{(m)} + (1-s)\mathbf{u}_{(m-1)}$, $\theta_s = s\tilde{\theta}_{(m)} + (1-s)\tilde{\theta}_{(m-1)}$. For the estimate $\|f(\mathbf{u}_{(m)},\theta_{(m)}) - f(\mathbf{u}_{(m-1)},\theta_{(m-1)})\|_{L_2(0,t;\dot{W}_2^{\frac{1}{2}+l}(\mathbf{R}^2))}$, it is sufficient to estimate the term

$$\int_0^1 \left[\frac{\partial}{\partial y_i}(|\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s)(y^{1\prime},p_0,t) - \frac{\partial}{\partial y_i}(|\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s)(y^{2\prime},p_0,t)\right]\mathrm{d}s\,.$$

Let

$$\begin{aligned}
G_i(t,y') &:= \frac{\partial}{\partial y_i}\int_0^1 |\mathbf{u}_s|^{\alpha-2}\,\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s\mathrm{d}s \\
&= \int_0^1 \left[(\alpha-2)\,|\mathbf{u}_s|^{\alpha-4}\left(\mathbf{u}_s\cdot\frac{\partial\mathbf{u}_s}{\partial y_i}\right)(\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s \right. \\
&\qquad\left. + |\mathbf{u}_s|^{\alpha-2}\,\frac{\partial}{\partial y_i}((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s)\right]\mathrm{d}s\,,
\end{aligned}$$

and estimate

$$\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \frac{|G_i(y^{1\prime},t) - G_i(y^{2\prime},t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime} \\
&\qquad = \int\int_{|y^{1\prime}-y^{2\prime}|>1} \frac{|G_i(y^{1\prime},t) - G_i(y^{2\prime},t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}
\end{aligned}$$

$$+\iint_{|y^{1\prime}-y^{2\prime}|\le 1} \frac{|G_i(y^{1\prime},t)-G_i(y^{2\prime},t)|^2}{|y^{1\prime}-y^{2\prime}|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}y^{2\prime}. \tag{6.10}$$

The estimate of the second term in (6.10) is more difficult, so that we first estimate it. Denoting $K_i(t,y') := |\mathbf{u}_s|^{\alpha-4}\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_i}\right)(t,y',p_0)$, we have

$$\begin{aligned}
&|G_i(y^{1\prime},t)-G_i(y^{2\prime},t)|^2\\
&\le C\Bigg[\int_0^1 \left|K_i(y^{1\prime},t)-K_i(y^{2\prime},t)\right|^2 |\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{1\prime},p_0,t)|^2|\theta_s(y^{1\prime},p_0,t)|^2\,\mathrm{d}s\\
&+\int_0^1 |K_i(y^{2\prime},t)|^2|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{1\prime},p_0,t)-\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{2\prime},p_0,t)|^2|\theta_s(y^{1\prime},p_0,t)|^2\,\mathrm{d}s\\
&+\int_0^1 |K_i(y^{2\prime},t)|^2|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}(y^{2\prime},p_0,t)|^2|\theta_s(y^{1\prime},p_0,t)-\theta_s(y^{2\prime},p_0,t)|^2\,\mathrm{d}s\\
&+\int_0^1 \left||\mathbf{u}_s|^{\alpha-2}(y^{1\prime},p_0,t)-|\mathbf{u}_s|^{\alpha-2}(y^{2\prime},p_0,t)\right|^2\left|\frac{\partial}{\partial y_i}\Big((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s\Big)(y^{1\prime},p_0,t)\right|^2\mathrm{d}s\\
&+\int_0^1 |\mathbf{u}_s(y^{2\prime},p_0,t)|^{2(\alpha-2)}\bigg|\frac{\partial}{\partial y_i}\Big((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s\Big)(y^{1\prime},p_0,t)\\
&-\frac{\partial}{\partial y_i}\Big((\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)})\theta_s\Big)(y^{2\prime},p_0,t)\bigg|^2\mathrm{d}s\Bigg].
\end{aligned} \tag{6.11}$$

Among the terms in the right-hand side of (6.11), the first term is most difficult due to its singularity, so that we show its estimate.

Take $0<\sigma<1$, which will be determined later. Then, by applying the mean value theorem, there exists $0<\hat{s}<1$ such that

$$\begin{aligned}
&\left|K_i(y^{1\prime},t)-K_i(y^{2\prime},t)\right|^2\\
&\quad=|\nabla K_i(\hat{s}y^{1\prime}+(1-\hat{s})y^{2\prime},t)|^{2\sigma}|y^{1\prime}-y^{2\prime}|^{2\sigma}\left|K_i(y^{1\prime},t)-K_i(y^{2\prime},t)\right|^{2-2\sigma}.
\end{aligned}$$

Taking into account

$$\frac{\partial}{\partial y_j}K_i=(\alpha-4)\,|\mathbf{u}_s|^{\alpha-6}\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_j}\right)\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_i}\right)+|\mathbf{u}_s|^{\alpha-4}\frac{\partial}{\partial y_j}\left(\mathbf{u}_s\cdot\frac{\partial \mathbf{u}_s}{\partial y_i}\right),$$

and putting $z' := y^{1\prime}-y^{2\prime}$ in (6.11), we have for the first term,

$$\int_0^t \mathrm{d}t\iint_{|z'|\le 1}\frac{|K_i(y^{1\prime},t)-K_i(y^{1\prime}+z',t)|^2|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s(y^{1\prime},p_0,t)|^2}{|z'|^{1+2l}}\,\mathrm{d}y^{1\prime}\mathrm{d}z'$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{3}\int_0^t \mathrm{d}t \iint_{|z'|\leq 1}\frac{1}{|z'|^{1+2l}} \\
&\quad\times\Bigg[|\mathbf{u}_s(y^{1\prime}+z', p_0, t)|^{2\sigma(\alpha-4)}\left|\frac{\partial \mathbf{u}_s}{\partial y_j}(y^{1\prime}+z', p_0, t)\right|^{2\sigma}\left|\frac{\partial \mathbf{u}_s}{\partial y_i}(y^{1\prime}+z', p_0, t)\right|^{2\sigma} \\
&\qquad+|\mathbf{u}_s(y^{1\prime}+z', p_0, t)|^{2\sigma(\alpha-3)}\left|\frac{\partial^2 \mathbf{u}_s}{\partial y_i\partial y_j}(y^{1\prime}+z', p_0, t)\right|^{2\sigma}\Bigg] \\
&\quad\times 2\bigg(\sup_{\mathbf{R}_t^2}|\mathbf{u}_s(\cdot, p_0, \cdot)|^{\alpha-3}\sup_{\mathbf{R}^2}\left|\frac{\partial \mathbf{u}_s}{\partial y_i}(\cdot, p_0, t)\right|\bigg)^{2-2\sigma} \\
&\quad\times |z'|^{2\sigma}|\mathbf{u}_s\cdot\tilde{\mathbf{u}}'_{(m)}\theta_s(y^{1\prime}, p_0, t)|^2\,\mathrm{d}z'\mathrm{d}y^{1\prime} \\
&\leq C\sum_{j=1}^{3}\sup_{\mathbf{R}_t^2}|\theta_s(\cdot, p_0, \cdot)|^2\int_{|z'|\leq 1}\frac{|z'|^{2\sigma}}{|z'|^{1+2l}}\,\mathrm{d}z' \\
&\quad\times\Bigg[\sup_{\mathbf{R}_t^2}|\mathbf{u}_s(\cdot, p_0, \cdot)|^{2(\alpha-\sigma-2)}\sup_t\int|\tilde{\mathbf{u}}'_{(m)}(y^{1\prime}, p_0, t)|^2\,\mathrm{d}y^{1\prime} \\
&\qquad\times\left(\int_0^t\sup_{\mathbf{R}^2}\left|\frac{\partial \mathbf{u}_s}{\partial y_i}(\cdot, p_0, t)\right|^{2p}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^t\sup_{\mathbf{R}^2}\left|\frac{\partial \mathbf{u}_s}{\partial y_j}(\cdot, p_0, t)\right|^{2\sigma p'}\mathrm{d}t\right)^{\frac{1}{p'}} \\
&\qquad+\sup_{\mathbf{R}_t^2}|\mathbf{u}_s(\cdot, p_0, \cdot)|^{2(\alpha-2)}\sup_{\mathbf{R}_t^2}|\tilde{\mathbf{u}}'_{(m)}(\cdot, p_0, \cdot)|^2 \\
&\qquad\times\left(\int_0^t\sup_{\mathbf{R}^2}\left|\frac{\partial \mathbf{u}_s}{\partial y_i}(\cdot, p_0, t)\right|^{(2-2\sigma)q}\mathrm{d}t\right)^{\frac{1}{q}}\left(\int_0^t\left\|\frac{\partial^2 \mathbf{u}_s}{\partial y_i\partial y_j}(\cdot, p_0, t)\right\|_{L_2(\mathbf{R}^2)}^{2\sigma q'}\mathrm{d}t\right)^{\frac{1}{q'}}\Bigg].
\end{aligned}
\tag{6.12}
$$

Here we applied Hölder inequality, and $1/p + 1/p' = 1/q + 1/q' = 1$. It is to be noted that the integral $\int_{|z'|\leq 1}\frac{|z'|^{2\sigma}}{|z'|^{1+2l}}\,\mathrm{d}z' = \int_0^1\frac{r^{2\sigma+1}}{r^{1+2l}}\,\mathrm{d}r$ determines as a finite value for $\sigma > l - 1/2$ and $W_2^{\frac{l}{2}-\frac{1}{4}}(0,t) \subset L_{2p}(0,t)\bigcap L_{2\sigma p'}(0,t)\bigcap L_{(2-2\sigma)q}(0,t)\bigcap L_{2\sigma q'}(0,t)$ with $\sigma \leq \frac{2\eta}{1-2\eta}$, $\eta = l/2 - 1/4$, $1 \leq p \leq \frac{1}{1-2\eta}$, $\frac{1}{1-2\sigma\eta'} \leq q < \frac{1}{(2-2\sigma)\eta'}$, $\eta' = 3/4 - l/2$. Now if we take σ such that $l - 1/2 < \sigma \leq \min\{\alpha - 2, \frac{2\eta}{1-2\eta}\}$, then the right most hand side of (6.12) is determined as a finite value. The first term in (6.10) can be estimated in the same manner by taking $\sigma = 0$. These give an estimate of $\| f(\mathbf{u}_{(m)}, \tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)},$

$\tilde{\theta}_{(m-1)})\|_{L_2(0,t;\dot{W}_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2))}$. Adding this to the lower order norms, which are easy to get, yields the estimate of $\|f(\mathbf{u}_{(m)},\tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)},\tilde{\theta}_{(m-1)})\|_{W_2^{\frac{1}{2}+l,0}(\mathbf{R}_t^2)}$. Similarly one can obtain the estimate $\|f(\mathbf{u}_{(m)},\tilde{\theta}_{(m)}) - f(\mathbf{u}_{(m-1)},\tilde{\theta}_{(m-1)})\|_{W_2^{0,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}$. Consequently, we have

$$\begin{aligned}
\|l_6^{(m+1)} - l_6^{(m)}\|_{W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} &\le (\varepsilon + C_\varepsilon t)\bigg[P\bigg(\big\|\big(\mathbf{u}'_{(m)},\mathbf{u}'_{(m-1)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)},\\
&\quad \big\|\big(\tilde{\theta}'_{(m)},\tilde{\theta}'_{(m-1)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}, \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\bigg)\\
&\quad \times\bigg(\|\tilde{\mathbf{u}}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{\theta}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\bigg)\\
&\quad + P\bigg(\|h'_{(m-1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\bigg)\bigg(\|\tilde{h}'_{(m+1)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)} + \|\tilde{\tilde{\theta}}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\bigg)\bigg].
\end{aligned}$$

$\Big\|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\Big\|_{W_2^{\frac{1}{2}+l,\frac{l}{2}+\frac{1}{4}}(\tilde{\Gamma}_{st})}$ is estimated in exactly the same way as above,

$$\begin{aligned}
\Big\|\mathbf{l}_2^{(m+1)} - \mathbf{l}_2^{(m)}\Big\|_{W_2^{\frac{1}{2}+l,\frac{l}{2}+\frac{1}{4}}(\tilde{\Gamma}_{st})} &\le (\varepsilon + C_\varepsilon t)\, P\bigg(\big\|\big(\mathbf{u}'_{(m)},\mathbf{u}'_{(m-1)}\big)\big\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)},\\
&\quad \|h'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\bigg)\bigg(\|\tilde{\mathbf{u}}'_{(m)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m)}\|_{W_2^{\frac{5}{2}+l,\frac{5}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\bigg).
\end{aligned}$$

In this case, we need only $\alpha > l + 1/2$.

Denoting

$$\tilde{E}_m(t) := \|(\tilde{\mathbf{u}}'_{(m)},\tilde{u}'_{3(m)},\tilde{\tilde{\theta}}'_{(m)},\tilde{\tilde{S}}'_{(m)},\tilde{h}'_{(m)})\|_{Z(t)},\quad \tilde{E}'_m(t) := \|(\tilde{\mathbf{u}}'_{(m)},\tilde{\tilde{\theta}}'_{(m)},\tilde{\tilde{S}}'_{(m)})\|_{Z'(t)}$$

we get for any $t \in (0, T_1]$,

$$\begin{aligned}
\tilde{E}'_{m+1}(t) &\le C_1(\varepsilon + C_\varepsilon t)\Big[\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}'_m(t)\\
&\quad + \phi_5(E_{m-1}(T_1))\tilde{E}'_{m+1}(t) + \phi_6(E_{m+1}(T_1) + E_m(T_1))\|\tilde{h}'_{(m)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\Big], \quad (6.13)
\end{aligned}$$

$$\begin{aligned}
&\|\tilde{u}'_{3(m+1)}\|_{\tilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|\tilde{h}'_{(m+1)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\\
&\quad \le \tilde{C}_2\bigg[\phi_7(E_m(T_1))\bigg(\|\tilde{\mathbf{u}}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)} + \|\tilde{\tilde{\theta}}'_{(m+1)}\|_{W_2^{2+l,1+\frac{l}{2}}(\tilde{\Omega}_t)}\bigg)\\
&\qquad + (\varepsilon + C_\varepsilon t)\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}_m(t)
\end{aligned}$$

$$+ (\varepsilon + C_\varepsilon t)\phi_5(E_{m-1}(T_1))\left\{\|\tilde{u}'_{3(m+1)}\|_{\widetilde{W}_2^{1+l,\frac{1+l}{2}}(\tilde{\Omega}_t)} + \|h'_{(m+1)}\|_{W_2^{\frac{3}{2}+l,\frac{3}{4}+\frac{l}{2}}(\mathbf{R}_t^2)}\right\}\Bigg], \tag{6.14}$$

where ϕ_i $(i = 4, 5, 6, 7)$ are monotonically increasing in their arguments.

Adding (6.13) and (6.14) multiplied by $1/(2\tilde{C}_2\phi_7(E_m(T_1)))$, we get the estimate

$$\begin{aligned}\tilde{E}_{m+1}(t) \le C_5(T_1)(\varepsilon + C_\varepsilon t)\big\{&\phi_4\big(E_{m+1}(T_1) + E_m(T_1) + E_{m-1}(T_1)\big)\tilde{E}_m(t)\\ &+\phi_5(E_{m-1}(T_1))\tilde{E}_{m+1}(t) + \phi_6(E_{m+1}(T_1) + E_m(T_1))\tilde{E}_m(t)\big\}\end{aligned} \tag{6.15}$$

for any $t \in (0, T_1]$ with $C_5(t)$ having the same property as $C_4(t)$. Take ε small enough again so that

$$\varepsilon C_5(T_1)\big[\phi_4(3M) + \phi_5(M) + \phi_6(2M)\big] < 1$$

holds, and then $T_2 \in (0, T_1]$ so that

$$C_5(T_1)C_\varepsilon\phi_5(M)T_2 < 1 - C_5(T_1)\varepsilon\phi_5(M),$$

$$C_5(T_1)C_\varepsilon\,[\phi_4(3M) + \phi_5(M) + \phi_6(2M)]\,T_2 < 1 - \varepsilon C_5(T_1)\,[\phi_4(3M) + \phi_5(M) + \phi_6(2M)]$$

hold. For these ε and T_2, we obtain

$$\tilde{E}_{m+1}(T_2) \le r\tilde{E}_m(T_2), \quad r = \frac{C_5(T_1)(\varepsilon + C_\varepsilon T_2)\big[\phi_4(3M) + \phi_6(2M)\big]}{1 - C_5(T_1)(\varepsilon + C_\varepsilon T_2)\phi_5(M)} \in (0, 1).$$

Then we can verify that $\{(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\bar{\theta}}'_{(m)}, \tilde{\bar{S}}'_{(m)}, \tilde{h}'_{(m)})\}_{m=0}^\infty$ is a Cauchy sequence in $Z(T_2)$. Therefore the limit

$$(\tilde{\mathbf{u}}', \tilde{u}'_3, \tilde{\bar{\theta}}', \tilde{\bar{S}}', \tilde{h}') = \lim_{m\to\infty}(\tilde{\mathbf{u}}'_{(m)}, \tilde{u}'_{3(m)}, \tilde{\bar{\theta}}'_{(m)}, \tilde{\bar{S}}'_{(m)}, \tilde{h}'_{(m)})$$

exists in $Z(T_2)$, which is our desired solution.

Now we shall show that $0 < \underline{\theta}_0/2 \le \tilde{\theta}(y, t) < \infty$ and $0 < \underline{S}_0 \le \tilde{S}(y, t) < \infty$ holds by taking the time interval small enough again. Since $\tilde{\theta}' = \tilde{\theta} - \tilde{\bar{\theta}}_0 \in W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_T)$, we have

$$\begin{aligned}\tilde{\theta}(y, t) &\ge \tilde{\bar{\theta}}_0\big|_{t=0}(y) - \Big(|\tilde{\theta}'(y, t)| + |\tilde{\bar{\theta}}_0(y, t) - \tilde{\bar{\theta}}_0(y, 0)|\Big)\\ &\ge \underline{\theta}_0 - t^\gamma\left(\sup_{y\in\tilde{\Omega}}|\tilde{\theta}'(y, \cdot)|_t^{(\gamma)} + \sup_{y\in\tilde{\Omega}}|\tilde{\bar{\theta}}_0(y, \cdot)|_t^{(\gamma)}\right),\end{aligned}$$

where $|f|_t^{(\gamma)}$ stands for the Hölder coefficient of f with respect to t with exponent $0 < \gamma < \frac{l}{2} - \frac{1}{4}$. Note that Sobolev embedding inequality leads to $\sup_{y\in\tilde{\Omega}}|\tilde{\theta}'(y, \cdot)|_t^{(\gamma)} \le \|\tilde{\theta}'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T_2})}$,

$\sup_{y\in\tilde{\Omega}} |\bar{\bar{\theta}}_0(y,\cdot)|_t^{(\gamma)} \le \|\bar{\bar{\theta}}'_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T_2})}$. If we take

$$T_3 = \left(\frac{\underline{\theta}_0}{2\left(\|\tilde{\theta}'\|_{W_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T_2})} + \|\bar{\bar{\theta}}'_0\|_{\overline{W}_2^{2+l,\frac{2+l}{2}}(\tilde{\Omega}_{T_2})} \right)} \right)^{\frac{1}{\gamma}},$$

then we have $\underline{\theta}_0/2 < \tilde{\theta}(y,t) < \infty$ on $[0, T_3]$. A similar argument holds for $\tilde{S}$, and we again denote the time interval by $[0, T_3]$ on which $\underline{\theta}_0/2 < \tilde{\theta}(y,t) < \infty$ and $\underline{S}_0/2 < \tilde{S}(y,t) < \infty$ hold. $T^* = \min\{T_2, T_3\}$ provides the desired result.

Uniqueness of the solution can be proved by virtue of an analogous inequality to (6.15). This completes the proof of the main theorem.

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