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Pictures and Littlewood-Richardson Crystals

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Abstract. We shall describe the one-to-one correspondence between the set of pictures and the set of Littlewood-Richardson crystals.

1. Introduction

The notion of pictures is initiated by James and Peel [6] and Zelevinsky [12], which is roughly a bijective map between two skew Young diagrams with certain conditions (See Sect.2). Let λ, μ, ν be Young diagrams with $|\mu| = |\nu \setminus \lambda|$ and denote the set of pictures from μ to $\nu \setminus \lambda$ by $\mathbf{P}(\mu, \nu \setminus \lambda)$. Then one has the following remarkable result:

$$\sharp\mathbf{P}(\mu, \nu \setminus \lambda) = c^{\nu}_{\lambda,\mu}, \tag{1.1}$$

where $c^{\nu}_{\lambda,\mu}$, is the usual Littlewood-Richardson number, which is shown in [4].

The theory of crystal bases is introduced by Kashiwara ([7], [8]), which is widely applied to many areas in mathematics and physics, in particular, combinatorial representation theory. In [10], it is revealed that crystal bases for classical Lie algebras are presented by 'Young tableaux' and in [11] by the first author it is shown that so-called Littlewood Richardson rule for tensor products of representations are described by crystal bases (see Sect.3). So, together with (3.1) we deduced certain one to one correspondence between pictures and crystal bases, which is given in Theorem 4.1.

This article is organized as follows. In Sect.2, we introduce pictures. In Sect.3, we review the crystal bases of type A_n and the description of Littlewood-Richardson rule in terms of crystal bases. In Sect.4, we shall state the main theorem, namely, we shall give an explicit one to one correspondence between pictures and Littlewood-Richardson crystals of type A_n. In the subsequent three sections, we shall give a proof of the theorem. In the last section, we shall generalize the notion of pictures and give certain conjecture on it.

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2. Young Tableaux and Pictures

2.1. Young Tableaux. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a Young diagram or a partition, which satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. Define $|\lambda| := \lambda_1 + \cdots + \lambda_m$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. We usually write a Young diagram by using square boxes:

EXAMPLE 2.1. For $\lambda = (2, 2, 1)$, write $\lambda = \ydiagram{2,2,1}$.

In this article we frequently use the following coordinated expression for a Young diagram, that is, we identify a Young diagram with a subset of $\mathbb{N} \times \mathbb{N}$:

$$\begin{ytableau} \none & \none[1] & \none[2] & \none[3] & \none[4] \\ \none[1] & & & & \\ \none[2] & & & a \\ \none[3] & & \\ \none[4] \end{ytableau}$$

. In this diagram, the coordinate of a is $(2, 3)$.

EXAMPLE 2.2. For a Young diagram $\lambda = \ydiagram{2,2,1}$, its coordinated expression is $\lambda = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1)\}$.

DEFINITION 2.3. A numbering of a Young diagram λ is called a *Young tableau* of shape λ if it satisfies.

(i) In each row, all entries weakly increase from left to right.
(ii) In each column, all entries increase from top to bottom.

Note that it is also called 'semi-standard tableau'. In this article, we prefer Young tableau to semi-standard tableau following [4].

For a Young tableau T of shape λ, we also consider a coordinate like as λ. Then an entry of T in (i, j) is denoted by $T_{i,j}$ and called (i, j)-entry. For $k > 0$, define

$$T^{(k)} = \{(l, m) \in \lambda | T_{l,m} = k\}. \tag{2.1}$$

REMARK. Note that in $T^{(k)}$, there is no two elements in one column. Thus, we can write

$$T^{(k)} = \{(a_1, b_1),\ (a_2, b_2), \ldots, (a_m, b_m)\},$$

with $a_1 \leq a_2 \leq \cdots a_m$ and $b_1 > b_2 > \cdots > b_m$. If (i, j)-entry in a tableau T is k and $(i, j) = (a_p, b_p)$ in $T^{(k)}$ as above, we define a function $p(T; i, j)$ by

$$p(T; i, j) = p, \tag{2.2}$$

that is, $p(T; i, j)$ is the number of (i, j)-entry from the right in $T^{(k)}$. It is immediate from the definition:

$$\text{If } T_{i,j} = T_{x,y} \text{ and } p(T; i, j) = p(T; x, y), \text{ then } (i, j) = (x, y). \tag{2.3}$$

EXAMPLE 2.4. For $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, we have $T_{1,1} = 1$, $T_{1,2} = 2$, $T_{2,1} = 2$, $T_{2,2} = 3$, $T_{3,1} = 4$, $p(T; 1, 2) = 1$, $p(T; 2, 1) = 2$.

DEFINITION 2.5. Let λ and μ be Young diagrams with $\mu \subset \lambda$. A *skew diagram* $\lambda \setminus \mu$ is obtained by removing μ from λ.

EXAMPLE 2.6. For $\lambda = (2, 2)$, $\mu = (1)$, we have $\lambda \setminus \mu = \begin{array}{cc} \cline{2-2} \multicolumn{1}{c|}{} & \\ \hline \multicolumn{1}{|c|}{} & \\ \hline \end{array}$.

2.2. Picture. Now, let us introduce the notion of pictures.

DEFINITION 2.7. (Orders $<_P$ and $<_J$) We define the following two kinds of orders on a subset $X \subset \mathbb{N} \times \mathbb{N}$: For (a, b), $(c, d) \in X$,
(1) $\leq_P$: $(a, b) \leq_P (c, d)$ iff $a \leq c$ and $b \leq d$.
(2) $\leq_J$: $(a, b) \leq_J (c, d)$ iff $a < c$, or $a = c$ and $b \geq d$.

Note that the order $\leq_P$ is a partial order and $\leq_J$ is a total order.

DEFINITION 2.8 ([12]). Let $X, Y \subset \mathbb{N} \times \mathbb{N}$.
(1) A map $f : X \to Y$ is said to be *PJ-standard* if it satisfies

$$\text{For } (a, b), (c, d) \in X, \text{ if } (a, b) \leq_P (c, d), \text{ then } f(a, b) \leq_J f(c, d)\,.$$

(2) A map $f : X \to Y$ is a *picture* if it is bijective and both f and f^{-1} are PJ-standard.

Taking three Young diagrams $\lambda, \mu, \nu \subset \mathbb{N} \times \mathbb{N}$, denote the set of pictures by:

$$\mathbf{P}(\mu, \nu \setminus \lambda) := \{f : \mu \to \nu \setminus \lambda \mid f \text{ is a picture}\}\,.$$

For a picture $f \in \mathbf{P}(\mu, \nu \setminus \lambda)$, we define f_1 (resp. f_2) to be the map from μ to $\mathbb{N}$ given as a row (resp. column) number of the image of f in $\nu \setminus \lambda$.

3. Crystal Bases and Young tableaux

We shall review the theory of crystal bases briefly (for more details, see [7, 8, 10]). Let $\mathfrak{g}$ be a semisimple Lie algebra and $U_q(\mathfrak{g})$ the associated quantum algebra generated by $\{e_i, f_i, t_i^{\pm} | i \in \{1, \ldots, n := \mathrm{rank}(\mathfrak{g})\}\}$ with certain commutation relations. Let M be a finite dimensional $U_q(\mathfrak{g})$-module. We can define linear maps $\tilde{e}_i$, $\tilde{f}_i \in \mathrm{End}(M)$ $(i \in \{1, \ldots, n\})$, which are called the *Kashiwara operators* ([7, 8]). Then, using the Kashiwara operators, we obtain the crystal base for arbitrary finite dimensional $U_q(\mathfrak{g})$-module. One of the most remarkable properties of crystal bases is the tensor product structure. More detailed statement is as follows: let M_1, M_2 be finite dimensional $U_q(\mathfrak{g})$-modules and B_1, B_2 their crystal bases. Then $B_1 \otimes B_2$ turns out to be the crystal bases of $M_1 \otimes M_2$.

Let $V(\lambda)$ be the irreducible finite-dimensional $U_q(\mathfrak{g})$-module with highest weight λ and $B(\lambda)$ the associated crystal base. Due to the tensor product structure, the crystal base $B(\lambda)$ of type A_n (and of other classical types) is realized in terms of Young tableaux ([10]) as follows.

Let Λ_i (resp. α_i, h_i)($i = 1, 2, \ldots, n$) be the fundamental weight (resp. simple root, simple coroot) of type A_n. Let $B_1 := \{\boxed{i} | i = 1, 2, \ldots, n+1\}$ be the crystal of type A_n for the fundamental weight Λ_1. A dominant, weight λ is identified with a Young diagram in usual way. Then, we use the same notation for a dominant weight and the corresponding Young diagram. Let λ be a Young diagram with a depth at most n and $|\lambda| = N$. Then the crystal $B(\lambda)$ is embedded in $B_1^{\otimes N}$ and realized by Young tableaux ([10]) . This embedding, say *reading*, is not unique. Now, we introduce two of them. One is the *middle-eastern reading* and the other is the *far-eastern reading* ([5]).

DEFINITION 3.1. Let T be a Young tableau of shape λ with entries $\{1, 2, \ldots, n+1\}$.

(i) We read the entries in T each row from right to left and from the top row to the bottom row. Then the resulting sequence of the entries $i_1, i_2, \ldots, i_N$ gives the embedding of crystals:

$$B(\lambda) \hookrightarrow B_1^{\otimes N} \quad (T \mapsto \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}),$$

which is called middle-eastern reading and denoted by ME.

(ii) We read the entries in T each column from the top to the bottom and from the right-most column to the left-most column. Then the resulting sequence of the entries $i_1, i_2, \ldots, i_N$ gives the embedding of crystals:

$$B(\lambda) \hookrightarrow B_1^{\otimes N} \quad (T \mapsto \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}),$$

which is called far-eastern reading and denoted by FE.

EXAMPLE 3.2. For a Young tableau $T = \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\ \hline 2&3&4 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array}$, we have

$$\mathrm{ME}(T) = \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{4} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{5},$$
$$\mathrm{FE}(T) = \boxed{3} \otimes \boxed{2} \otimes \boxed{4} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{5}.$$

DEFINITION 3.3. (Addition) For $i \in \{1, 2, \ldots, n+1\}$ and a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we define

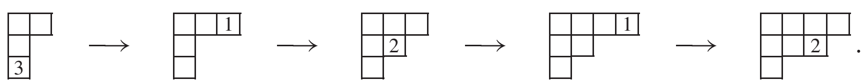
$$\lambda[i] := (\lambda_1, \lambda_2, \ldots, \lambda_i + 1, \ldots, \lambda_n)$$

which is said to be an *addition* of i to λ. In general, for $i_1, i_2, \ldots, i_N \in \{1, 2, \ldots, n+1\}$ and a Young diagram λ, we define

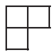
$$\lambda[i_1, i_2, \ldots, \imath_N] := (\cdots((\lambda[i_1])[i_2])\cdots)[i_N],$$

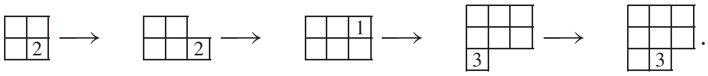
which is called an *addition* of $i_1, \ldots, i_N$ to λ.

EXAMPLE 3.4. For a sequence $\mathbf{i} = 31212$, the addition of $\mathbf{i}$ to $\lambda = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$ is:



REMARK. For a Young diagram λ, an addition $\lambda[i_1, \dots, i_N]$ is not necessarily a Young diagram.

EXAMPLE 3.5. For a sequence $\mathbf{i}' = 22133$, the addition of $\mathbf{i}'$ to $\lambda =$  is



Then we see that $\lambda[2, 2]$ is not a Young digram.

3.1. Littlewood-Richardson rule. As an application of the description of crystal bases of type A_n, we see so-called "Littlewood-Richardson rule" of type A_n.

For a sequence $i_1, i_2, \dots, i_N \in \{1, 2, \dots, n+1\}$ and a Young diagram λ, let $\tilde{\lambda} := \lambda[i_1, i_2, \dots, i_N]$ be an addition of $i_1, i_2, \dots, i_N$ to λ. Then set

$$\mathbf{B}(\tilde{\lambda}) = \begin{cases} B(\tilde{\lambda}) & \text{if } \lambda[i_1, \dots, i_k] \text{ is a Young diagram for any } k = 1, 2, \dots, N\,, \\ \emptyset & \text{otherwise}\,. \end{cases}$$

THEOREM 3.6 ([11]). *Let λ and μ be Young diagrams with at most n rows. Then we have*

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) \cong \bigoplus_{\substack{T \in \mathbf{B}(\mu),\ \text{where} \\ \mathrm{FE}(T) = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}}} \mathbf{B}(\lambda[i_1, i_2, \dots, i_N])\,. \tag{3.1}$$

Note that this also holds for ME.

Let $c^{\nu}_{\lambda,\mu}$ be the multiplicity of $\mathbf{B}(\nu)$ in $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$, which is denoted by $c^{\nu}_{\lambda,\mu}$ and called the Littlewood-Richardson number. We have the following:

THEOREM 3.7 ([4]). $\sharp\mathbf{P}(\mu, \nu \setminus \lambda) = c^{\nu}_{\lambda,\mu}$.

For Young diagrams λ, μ, ν, we define

$$\mathbf{B}(\mu)^{\nu}_{\lambda} := \left\{ T \in \mathbf{B}(\mu) \;\middle|\; \begin{array}{l} \text{For any } k = 1, \dots, N, \\ \lambda[i_1, \dots, i_k] \text{ is a Young diagram and} \\ \lambda[i_1, \dots, i_N] = \nu, \text{ where} \\ \mathrm{ME}(T) = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k} \otimes \cdots \otimes \boxed{i_N}\,. \end{array} \right\},$$

whose element is called a *Littlewood-Richardson crystal* with respect to a triplet (λ, μ, ν). Then by Theorem 3.6, we have

COROLLARY 3.8. $\sharp\mathbf{P}(\mu, \nu \setminus \lambda) = \sharp\mathbf{B}(\mu)^{\nu}_{\lambda}$.

We shall see an explicit one-to-one correspondence between $\mathbf{P}(\mu, \nu \setminus \lambda)$ and $\mathbf{B}(\mu)^{\nu}_{\lambda}$ in the next section.

4. Main Theorem

For Young diagrams λ, μ, ν, we have two sets: $\mathbf{P}(\mu, \nu \setminus \lambda)$ and $\mathbf{B}(\mu)^{\nu}_{\lambda}$. In case $|\lambda|+|\mu| = |\nu|$, we define the following map $\Phi : \mathbf{P}(\mu, \nu \setminus \lambda) \to \mathbf{B}(\mu)^{\nu}_{\lambda}$: For $f = (f_1, f_2) \in \mathbf{P}(\mu, \nu \setminus \lambda)$, set

$$\Phi(f)_{i,j} := f_1(i, j)\,,$$

that is, $\Phi(f)$ is a filling of shape μ and its (i, j)-entry is given as $f_1(i, j)$.

Furthermore, for a crystal $T \in \mathbf{B}(\mu)^{\nu}_{\lambda}$, define a map $\Psi : \mathbf{B}(\mu)^{\nu}_{\lambda} \to \mathbf{P}(\mu, \nu \setminus \lambda)$ by

$$\Psi(T) : (i, j) \in \mu \mapsto (T_{i,j}, \lambda_{T_{i,j}} + p(T; i, j)) \in \nu \setminus \lambda\,,$$

where $p(T; i, j)$ as in (2.2).

The following is the main theorem in this article.

THEOREM 4.1. *For Young diagrams λ, μ, ν as above, the map $\Phi : \mathbf{P}(\mu, \nu \setminus \lambda) \to \mathbf{B}(\mu)^{\nu}_{\lambda}$ is a bijection and the map Ψ is the inverse of Φ.*

EXAMPLE 4.2. Take $\lambda = (3, 1, 1) = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array}$, $\mu = (3, 2) = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$ and $\nu = (4, 3, 2, 1) = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ \\ \cline{1-3} \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}$. As subsets in $\mathbb{N} \times \mathbb{N}$, we have

$\mu = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2)\}$, $\nu \setminus \lambda = \{(1, 4), (2, 2), (2, 3), (3, 2), (4, 1)\}$.

In this case $\sharp\mathbf{P}(\mu, \nu \setminus \lambda) = 2$. Set $\mathbf{P}(\mu, \nu \setminus \lambda) = \{f, f'\}$ and their explicit forms are

$$f = \begin{array}{c||c|c|c|c|c} \mu & (1,1) & (1,2) & (1,3) & (2,1) & (2,2) \\ \hline \nu \setminus \lambda & (1,4) & (2,3) & (2,2) & (3,2) & (4,1) \end{array}$$

$$f' = \begin{array}{c||c|c|c|c|c} \mu & (1,1) & (1,2) & (1,3) & (2,1) & (2,2) \\ \hline \nu \setminus \lambda & (1,4) & (2,2) & (4,1) & (2,3) & (3,2) \end{array}.$$

We have

$$\mathbf{B}(\mu)^{\nu}_{\lambda} = \{T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2}\end{array},\ T' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}\}\,,$$

and $\Phi(f) = T$, $\Phi(f') = T'$.

In the subsequent sections, let us give the proof of Theorem 4.1, which consists in the following three steps:

(i) Well-definedness of the map Φ.
(ii) Well-definedness of the map Ψ.
(iii) Bijectivity of Φ and $\Psi = \Phi^{-1}$.

5. Well-definedness of Φ

For the well-definedness of Φ, it suffices to show:

PROPOSITION 5.1. *Let λ, μ and ν be Young diagrams with $|\lambda| + |\mu| = |\nu|$.*

(i) *For any $f \in \mathbf{P}(\mu, \nu \backslash \lambda)$, $\Phi(f)$ is a Young tableau of shape μ, that is, $\Phi(f) \in \mathbf{B}(\mu)$.*

(ii) *Writing* $\mathrm{ME}(\Phi(f)) = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k} \otimes \cdots \otimes \boxed{i_N}$, *for any $k = 1, \ldots, N$, $\lambda[i_1, i_2, \ldots, i_k]$ is a Young diagram and $\lambda[i_1, \ldots, i_N] = \nu$.*

5.1. Proof of Proposition 5.1(i). For $f \in \mathbf{P}(\mu, \nu \setminus \lambda)$, it is immediate from the definition of Φ that the shape of $\Phi(f)$ is μ. Next, in order to see that $\Phi(f)$ is a Young tableau, we may show

(a) $\Phi(f)_{i,j} \leq \Phi(f)_{i,j+1}$.

(b) $\Phi(f)_{i,j} < \Phi(f)_{i+1,j}$.

By the definition of Φ, one has

$$\Phi(f)_{i,j} = f_1(i, j), \quad \Phi(f)_{i,j+1} = f_1(i, j+1).$$

Since $(i, j) <_P (i, j+1)$ and f is a picture,

$$(f_1(i, j), f_2(i, j)) <_J (f_1(i, j+1), f_2(i, j+1)).$$

Then, by the definition of $<_J$ one gets

$$\Phi(f)_{i,j} = f_1(i, j) \leq f_1(i, j+1) = \Phi(f)_{i,j+1}$$

which shows (a).

By the definition of Φ again, one has

$$\Phi(f)_{i,j} = f_1(i, j) \text{ and } \Phi(f)_{i+1,j} = f_1(i+1, j).$$

Since $(i, j) <_P (i+1, j)$ and f is a picture,

$$(f_1(i, j), f_2(i, j)) <_J (f_1(i+1, j), f_2(i+1, j)),$$

which implies $f_1(i, j) \leq f_1(i+1, j)$. Here, suppose that $f_1(i, j) = f_1(i+1, j)$. It follows from the definition of $<_J$ that

$$f_2(i, j) > f_2(i+1, j).$$

This means

$$(f_1(i, j), f_2(i, j))_P > (f_1(i+1, j), f_2(i+1, j)).$$

Since f is a picture, applying f^{-1} to this one has

$$(i, j)_J > (i+1, j),$$

which derives a contradiction. Thus, one gets $f_1(i, j) < f_1(i+1, j)$, that is, $\Phi(f)_{i,j} < \Phi(f)_{i+1,j}$, or equivalently, (b). Now, we obtain $\Phi(f) \in B(\mu)$.

5.2. Addition and Picture. Before showing Proposition 5.1(2), we prepare the lemma as below:

LEMMA 5.2. *Let $f : \mu \to \nu \setminus \lambda$ be a picture and set* $\mathrm{ME}(\Phi(f)) = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k} \otimes \cdots \otimes \boxed{i_N}$. *Let $(p_k, q_k) \in \mu$ be the place of $\boxed{i_k}$ in $\Phi(f) \in B(\mu)$ and $(a_k, b_k) \in \nu$ the place of the k-th addition in $\lambda[i_1, \ldots, i_N]$. Then we have $f(p_k, q_k) = (a_k, b_k)$ for any $k = 1, \ldots, N$.*

EXAMPLE 5.3. For a picture $f =$

μ	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$
$\nu \setminus \lambda$	$(1,4)$	$(2,3)$	$(2,2)$	$(3,2)$	$(4,1)$

as in Example 4.2, we have $\Phi(f) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}$ and $\mathrm{ME}(\Phi(f)) = \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{3}$. Now, let us see the second $\boxed{i_2} = \boxed{2}$. This is added to the second row of λ by the addition:

$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & 2 \\ \cline{1-3} & \\ \cline{1-2}\end{array}$, and then it is placed in $(2,3) \in \nu$.

The place of $\boxed{i_2} = \boxed{2}$ in μ is $(1,2)$ and $f(1,2) = (2,3)$.

PROOF. Set $m := i_k$. List the m-th row in $\nu \backslash \lambda$ according to the order $<_P$:

$$(m, \lambda_m + 1) <_P (m, \lambda_m + 2) <_P \cdots <_P (m, \lambda + c_m) = (m, \nu_m)\,.$$

Since f is a picture, one has

$$f^{-1}(m, \lambda_m + 1) <_J f^{-1}(m, \lambda_m + 2) <_J \cdots <_J f^{-1}(m, \nu_m)\,.$$

Since the middle-eastern reading follows the order $<_J$, $(p, q) := f^{-1}(m, \lambda_m + j)$ $(j = 1, \ldots, \nu_m - \lambda_m)$ is added j-th to the m-th row of λ, which implies that the entry in $(p, q) \in \mu$ is added to $(m, \lambda_m + j) \in \nu$.

On the other-hand $f(p, q) = f(f^{-1}(m, \lambda_m + j)) = (m, \lambda_m + j)$, which completed the proof of the lemma. □

5.3. Proof of Proposition 5.1 (ii). Due to the definition of Φ, it is easy to see that the number of entry i $(i = 1, \ldots, n+1)$ is equal to $\nu_i - \lambda_i$, which implies $\lambda[i_1, \ldots, i_N] = \nu$.

Writing $\mathrm{ME}(\Phi(f)) = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k} \otimes \cdots \otimes \boxed{i_N}$, let us show that $\lambda[i_1, \ldots, i_k]$ is a Young diagram for any k by the induction on k.

In case $k = 1$. Denote $\Phi(f)$ by T. Let us show that $\lambda[i_1] = \lambda[T_{1,\mu_1}]$ is a Young diagram. Since $(1, \mu_1)$ is the minimum element in μ with respect to the order $<_J$ and f is a picture, $f(1, \mu_1) = (f_1(1, \mu_1), f_2(1, \mu_1))$ must be minimal with respect to the order $<_P$. Set $s := f_1(1, \mu_1)$. Then, by Lemma 5.2 we have $f_2(1, \mu_1) = \lambda_s + 1$. Assume that there is no box above $(s, \lambda_s + 1)$ in $\lambda[i_1]$. $\nu = \lambda[i_1, \ldots, i_N]$ is a Young diagram, which means that there is some j such that i_j is added above $(s, \lambda_s + 1)$. Since $f(1, \mu_1)$ is minimal in $\nu \backslash \lambda$ with respect to $<_P$, in the addition of $\mathrm{ME}(\Phi(f))$ to λ, nothing is added above $(s, \lambda_s + 1)$ after i_1, which derives a contradiction. Then, we know that there is originally a box above $(s, \lambda_s + 1)$ and then shows

$$\lambda_{s-1} - \lambda_s > 0\,.$$

Therefore, $\lambda[i_1]$ is a Young diagram.

In case $k = m > 1$. Suppose $\lambda' := \lambda[i_1, i_2, \dots, i_{m-1}]$ to be a Young diagram and set $i_m := T_{x,y}$, namely, i_m is the (x, y)-entry in T. By considering similarly to the case $k = 1$, $f(x, y)$ must be minimal in $\nu\backslash\lambda'$ with respect to the order $<_P$. By Lemma 5.2, the destination of i_m by the addition is $f(x, y) = (i_m, \lambda'_{i_m} + 1)$. Then, nothing comes above $f(x, y)$ after i_m. Thus, by arguing similarly to the case $k = 1$, we have

$$\lambda'_{i_m-1} - \lambda'_{i_m} > 0\,,$$

and then $\lambda'[i_m]$ is a Young diagram. □

6. Well-definedness of Ψ

In this section, we shall show the well-definedness of Ψ, that is, the image $\Psi(\mathbf{B}(\mu)^{\nu}_{\lambda})$ is in $\mathbf{P}(\mu, \nu \setminus \lambda)$. Let λ, μ, ν be as above.

PROPOSITION 6.1. *For $T \in \mathbf{B}(\mu)^{\nu}_{\lambda}$, we have*
(i) *$\Psi(T)$ is a map from μ to $\nu\backslash\lambda$ and $\Psi(T)(\mu) = \nu \setminus \lambda$.*
(ii) *$\Psi(T)$ is a bijection.*
(iii) *Both $\Psi(T)$ and $\Psi(T)^{-1}$ are PJ-standard.*

Before starting the proof, we prepare one lemma:

LEMMA 6.2. *For $T \in \mathbf{B}(\mu)^{\nu}_{\lambda}$ and $(i, j) \in \mu$, define $(p, q) := \Psi(T)(i, j)$. Then we have that the destination of (i, j) by the addition of* ME(T) *is equal to (p, q).*

PROOF. Set $m := T_{i,j}$ and let (i, j) be the p-th element in $T^{(m)}$ from the right, where $T^{(m)}$ is as in (2.1). Then, by the addition, $T_{i,j}$ is added p-th to the m-th row in ν. By the definition of Ψ, one has $\Psi(T)(i, j) = (m, \lambda_m + p)$. This shows the lemma. □

6.1. Proof of Proposition 6.1 (i).

It is clear from the definition of Ψ that $\Psi(T)$ is a map from μ. Since $T \in \mathbf{B}(\mu)^{\nu}_{\lambda}$, one has that for any $j = 1, \dots, n$ the number of j in T is equal to $\nu_j - \lambda_j$. Then it follows from Lemma 6.2 that $\Psi(T)(\mu) = \nu\backslash\lambda$. Thus, we have (1).

6.2. Proof of proposition 6.1 (ii).

Since $|\mu| = |\nu \setminus \lambda|$ and $\Psi(T) = \nu \setminus \lambda$ by Proposition 6.1 (1), it suffices to show that $f := \Psi(T)$ is injective. By the definition of Ψ, for $(i, j), (x, y) \in \mu$ there are some p and q such that

$$f(i, j) = (T_{i,j}, \lambda_{T_{i,j}} + p)\,, \qquad f(x, y) = (T_{x,y}, \lambda_{T_{x,y}} + q)\,.$$

Indeed, $p = p(T; , i, j)$ and $q = p(T; x, y)$. Suppose that $f(i, j) = f(x, y)$. One has

$$T_{i,j} = T_{x,y}, \quad \lambda_{T_{i,j}} + p = \lambda_{T_{x,y}} + q\,.$$

Then $p = q$. Hence, by (2.3) one has $(i, j) = (x, y)$ and then f is injective.

6.3. Proof of Proposition 6.1 (iii). First, let us see $f = \Psi(T)$ to be PJ-standard. For the purpose, we may show for any $(i, j) \in \mu$,

(a) $f(i, j) <_J f(i, j+1)$.

(b) $f(i, j) <_J f(i+1, j)$.

(a) For $(i, j), (i, j+1) \in \mu$, there are some p and q such that

$$f(i, j) = (T_{i,j}, \lambda_{T_{i,j}} + p), \qquad f(i, j+1) = (T_{i,j+1}, \lambda_{T_{i,j+1}} + q)\,.$$

Since T is a Young tableau, one has

$$T_{i,j} \le T_{i,j+1}\,.$$

If $T_{i,j} < T_{i,j+1}$, this implies $f(i, j) <_J f(i, j+1)$ and then there is nothing to show. So, assume $T_{i,j} = T_{i,j+1} =: m$. In this case, $(i, j), (i, j+1) \in T^{(m)}$ and they are neighboring each other. Thus, we have $p = q + 1$ and then

$$\lambda_{T_{i,j}} + p > \lambda_{T_{i,j+1}} + q\,.$$

This shows $f(i, j) <_J f(i, j+1)$.

(b) For $(i, j), (i+1, j) \in \mu$, there are some p and r such that

$$f(i, j) = (T_{i,j}, \lambda_{T_{i,j}} + p), \qquad f(i+1, j) = (T_{i+1,j}, \lambda_{T_{i+1,j}} + r)\,.$$

Since T is a Young tableau, we have

$$T_{i,j} < T_{i+1,j}\,,$$

which means $f(i, j) <_J f(i+1, j)$ and then f is PJ-standard.

Next, let us show f^{-1} to be PJ-standard. It is sufficient to see that for $(a, b), (a, b+1), (a+1, b) \in \nu\backslash\lambda$:

(c) $f^{-1}(a, b) <_J f^{-1}(a, b+1)$.

(d) $f^{-1}(a, b) <_J f^{-1}(a+1, b)$.

Set

$$(i, j) := f^{-1}(a, b), \quad (x, y) := f^{-1}(a, b+1), \quad (s, t) := f^{-1}(a+1, b)\,.$$

(c) There exist p and q such that

$$(a, b) = f(i, j) = (T_{i,j}, \lambda_{T_{i,j}} + p), \qquad (a, b+1) = f(x, y) = (T_{x,y}, \lambda_{T_{x,y}} + q)\,.$$

Thus, we have

$$T_{i,j} = T_{x,y} = a, \qquad \lambda_a + p = b, \qquad \lambda_a + q = b + 1\,,$$

which implies $q = p + 1$. Then we know that (i, j) and (x, y) are neighboring in $T^{(a)}$ and then $i = x$ and $j > y$, or $i < x$. Therefore,

$$f^{-1}(a, b) = (i, j) <_J (x, y) = f^{-1}(a, b+1)\,,$$

and then we show (c).
(d) There is (a, b) just above $(a+1, b)$ in the same column in $\nu\backslash\lambda$. It follows from Lemma 6.2 that in the addition of $\mathrm{ME}(T)$, $T_{i,j}$ is added earlier than $T_{s,t}$. Since the middle-eastern reading follows the order $<_J$, we have

$$f^{-1}(a, b) = (i, j) <_J (s, t) = f^{-1}(a+1, b)\,,$$

which implies (d). Hence, both f and f^{-1} are PJ-standard and then $f = \Psi(T) \in \mathbf{P}(\mu, \nu \setminus \lambda)$. Now, we have completed the proof of Proposition 6.1. □

7. Bijectivity of Φ and Ψ

In order to show Φ and Ψ to be bijective, we shall prove
(e) $\Psi \circ \Phi = \mathrm{id}_{\mathbf{P}(\mu,\nu\backslash\lambda)}$.
(f) $\Phi \circ \Psi = \mathrm{id}_{\mathbf{B}(\mu)^{\nu}_{\lambda}}$.
(e) For $f = (f_1, f_2) \in \mathbf{P}(\mu, \nu \setminus \lambda)$, set $g := \Psi \circ \Phi(f)$. $\Phi(f)$ is a Young tableau whose (s, t)-entry $\Phi(f)_{s,t}$ is equal to $f_1(s, t)$. Let $m := \Phi(f)_{s,t}$ be the p-th entry from the right in $\Phi(f)^{(m)}$ and then

$$g(s, t) = (\Phi(f)_{s,t}, \lambda_{\Phi(f)_{s,t}} + p) = (f_1(s, t), \lambda_{f_1(s,t)} + p)\,.$$

We can easily see from Lemma 5.2 that $f(s, t) = (\Phi(f)_{s,t}, \lambda_{\Phi(f)_{s,t}} + p) = (f_1(s, t), \lambda_{f_1(s,t)} + p)$. Hence, we have $g = f$ and then $\Psi \circ \Phi = \mathrm{id}_{\mathbf{P}}(\mu, \nu \setminus \lambda)$.

(f) Take $T \in \mathbf{B}(\mu)^{\nu}_{\lambda}$. By the definition of Ψ, $\Psi(T)$ is a map which sends (i, j) to $(T_{i,j}, \nu_{T_{i,j}} + p)$, where $p = p(T; i, j)$. Furthermore, by the definition of Φ, $\Phi \circ \Psi(T)$ is a Young tableau in the shape μ with a entry $T_{i,j}$ in a box (i, j). This means $T = \Phi \circ \Psi(T)$ and then $\Phi \circ \Psi = \mathrm{id}_{\mathbf{B}(\mu)^{\nu}_{\lambda}}$.
Now, we have completed the proof of Theorem 4.1. □

EXAMPLE 7.1. Set $f :=$

μ	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)
$\nu \setminus \lambda$	(1, 4)	(2, 2)	(4, 1)	(2, 3)	(3, 2)

$\in \mathbf{P}(\mu, \nu \setminus \lambda)$. We have $\Phi(f) = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2&3\\\cline{1-2}\end{array}$. Let us apply Ψ to this. The number of entries 1, 3, 4 in $\Phi(f)$ is one and then their destinations are determined uniquely: $1 \mapsto (1, 4)$, $3 \mapsto (3, 2)$ and $4 \mapsto (4, 1)$. There are two entries 2 in $\Phi(f)$. Since 2 in (1, 2) is right to the one in (2, 1), it goes to (2, 2) and the other goes to (2, 3). Hence we have,

$\Psi \circ \Phi(f) =$

μ	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)
$\nu \setminus \lambda$	(1, 4)	(2, 2)	(4, 1)	(2, 3)	(3,2)

$= f$.

This shows $\Psi \circ \Phi = \mathrm{id}_{\mathbf{P}(\mu,\nu\backslash\lambda)}$.

EXAMPLE 7.2. Set $T := \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2&3\\\cline{1-2}\end{array} \in \mathbf{B}(\mu)^{\nu}_{\lambda}$. We have

$$\Psi(T) = \begin{array}{c||c|c|c|c|c} \mu & (1,1) & (1,2) & (1,3) & (2,1) & (2,2) \\ \hline \nu \setminus \lambda & (1,4) & (2,2) & (4,1) & (2,3) & (3,2) \end{array}.$$

By the definition of Φ, we obtain: $\Phi \circ \Psi(T) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$. Hence, $\Phi \circ \Psi = \mathrm{id}_{\mathbf{B}(\mu)^{\nu}_{\lambda}}$.

8. Conjecture

We define a total order on a subset X in $\mathbb{N} \times \mathbb{N}$, called "*admissible order*" and denoted by $<_A$.

DEFINITION 8.1.

(i) A total order $<_A$ on $X \subset \mathbb{N} \times \mathbb{N}$ is called *admissible* if it satisfies:

$$\text{For any } (a,b),\ (c,d) \in X \text{ if } a \le c \text{ and } b \ge d \text{ then } (a,b) <_A (c,d)\,.$$

(ii) For $X, Y \subset \mathbb{N} \times \mathbb{N}$ and a map $f : X \to Y$, if f satisfies that if $(a,b) <_P (c,d)$, then $f(a,b) <_A f(c,d)$ for any $(a,b),\ (c,d) \in X$, then f is called PA-standard.

REMARK. Note that for fixed $X \subset \mathbb{N} \times \mathbb{N}$, there can be several admissible orders on X. For example, the order $<_J$ is one of admissible orders on X. If we define the total order $<_F$ by

$$(a,b) <_F (c,d) \text{ iff } b > d,\ \text{ or } b = d \text{ and } a < c\,,$$

then this is also admissible.

Let λ, μ, ν be Young diagrams as above and $<_A$ (resp. $<_{A'}$) an admissible order on $\nu \setminus \lambda$ (resp. μ). Note that we do not assume $<_A \ = \ <_{A'}$. We define a set (A, A')-pictures $\mathbf{P}(\mu, \nu \setminus \lambda : A, A')$ by

$$\mathbf{P}(\mu, \nu \setminus \lambda : A, A') := \left\{ f : \mu \to \nu \setminus \lambda \ \middle|\ \begin{array}{l} f \text{ is PA-standard and bijective,} \\ \text{and } f^{-1} \text{ is PA}'\text{-standard.} \end{array} \right\}.$$

DEFINITION 8.2. Let A be an admissible order on a Young diagram μ with $|\mu| = N$. For $T \in B(\mu)$, by reading the entries in T according to A, we obtain the map

$$R_A : B(\mu) \longrightarrow B^{\otimes N} \quad (T \mapsto \boxed{i_1} \otimes \cdots \otimes \boxed{i_N})\,,$$

which is called an admissible reading associated with the order A. It is known that the map R_A is an embedding of crystals([5]).

Here note that Theorem 3.6 is valid for an arbitrary reading R_A, that is, in (3.1) we can replace FE(T) with $R_A(T)$. Define

$$\mathbf{B}(\mu)^{\nu}_{\lambda}[A] := \left\{ T \in \mathbf{B}(\mu) \ \middle|\ \begin{array}{l} \text{For any } k = 1, \ldots, N, \\ \lambda[i_1, \ldots, i_k] \text{ is a Young diagram and} \\ \lambda[i_1, \ldots, i_N] = \nu, \text{ where} \\ R_A(T) = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k} \otimes \cdots \otimes \boxed{i_N}. \end{array} \right\}.$$

It is shown in [5] that for any admissible order on μ,

$$\mathbf{B}(\mu)^{\nu}_{\lambda}[A] = \mathbf{B}(\mu)^{\nu}_{\lambda}\,. \tag{8.1}$$

CONJECTURE 8.3. *Let A (resp. A') be an admissible order on $\nu \setminus \lambda$ (resp. μ). There exists a bijection*

$$\Psi : \mathbf{B}(\mu)^{\nu}_{\lambda}[A'] \longrightarrow \mathbf{P}(\mu, \nu \setminus \lambda : A, A')\,,$$

where Ψ is the same as in 4.1.

If we show the conjecture, together with (8.1), we have

COROLLARY 8.4. *For arbitrary admissible orders A on $\nu \setminus \lambda$ and A' on μ,*

$$\mathbf{P}(\mu, \nu \setminus \lambda) = \mathbf{P}(\mu, \nu \setminus \lambda : A, A')\,.$$

This has been shown in [2] and [3] by some purely combinatorial way.

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