TOKYO J. MATH.
VOL. 34, NO. 2, 2011

Some Relations among Apostol-Vu Double Zeta Values for Coordinatewise Limits at Non-positive Integers

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(Communicated by M. Tsuzuki)

Abstract. We consider Apostol-Vu double zeta values for coordinatewise limits at non-positive integers, and we give some relations among Riemann's zeta values, Euler-Zagier double zeta values and Apostol-Vu double zeta values for all coordinatewise limits at non-positive integers. Using the relations, we also give relations among multiple Bernoulli numbers.

1. Introduction

Apostol and Vu [3] introduced the following sum:

$$\sum_{m_1=1}^{\infty}\sum_{m_2\geq m_1} m_1^{-s_1}m_2^{-s_2}(m_1+m_2)^{-1}\,,$$

where $s_1, s_2 \in \boldsymbol{C}$ with $\Re(s_1+s_2) > 2$, $\Re s_2 > 1$. They proved that this can be expressed as a rational linear combination of products of zeta values at positive integers when $s_1 = s_2 = a$ where $a \in \boldsymbol{N}$. For complex variables $s_1, \ldots, s_{r+1}$, Matsumoto [12], [13] introduced r-ple Apostol-Vu multiple zeta functions

$$\begin{aligned}
&\zeta_{AV,r}(s_1,\ldots,s_r;s_{r+1})\\
&\quad=\sum_{1\leq m_1<\cdots<m_r<\infty}\cdots\sum m_1^{-s_1}m_2^{-s_2}\cdots m_r^{-s_r}(m_1+m_2+\cdots+m_r)^{-s_{r+1}}\\
&\quad=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-s_1}(m_1+m_2)^{-s_2}\times\cdots\times(m_1+\cdots+m_r)^{-s_r}\\
&\qquad\times(rm_1+(r-1)m_2+\cdots+m_r)^{-s_{r+1}}, \qquad (1.1)
\end{aligned}$$

including the sum of Apostol and Vu. He also proved the meromorphic continuation to the whole space by the Mellin-Barnes integral formula (see Lemma 2.3 in the present paper).

Received April 15, 2010
2000 *Mathematics Subject Classification*: 40B05 (Primary), 11M41 (Secondary)
Key words and phrases: Apostol-Vu multiple zeta function, true singularity, coordinatewise limit

On the other hand, Zagier [19] first treated Witten zeta functions defined by

$$\zeta_{\mathfrak{g}}(s) = \sum \frac{1}{\dim(\rho)^s},$$

where $s \in \boldsymbol{C}$ and ρ runs over all finite-dimensional representations of a certain semi-simple Lie algebra $\mathfrak{g}$. In particular,

$$\zeta_{\mathfrak{sl}(3)}(s) = \sum_{m,n=1}^{\infty} \frac{2^s}{m^s n^s (m+n)^s}, \tag{1.2}$$

$$\zeta_{\mathfrak{so}(5)}(s) = \sum_{m,n=1}^{\infty} \frac{6^s}{m^s n^s (m+n)^s (m+2n)^s}. \tag{1.3}$$

Matsumoto [13] considered a generalization of (1.3), that is

$$\zeta_{\mathfrak{so}(5)}(s_1, s_2, s_3, s_4) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}}, \tag{1.4}$$

where s_1, s_2, s_3, s_4 are complex variables. Also, Matsumoto and Tsumura [14] define multi-variable Witten zeta functions for $\mathfrak{g} = \mathfrak{sl}(n)$, including the function (1.2). We see that the Apostol-Vu double zeta function is a special case of (1.4) by the following relation:

$$\zeta_{AV,2}(s_1, s_2; s_3) = \zeta_{\mathfrak{so}(5)}(0, s_1, s_2, s_3).$$

For $r \in \boldsymbol{N}$, the r-ple Euler-Zagier multiple zeta function is defined by

$$\zeta_{EZ,r}(s_1, \ldots, s_r) = \sum \cdots \sum_{1 \le m_1 < \cdots < m_r < \infty} m_1^{-s_1} m_2^{-s_2} \cdots m_r^{-s_r}, \tag{1.5}$$

where $s_1, \ldots, s_r \in \boldsymbol{C}$. Akiyama, Egami and Tanigawa [1] and Zhao [20], independently of esch other, and later Matsumoto [11] also, proved the meromorphic continuation to the whole space. When $s_1, \ldots, s_r$ are positive integers with $s_r \ge 2$, the r-ple Euler-Zagier multiple zeta function is absolutely convergent. Its values are called multiple zeta values. The multiple zeta function, especially some relation among multiple zeta values, has been studied extensively by many mathematicians. For example, the sum formula [4], [5], Hoffman's relation [6], Ohno's relation [15], Kawashima's relation [7], etc.

Also, there are studies about multiple zeta values at non-positive integers. When $s_1, \ldots, s_r$ are non-positive integers, the point can be singularity of $\zeta_{EZ,r}$. In fact, we have the following theorem:

THEOREM 1.1 ([1], [11]). *The function* (1.5) *is holomorphic except for the singularities located only on*

$$s_r = 1,$$

$$s_r + s_{r-1} = 2, 1, -2n \quad (n \in \boldsymbol{N}_0),$$

$$s_r + s_{r-1} + s_{r-2} = 3 - n \quad (n \in \boldsymbol{N}_0)\,,$$

$$\cdots$$

$$s_r + \cdots + s_1 = r - n \quad (n \in \boldsymbol{N}_0)\,.$$

Here, *we put* $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$.

Also, it is well-known that each point $(s_1, \dots, s_r) = (-l_1, \dots, -l_r) \in \boldsymbol{Z}^r_{\le 0}$ is a point of indeterminacy of $\zeta_{EZ,r}$ (see [1]). Here, $\boldsymbol{Z}_{\le 0}$ is the set of non-positive integers. The values of $\zeta_{EZ,r}$ at $(-l_1, \dots, -l_r) \in \boldsymbol{Z}^r_{\le 0}$depend on the limiting process. Akiyama and Tanigawa [2] defined regular and reverse values by

$$\zeta_{EZ,r}(-l_1, \dots, -l_r) := \lim_{s_1 \to -l_1} \cdots \lim_{s_r \to -l_r} \zeta_{EZ,r}(s_1, \dots, s_r)\,,$$

$$\zeta^R_{EZ,r}(-l_1, \dots, -l_r) := \lim_{s_r \to -l_r} \cdots \lim_{s_1 \to -l_1} \zeta_{EZ,r}(s_1, \dots, s_r)\,,$$

respectively. They also gave recursive formulas for these values. For example, in the case $r = 2$, they gave

$$\zeta_{EZ,2}(-l_1, -l_2) = \sum_{q=-1}^{l_2} (-l_2)^+_q a_q \zeta(-l_1 - l_2 + q)\,, \tag{1.6}$$

$$\zeta^R_{EZ,2}(-l_1, -l_2) = -\sum_{q=-1}^{l_1} (-l_1)^-_q a_q \zeta(-l_1 - l_2 + q) + \zeta(-l_1)\zeta(-l_2) \tag{1.7}$$

(see [1] and [2]). Here, we put

$$(s)^{\pm}_q = \begin{cases} s(s+1)\cdots(s+q-1) & \text{if } q \text{ is a positive integer}\,, \\ \pm 1 & \text{if } q = 0\,, \\ 1/(s-1) & \text{if } q = -1\,, \end{cases}$$

$a_q := B_{q+1}/(q+1)!$ and B_n is the nth Bernoulli number. The case $r = 3$, Sasaki [17] gave some relations among multiple zeta values for all coordinatewise limits. Komori [9] treated more general multiple zeta values at non-negative integers by using generalizations of Bernoulli numbers.

One of the purposes of the present paper is to determine true singularities of (1.1). Our proof is based on two methods; one is the method of Matsumoto [11] which uses the Mellin-Barnes integral formula (Lemma 2.3), and the other is the technique of changing variables which is introduced by Akiyama, Egami and Tanigawa [1].

The Apostol-Vu double zeta function is a special case of the Witten zeta function for $\mathfrak{g} = \mathfrak{so}(5)$ (see (1.4)) and also a generalization of the Euler-Zagier double zeta function (see (1.5)). Therefore it is important that we study Apostol-Vu double zeta values at integers. The major purpose of the present paper is to give some relations among Riemann's zeta values, Euler-Zagier double zeta values and Apostol-Vu double zeta values for all coordinatewise limits at non-positive integers (Theorem 3.1). By these relations and a certain functional relation, we will also give some relations among Mordell-Tornheim double zeta values and Euler-Zagier double zeta values for all coordinatewise limits at non-positive integers (Corollary 4.1). We note that the Mordell-Tornheim double zeta function is defined by

$$\zeta_{MT,2}(s_1, s_2; s_3) = \sum_{m,n=1}^{\infty} m^{-s_1} n^{-s_2} (m+n)^{-s_3} ,$$

where $s_1, s_2, s_3 \in \boldsymbol{C}$.

2. The true singularities of Apostol-Vu multiple zeta functions

In this section, we determine the true singularities of (1.1). First we recall analytic properties of (1.1).

THEOREM 2.1 ([13], [16]). *Let r be a positive integer. Then the following assertions hold*:

(1) *The series* (1.1) *can be continued meromorphically*, *as a function in* $s_1, \dots, s_r, s_{r+1}$, *to the whole* $\boldsymbol{C}^{r+1}$*-space.*

(2) *The function* $\zeta_{AV,r}$ *is holomorphic except for the possible singularities located only on*

$$s_r + s_{r+1} = 1 - n \quad (n \in \boldsymbol{N}_0) ,$$

$$s_{r-1} + s_r + s_{r+1} = 2 - n \quad (n \in \boldsymbol{N}_0) ,$$

$$s_{r-2} + s_{r-1} + s_r + s_{r+1} = 3 - n \quad (n \in \boldsymbol{N}_0) ,$$

$$\dots$$

$$s_1 + \cdots + s_{r+1} = r - n \quad (n \in \boldsymbol{N}_0) .$$

The possible singularities of (1.1) is given by the above list ((2) of Theorem 2.1). By a certain recursive structure which can be expressed as a Mellin-Barnes integral and the technique of changing variables, we can determine the true singularities of (1.1) as follows.

THEOREM 2.2. *The function* $\zeta_{AV,2}$ *is holomorphic except for the true singularities located only on*

$$s_2 + s_3 = 1 - n, \quad (n \in \boldsymbol{N}_0)$$

$$s_1 + s_2 + s_3 = 2, 1, -2n \quad (n \in \boldsymbol{N}_0) .$$

Also, *for* $r \geq 3$, *the function* $\zeta_{AV,r}$ *is holomorphic except for the true singularities located only on*

$$s_r + s_{r+1} = 1 - n \quad (n \in \boldsymbol{N}_0) ,$$

$$s_{r-1} + s_r + s_{r+1} = 2 - n \quad (n \in \boldsymbol{N}_0)\,,$$
$$s_{r-2} + s_{r-1} + s_r + s_{r+1} = 3 - n \quad (n \in \boldsymbol{N}_0)\,,$$
$$\cdots$$
$$s_1 + \cdots + s_{r+1} = r - n \quad (n \in \boldsymbol{N}_0)\,.$$

Our proof of Theorem 2.2 uses the following Mellin-Barnes integral formula.

LEMMA 2.3. *Let* s, λ *be complex numbers*, $\Re s > 0$, $|\arg\lambda| < \pi$ *and* $\lambda \neq 0$. *The Mellin-Barnes integral formula*

$$\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \Gamma(s+z)\Gamma(-z)\lambda^z dz \tag{2.1}$$

is classically known ([18], *Section*14.51, *p.*289, *Corollary*), *where* $-\Re s < c < 0$ *and the path of integration is the vertical line* $\Re z = c$.

PROOF OF THEOREM 2.2. In the case $r = 2$, we assume $\Re s_i > 1 (i = 1, 2)$, $\Re s_3 > 0$. Using the Mellin-Barnes integral formula and shifting the path of integration to $\Re z = N - \eta$, where N is a positive integer and η is a small positive number, we have

$$\begin{aligned}\zeta_{AV,2}(s_1, s_2; s_3) &= \sum_{k=0}^{N-1} \binom{-s_3}{k} \zeta_{EZ,2}(s_1 - k, s_2 + s_3 + k) \\ &\quad + \frac{1}{2\pi i}\int_{(N-\eta)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)} \zeta_{EZ,2}(s_1 - z, s_2 + s_3 + z) dz\,. \end{aligned}\tag{2.2}$$

Here, for $v \in \boldsymbol{C}$ and a nonpositive integer n, we put

$$\binom{v}{n} = \begin{cases} v(v-1)\cdots(v-n+1)/n! & \text{if } n \text{ is a positive integer}, \\ 1 & \text{if } \mathrm{n} = 0\,. \end{cases}$$

Then, by Theorem 1.1 and the technique of changing variables, we determine the true singularities of $\zeta_{AV,2}$ as follows:

$$s_2 + s_3 = 1 - n, \quad (n \in \boldsymbol{N}_0)$$
$$s_1 + s_2 + s_3 = 2, 1, -2n \quad (n \in \boldsymbol{N}_0)\,.$$

In fact, by Theorem 1.1 and (2.2), we see that

$$s_2 + s_3 = 1 - n, \quad (n \in \boldsymbol{N}_0)$$
$$s_1 + s_2 + s_3 = 2, 1, -2n \quad (n \in \boldsymbol{N}_0)$$

determine the possible singularities of $\zeta_{AV,2}$. We also put

$$u_1 = s_1, \ \ u_2 = s_2 + s_3, \ \ u_3 = s_3$$

(this is the technique of changing variables) and substitute them into the first term on the right hand side of (2.2), to find that the degree of each term in the sum with respect to u_3 is different from each other. Therefore the singularities will not vanish identically. Hence

$$s_2 + s_3 = 1 - n, \quad (n \in \boldsymbol{N}_0)$$
$$s_1 + s_2 + s_3 = 2, 1, -2n \quad (n \in \boldsymbol{N}_0)$$

determine the true singularities of $\zeta_{AV,2}$.

When $r = 3$, we use the recursive structure of Theorem 5 in [16]. First we define

$$\widetilde{\zeta}_{AV,3}(s_1, s_2, s_3, s_4) := \sum_{m_1,m_2,m_3=1}^{\infty} m_1^{-s_1}(m_1 + m_2)^{-s_2}(2m_1 + m_2)^{-s_3}(m_1 + m_2 + m_3)^{-s_4} .$$

We assume $\Re s_i > 1 (i = 1, 2, 3)$, $\Re s_4 > 0$. Using the Mellin-Barnes integral formula and shifting the path of integration to $\Re z = N - \eta$, we have

$$\begin{aligned}\widetilde{\zeta}_{AV,3}(s_1, s_2, s_3, s_4) &= \frac{1}{s_4 - 1}\zeta_{AV,2}(s_1, s_2 + s_4 - 1; s_3) \\ &+ \sum_{k=0}^{N-1} \binom{-s_4}{k} \zeta(-k)\zeta_{AV,2}(s_1, s_2 + s_4 + k; s_3) \\ &+ \frac{1}{2\pi i} \int_{(N-\eta)} \frac{\Gamma(s_4 + z)\Gamma(-z)}{\Gamma(s_4)} \zeta(-z)\zeta_{AV,2}(s_1, s_2 + s_4 + z; s_3)dz . \end{aligned} \tag{2.3}$$

Then, applying Theorem 2.2 for $\zeta_{AV,2}$ and the technique of changing variables to (2.3). Indeed we put

$$u_1 = s_1, \quad u_2 = s_2 + s_4, \quad u_3 = s_3, \quad u_4 = s_4$$

and substitute them into the first term on the right hand side of (2.3), to find that the degree of each term in the sum with respect to u_4 is different from each other. Hence we see that

$$s_4 = 1 ,$$
$$s_2 + s_3 + s_4 = 2 - n , \quad (n \in \boldsymbol{N}_0)$$
$$s_1 + s_2 + s_3 + s_4 = 3 - n \quad (n \in \boldsymbol{N}_0)$$

determine the true singularities of $\widetilde{\zeta}_{AV,3}(s_1, s_2, s_3, s_4)$. By this result, we obtain Theorem 2.2 in the case $r = 3$.

In fact, using the Mellin-Barnes integral formula and shifting the path of integration to $\Re z = N - \eta$, we have

$$\zeta_{AV,3}(s_1, s_2, s_3; s_4) = \sum_{k=0}^{N-1} \binom{-s_4}{k} \widetilde{\zeta}_{AV,3}(s_1, s_2, -k, s_3 + s_4 + k)$$

$$+\frac{1}{2\pi i}\int_{(N-\eta)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\widetilde{\zeta}_{AV,3}(s_1,s_2,-z,s_3+s_4+z)dz\,. \tag{2.4}$$

By the above result on the true singularities of $\widetilde{\zeta}_{AV,3}$ and the technique of changing variables ($u_1=s_1, u_2=s_2, u_3=s_3+s_4, u_4=s_4$), we have Theorem 2.2 in the case $r=3$.

Lastly we assume the validity of Theorem 2.2 for $\zeta_{AV,r-1}$, and prove Theorem 2.2 for $\zeta_{AV,r}$. The argument is similar to the case $r=3$ and we omit the details.

We only note that we define $\widetilde{\zeta}_{AV,r}$ by the following function:

$$\begin{aligned}
&\widetilde{\zeta}_{AV,r}(s_1,\dots,s_r,s_{r+1})\\
&:=\sum_{m_1,\dots,m_r=1}^{\infty} m_1^{-s_1}(m_1+m_2)^{-s_2}\dots((m_1+\cdots+m_{r-1})^{-s_{r-1}}\\
&\qquad\times((r-1)m_1+(r-2)m_2+\cdots+m_{r-1})^{-s_r}(m_1+\cdots+m_r)^{-s_{r+1}}\,,
\end{aligned}$$

where $s_1\dots s_{r+1}\in\boldsymbol{C}$. □

3. Apostol-Vu double zeta values for coordinatewise limits at non-positive integers

In this section, we will give some relations among Apostol-Vu double zeta values for all coordinatewise limits, (1.6) and (1.7) at non-positive integers.

THEOREM 3.1. *For any non-negative integers l_i $(i=1,2,3)$, we have*

$$\begin{aligned}
&\lim_{s_1\to -l_1}\lim_{s_2\to -l_2}\lim_{s_3\to -l_3}\zeta_{AV,2}(s_1,s_2;s_3)\\
&\quad=\lim_{s_1\to -l_1}\lim_{s_3\to -l_3}\lim_{s_2\to -l_2}\zeta_{AV,2}(s_1,s_2;s_3)\\
&\quad=\sum_{k=0}^{l_3}\zeta_{EZ,2}(-l_1-k,-l_2-l_3+k)\,,
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
&\lim_{s_2\to -l_2}\lim_{s_1\to -l_1}\lim_{s_3\to -l_3}\zeta_{AV,2}(s_1,s_2;s_3)\\
&\quad=\lim_{s_2\to -l_2}\lim_{s_3\to -l_3}\lim_{s_1\to -l_1}\zeta_{AV,2}(s_1,s_2;s_3)\\
&\quad=\sum_{k=0}^{l_3}\zeta^{R}_{EZ,2}(-l_1-k,-l_2-l_3+k)\,,
\end{aligned}\tag{3.2}$$

$$\begin{aligned}
&\lim_{s_3\to -l_3}\lim_{s_1\to -l_1}\lim_{s_2\to -l_2}\zeta_{AV,2}(s_1,s_2;s_3)\\
&=\sum_{j=0}^{l_2}(-1)^j\binom{l_2}{j}\Big\{\zeta(-l_1-j)\zeta(-l_2-l_3+j)-2^{l_2+l_3-j}\zeta(-l_1-l_2-l_3)
\end{aligned}$$

$$- 2^{l_2+l_3-j}\zeta_{EZ,2}(-l_2-l_3+j,-l_1-j)$$
$$+ \frac{2^{-l_1-j-1}}{l_1+j+1}(1-2^{l_1+l_2+l_3+1})\zeta(-l_1-l_2-l_3-1)$$
$$- 2^{-l_1-j}\sum_{k=1}^{l_1+j}\binom{l_1+j}{k}(1-2^k)(1-2^{l_1+l_2+l_3-k})$$
$$\times \zeta(-l_1-l_2-l_3+k)\zeta(-k)\Bigg\}, \tag{3.3}$$

$$\lim_{s_3\to -l_3}\lim_{s_2\to -l_2}\lim_{s_1\to -l_1}\zeta_{AV,2}(s_1,s_2;s_3)$$
$$= \sum_{j=0}^{l_1}(-1)^j\binom{l_1}{j}\Bigg\{2^{l_1+l_3-j}\zeta_{EZ,2}(-l_1-l_3+j,-l_2-j)$$
$$+ \zeta^R_{EZ,2}(-l_2-j,-l_1-l_3+j)-\zeta(-l_2-j)\zeta(-l_1-l_3+j)$$
$$- \frac{2^{-l_2-j-1}}{l_2+j+1}(1-2^{l_1+l_2+l_3+1})\zeta(-l_1-l_2-l_3-1)$$
$$+ 2^{-l_2-j}\sum_{k=1}^{l_2+j}\binom{l_2+j}{k}(1-2^k)(1-2^{l_1+l_2+l_3-k})$$
$$\times \zeta(-l_1-l_2-l_3+k)\zeta(-k)\Bigg\}. \tag{3.4}$$

PROOF. First we show (3.1) and (3.2). By (5.9) in [10], we have

$$\zeta_{AV,2}(s_1,s_2;s_3) = \sum_{k=0}^{N-1}\binom{-s_3}{k}\zeta_{EZ,2}(s_1-k,s_2+s_3+k)$$
$$+ \zeta(s_1+s_2+s_3-1)\frac{1}{2\pi i}\int_{(N-\eta)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\frac{dz}{s_2+s_3+z-1}$$
$$+ \sum_{j=0}^{N-1}\zeta(s_1+s_2+s_3+j)\zeta(-j)$$
$$\times \frac{1}{2\pi i}\int_{(N-\eta)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\binom{-s_2-s_3-z}{j}dz$$
$$+ \frac{1}{(2\pi i)^2}\int_{(N-\eta)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\int_{(N-\eta)}\frac{\Gamma(s_2+s_3+z+z')\Gamma(-z')}{\Gamma(s_2+s_3+z)}$$
$$\times \zeta(s_1+s_2+s_3+z')\zeta(-z')dz'dz\,. \tag{3.5}$$

By (3.5), $\zeta_{AV,2}(s_1,s_2;s_3)$ can be continued meromorphically to $\Re s_3 > -N+\eta$, $\Re(s_2+s_3) >$

$1-N+\eta$ and $\Re(s_1+s_2+s_3)>1-N+\eta$ (see [10]). We note that $1/\Gamma(-n)=0$ for non-negative integer n. Hence, by (3.5), we have (3.1) and (3.2).

We next show (3.3). We fix $l_2\in N_0$ and consider $\lim_{s_2\to -l_2}\zeta_{AV,2}(s_1,s_2;s_3)$. We note that

$$\zeta_{AV,2}(s_1,s_2;s_3)=\sum_{m_1,m_2=1}^{\infty} m_1^{-s_1}(m_1+m_2)^{-s_2}(2m_1+m_2)^{-s_3}$$

is absolutely convergent in the region

$$\Re s_3>0,\ \ \Re(s_2+s_3)>1,\ \ \Re(s_1+s_2+s_3)>2\,.$$

We assume $\Re s_3>1+l_2$, $\Re(s_1+s_3)>2+l_2$. We have

$$\begin{aligned}\lim_{s_2\to -l_2}\zeta_{AV,2}(s_1,s_2;s_3)&=\sum_{m_1,m_2=1}^{\infty} m_1^{-s_1}(m_1+m_2)^{l_2}(2m_1+m_2)^{-s_3}\\ &=\sum_{j=0}^{l_2}(-1)^j\binom{l_2}{j}\sum_{m_1,m_2=1}^{\infty}\frac{1}{m_1^{s_1-j}(2m_1+m_2)^{s_3-l_2+j}}\,. \end{aligned}\tag{3.6}$$

Here, we put

$$\widehat{\zeta}(s_1,s_2):=\sum_{m_1,m_2=1}^{\infty}\frac{1}{m_1^{s_1}(2m_1+m_2)^{s_2}}\,.$$

We easily see that $\hat{\zeta}(s_1,s_2)$ is absolutely convergent in the region $\Re s_1>1$, $\Re s_2>1$. Then we have

$$\zeta(s_1)\zeta(s_2)=\widehat{\zeta}(s_1,s_2)+\sum_{\substack{m_1,m_2=1\\ m_2<2m_1}}^{\infty}\frac{1}{m_1^{s_1}m_2^{s_2}}+2^{-s_2}\zeta(s_1+s_2)\,,\tag{3.7}$$

which can be easily proved by a simple transformation of the sum. We have to study the second term on the right hand of (3.7). We have

$$\begin{aligned}\sum_{\substack{m_1,m_2=1\\ m_2<2m_1}}^{\infty}\frac{1}{m_1^{s_1}m_2^{s_2}}&=2^{-s_2}\zeta_{EZ,2}(s_2,s_1)\\ &\quad+2^{s_1}\sum_{m_1,m_2=1}^{\infty}\frac{1}{(2m_1+2m_2-2)^{s_1}(2m_2-1)^{s_2}}\,,\end{aligned}\tag{3.8}$$

which can be easily proved by replacing $2m_1$ with $m_1'+m_2$ ($m_1'\in N$). Then we assume $\Re s_1>1$, $\Re s_2>1$. Applying the Mellin-Barnes integral formula for the second term on the right hand side of (3.8), we have

$$\sum_{m_1,m_2=1}^{\infty} \frac{1}{(2m_1+2m_2-2)^{s_1}(2m_2-1)^{s_2}}$$
$$= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_1+z)\Gamma(-z)}{\Gamma(s_1)}(1-2^z)(1-2^{-(s_1+s_2+z)})$$
$$\times \zeta(s_1+s_2+z)\zeta(-z)dz\,,$$

where $-\Re s_1 < c < -1$. Shifting the path of integration to $\Re z = N-\eta$, where N is a positive integer and η is a small positive number, we have

$$\sum_{m_1,m_2=1}^{\infty} \frac{1}{(2m_1+2m_2-2)^{s_1}(2m_2-1)^{s_2}}$$
$$= \frac{1}{2(s_1-1)}(1-2^{-(s_1+s_2-1)})\zeta(s_1+s_2-1)$$
$$+ \sum_{k=0}^{N-1}\binom{-s_1}{k}(1-2^k)(1-2^{-(s_1+s_2+k)})\zeta(s_1+s_2+k)\zeta(-k)$$
$$+ \frac{1}{2\pi i}\int_{(N-\eta)} \frac{\Gamma(s_1+z)\Gamma(-z)}{\Gamma(s_1)}(1-2^z)(1-2^{-(s_1+s_2+z)})$$
$$\times \zeta(s_1+s_2+z)\zeta(-z)dz\,. \tag{3.9}$$

By (3.9),

$$\sum_{m_1,m_2=1}^{\infty} \frac{1}{(2m_1+2m_2-2)^{s_1}(2m_2-1)^{s_2}}$$

can be continued meromorphically to $\Re s_1 > -N+\eta$, $\Re(s_1+s_2) > -N+\eta$. Hence, by (3.7), (3.8) and (3.9), we have

$$\lim_{s_2\to -r_2}\lim_{s_1\to -r_1} \widehat{\zeta}(s_1,s_2)$$
$$=\zeta(-r_1)\zeta(-r_2) - 2^{r_2}\zeta_{EZ,2}(-r_2,-r_1) - 2^{r_2}\zeta(-r_1-r_2)$$
$$- 2^{-r_1}\sum_{k=1}^{r_1}\binom{r_1}{k}(1-2^k)(1-2^{r_1+r_2-k})\zeta(-r_1-r_2+k)\zeta(-k)$$
$$+ \frac{2^{-r_1-1}}{r_1+1}(1-2^{r_1+r_2+1})\zeta(-r_1-r_2-1) \tag{3.10}$$

for $r_1, r_2 \in \boldsymbol{N}_0$. By (3.6) and (3.10), we have (3.3).

Lastly, we show (3.4). We fix $l_1 \in \boldsymbol{N}_0$ and we assume $\Re s_3 > 1$, $\Re(s_2+s_3) > 3+l_1$.

Then we have

$$\lim_{s_1\to -l_1}\zeta_{AV,2}(s_1,s_2;s_3)=\sum_{m_1,m_2=1}^{\infty} m_1^{l_1}(m_1+m_2)^{-s_2}(2m_1+m_2)^{-s_3}$$

$$=\sum_{j=0}^{l_1}(-1)^j\binom{l_1}{j}\sum_{m_1,m_2=1}^{\infty}\frac{1}{(m_1+m_2)^{s_2-j}(2m_1+m_2)^{s_3-l_1+j}}\,. \tag{3.11}$$

Here, we put

$$\widetilde{\zeta}(s_1,s_2):=\sum_{m_1,m_2=1}^{\infty}\frac{1}{(m_1+m_2)^{s_1}(2m_1+m_2)^{s_2}}\,.$$

We note that $\widetilde{\zeta}(s_1,s_2)$ is absolutely convergent in the region $\Re s_1>1$, $\Re s_2>1$. Then we have

$$\zeta_{EZ,2}(s_1,s_2)=\widehat{\zeta}(s_1,s_2)+\widetilde{\zeta}(s_1,s_2)+2^{-s_2}\zeta(s_1+s_2)\,, \tag{3.12}$$

which can be easily proved by a simple transformation of the sum. By (3.10), (3.11) and (3.12), we have (3.4). □

4. Mordell-Tornheim double zeta values for coordinatewise limits at non-positive integers

In this section, we will give some relations among Mordell-Tornheim double zeta values for all coordinatewise limits, (1.6) and (1.7) at non-positive integers by Theorem 3.1 and certain functional relations.

COROLLARY 4.1. *For any non-negative integers l_i $(i=1,2,3)$, we have*

$$\lim_{s_3\to -l_3}\lim_{s_1\to -l_1}\lim_{s_2\to -l_2}\zeta_{MT,2}(s_1,s_2;s_3)$$
$$=\sum_{j=0}^{l_2}(-1)^j\binom{l_2}{j}\zeta^{R}_{EZ,2}(-l_1-j,-l_2-l_3+j)$$

$$\lim_{s_3\to -l_3}\lim_{s_2\to -l_2}\lim_{s_1\to -l_1}\zeta_{MT,2}(s_1,s_2;s_3)$$
$$=\sum_{j=0}^{l_1}(-1)^j\binom{l_1}{j}\zeta^{R}_{EZ,2}(-l_2-j,-l_1-l_3+j)$$

$$\lim_{s_1\to -l_1}\lim_{s_2\to -l_2}\lim_{s_3\to -l_3}\zeta_{MT,2}(s_1,s_2;s_3)$$
$$=\lim_{s_1\to -l_1}\lim_{s_3\to -l_3}\lim_{s_2\to -l_2}\zeta_{MT,2}(s_1,s_2;s_3)$$
$$=\sum_{k=0}^{l_3}\left\{\zeta_{EZ,2}(-l_1-k,-l_2-l_3+k)+\zeta^{R}_{EZ,2}(-l_2-k,-l_1-l_3+k)\right\}$$

$$+ 2^{l_3}\zeta(-l_1 - l_2 - l_3)$$

$$\lim_{s_2\to -l_2}\lim_{s_1\to -l_1}\lim_{s_3\to -l_3}\zeta_{MT,2}(s_1, s_2; s_3)$$
$$= \lim_{s_2\to -l_2}\lim_{s_3\to -l_3}\lim_{s_1\to -l_1}\zeta_{MT,2}(s_1, s_2; s_3)$$
$$= \sum_{k=0}^{l_3}\left\{\zeta_{EZ,2}(-l_2 - k, -l_1 - l_3 + k) + \zeta^{R}_{EZ,2}(-l_1 - k, -l_2 - l_3 + k)\right\}$$
$$+ 2^{l_3}\zeta(-l_1 - l_2 - l_3)\,.$$

PROOF. Corollary 4.1 can be easily proved by Theorem 3.1 and the following two functional relations:

$$\zeta_{MT,2}(s_1, s_2; s_3) = \zeta_{MT,2}(s_2, s_1; s_3)\,,$$
$$\zeta_{MT,2}(s_1, s_2; s_3)$$
$$= \zeta_{AV,2}(s_1, s_2; s_3) + \zeta_{AV,2}(s_2, s_1; s_3) + 2^{-s_3}\zeta(s_1 + s_2 + s_3)\,. \qquad \square$$

5. Application

By Theorem 3.1 and Corollary 4.1, we can give relations among multiple Bernoulli numbers which is defined by Komori [9]. In this section, we will give an example.

By Theorem 3.21 in [9], for $l_1, l_2, l_3 \in \boldsymbol{N}_0$ we have

$$\zeta_{EZ,2}(-l_1, -l_2) = (-1)^{l_1+l_2} l_1! l_2! B_{EZ,2(l_1,l_2)}, \tag{5.1}$$

$$\lim_{s_1\to -l_1}\lim_{s_2\to -l_2}\lim_{s_3\to -l_3}\zeta_{AV,2}(s_1, s_2; s_3) = (-1)^{l_1+l_2+l_3} l_1! l_2! l_3! B_{AV,2(l_1,l_2,l_3)}\,, \tag{5.2}$$

where $B_{EZ,2(l_1,l_2)}$ is the coefficient of $\alpha_1^{l_2}\alpha_2^{l_1+l_2}$ in the Laurent expansion of

$$\frac{1}{(e^{\alpha_1(\alpha_2+1)} - 1)(e^{\alpha_1\alpha_2} - 1)}$$

and $B_{AV,2(l_1,l_2,l_3)}$ is the coefficient of $\alpha_1^{l_1+l_2+l_3}\alpha_2^{l_2+l_3}\alpha_3^{l_3}$ in the Laurent expansion of

$$\frac{1}{(e^{\alpha_1(1+\alpha_2+2\alpha_2\alpha_3)} - 1)(e^{\alpha_1\alpha_2(1+\alpha_3)} - 1)}\,.$$

By (3.1), (5.1) and (5.2), we have the following relation among multiple Bernoulli numbers:

$$l_1! l_2! l_3! B_{AV,2(l_1,l_2,l_3)} = \sum_{k=0}^{l_3}(l_1 + k)!(l_2 + l_3 - k)! B_{EZ,2(l_1+k,l_2+l_3-k)}\,.$$

We note that our definitions of multiple Bernoulli numbers $B_{EZ,2(l_1,l_2)}$ and $B_{AV,2(l_1,l_2,l_3)}$ are a little different from the definition in [9].

ACKNOWLEDGEMENTS. The author sincerely thanks Professor Kohji Matsumoto and Professor Yasushi Komori for discussions and encouragement. This paper was improved by their valuable comments.

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