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On Some Functional-differential Inequalities Related to the Exponential Mapping

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Abstract. We consider real-valued twice differentiable functions defined on an open interval. Our main result states that if a function f satisfies some inequalities then a map $x \mapsto f(x)\exp(-cx)$ is convex, where c is an arbitrary point of $\mathbf{R}$ or of $\mathbf{R} \setminus (c_1, c_2)$ for some real c_1, c_2.

1. Introduction

The exponential mapping is one of the most important functions in mathematics and in the literature one can find several different approaches how to define formally the exponential mapping on an undergraduate course of advanced calculus. One of the possibilities is to consider the following equation:

$$f'(x) = f(x) \tag{1}$$

with the initial condition $f(0) = 1$; S. Lang's book [8] may serve as a possible illustration of this approach.

Since the significance of equation (1) is self-evident there is no need to justify the importance of examining its behavior and, in particular, its asymptotic and stability properties. In 1998 C. Alsina and R. Ger [1] studied the Hyers-Ulam stability questions for equation (1). This work proved to be of use for future research since it builds a bridge between two significant and rapidly developing branches of mathematics, namely the theory of differential and functional-differential equations (broadly understood) and the theory of the Hyers-Ulam stability of functional equations. Paper [1] played an important role being a motivation for the recent research of a numbers of authors. Several stability problems for differential and functional-differential equations has been considered recently by H. Choda, S.-M. Jung, T. Miura, S. Miyajima, H. Oka, Th. M. Rassias, H. Takagi, S.-E. Takahasi (see [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14], [15]), among others.

The present work yields a contribution to this stream of research. Our goal is to follow and modify the approach from [1]. Some characterizations of solutions of certain inequalities

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in terms of convexity are obtained. Equivalently, it can be said that we examine "how convex" is a solution of the inequality studied in comparison to the exponential mapping.

Throughout the paper **R** stands for the set of all real numbers, $I \subset \mathbf{R}$ is a nonempty open interval, round and square brackets are used to denote open and closed intervals, respectively. Finally denote by $\mathcal{C}^2$ the set of all twice differentiable real functions from I and by $\mathcal{C}^2_+$ the set of all nonnegative functions from $\mathcal{C}^2$.

2. Results

We begin with an elementary result which is a slight modification of [1, Lemma 1].

PROPOSITION 1. *Let $f\colon I \to \mathbf{R}$ be a differentiable function and $M \in \mathbf{R}$ an arbitrary constant. Then*

$$f'(x) \le Mf(x) \quad \textit{for all}\ \ x \in I\,,$$

if and only if there exists a non-increasing and differentiable map $d\colon I \to \mathbf{R}$ such that

$$f(x) = d(x)\exp(Mx) \quad \textit{for all} \quad x \in I.$$

PROOF. To prove the sufficiency define a map $d\colon I \to \mathbf{R}$ by the formula

$$d(x) = f(x)\exp(-Mx) \quad \text{for all}\ \ x \in I. \tag{2}$$

Clearly, d is differentiable and for every $x \in I$ we have

$$\begin{aligned} d'(x) &= f'(x)\exp(-Mx) - Mf(x)\exp(-Mx) \\ &= (f'(x) - Mf(x))\exp(-Mx) \le 0\,. \end{aligned}$$

Thus, d is non-increasing and obviously $f(x) = d(x)\exp(Mx)$ for every $x \in I$. The necessity is straightforward. □

Assume that $f\colon I \to \mathbf{R}$ is a twice differentiable mapping. If we define $d\colon I \to \mathbf{R}$ by (2) we get

$$\begin{aligned} d''(x) &= (f'(x) - Mf(x))'\exp(-Mx) - M(f'(x) - Mf(x))\exp(-Mx) \\ &= (f''(x) - 2Mf'(x) + M^2 f(x))\exp(-Mx)\,. \end{aligned}$$

This means that d is convex if and only if the following inequality holds true:

$$f''(x) \ge 2Mf'(x) - M^2 f(x) \quad \text{for all}\ \ x \in I. \tag{3}$$

On the other hand, assume that $M \in \mathbf{R}$, $f\colon I \to \mathbf{R}$ is twice differentiable, $f \ge 0$, f satisfies (3) and

$$f'(x) \ge Mf(x) \quad \text{for all}\ \ x \in I\,.$$

By Proposition 1 applied for $-f$ we get that $x \mapsto f(x)\exp(-Mx)$ is non-decreasing and then we may calculate that for each $c \le M$ the map $x \mapsto f(x)\exp(-cx)$ is convex (or equivalently: for $c \le M$ there exists a convex map f_c such that $f(x) = f_c(x)\exp(cx)$). Indeed, denote $f_M(x) := f(x)\exp(-Mx)$. We have to show that for every $t \ge 0$ the map $x \mapsto f_M(x)\exp(tx)$ is convex. Using (3) we can observe that $f_M''(x) \ge 0$ for all $x \in I$. Then, using this jointly with the fact that f_M is nondecreasing and with $f \ge 0$ we deduce the estimation

$$[f_M(x)\exp(tx)]'' = f_M''(x)\exp(tx) + 2tf_M'(x)\exp(tx) + t^2 f_M(x)\exp(tx) \ge 0\,,$$

for each $x \in I$ as claimed.

The purpose of the present paper is to obtain an analogous effect under some more flexible assumptions.

THEOREM 2. *Let $f \in \mathcal{C}_+^2$ and denote $I_+ = \{x \in I : f(x) > 0\}$. Suppose that there exist constants $M_1, M_2 \in \mathbf{R}$ such that*

$$\left|\frac{f'(x)}{f(x)}\right| \le M_1 \quad \text{and} \quad M_2 \le \frac{f''(x)}{f(x)} \quad \text{for all } x \in I_+\,. \tag{4}$$

(a) *If $M_1^2 - M_2 \le 0$, then for each $c \in \mathbf{R}$ there exists a convex map $f_c \in \mathcal{C}_+^2$ such that $f(x) = f_c(x)\exp(cx)$ for all $x \in I$.*
(b) *If $M_1^2 - M_2 > 0$, then there exist constants $c_1, c_2 \in \mathbf{R}$ with $-T \le c_1 \le c_2 \le T$ such that for each $c \in \mathbf{R} \setminus (c_1, c_2)$ there exists a convex map $f_c \in \mathcal{C}_+^2$ satisfying $f(x) = f_c(x)\exp(cx)$ for all $x \in I$, where $T = M_1 + \sqrt{M_1^2 - M_2}$.*

PROOF. Let $F\colon I \times \mathbf{R} \to \mathbf{R}$ be a function defined by $F(x,t) := f(x)\exp(-tx)$. Then $f(x) = F(x,t)\exp(tx)$ for all $x \in I$ and $t \in \mathbf{R}$. By a straightforward calculation we see that

$$\frac{\partial^2 F}{\partial x^2}(x,t)\exp(tx) = f(x)t^2 - 2f'(x)t + f''(x)\,.$$

We will prove that:

(a) if $M_1^2 - M_2 \le 0$, then $F(x,c)$ is convex for every $c \in \mathbf{R}$, and
(b) if $M_1^2 - M_2 > 0$, then there exist constants $c_1, c_2 \in \mathbf{R}$ such that $F(x,c)$ is convex for every $c \in \mathbf{R} \setminus (c_1, c_2)$.

To prove the convexity of $F(x,c)$ it is enough to show that $\frac{\partial^2 F}{\partial x^2} \ge 0$, or equivalently

$$f(x)c^2 - 2f'(x)c + f''(x) \ge 0\,. \tag{5}$$

We first show that if $x \in I \setminus I_+$, then (5) holds for all $c \in \mathbf{R}$. Let $x \in I \setminus I_+$. Since $f(x) \ge 0$, we have $f(x) = 0$. First, we will show that $f'(x) = 0$. Let $n \in \mathbf{N}$. Since $f \ge 0$ and $f(x) = 0$, we deduce that

$$0 \le \frac{f\big(x + \frac{1}{n}\big)}{\frac{1}{n}} = \frac{f\big(x + \frac{1}{n}\big) - f(x)}{\frac{1}{n}} \overset{n\to\infty}{\longrightarrow} f'(x)$$

and thus $f'(x) \geq 0$. Analogously

$$0 \geq \frac{f\left(x - \frac{1}{n}\right)}{-\frac{1}{n}} = \frac{f\left(x - \frac{1}{n}\right) - f(x)}{-\frac{1}{n}} \overset{n\to\infty}{\longrightarrow} f'(x)$$

and consequently, $f'(x) = 0$ as claimed. Finally, we will prove that $f''(x) \geq 0$. By the mean value theorem, there exists a_n such that $x < a_n < x + \frac{1}{n}$ and that

$$f'(a_n) = \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}} = \frac{f\left(x + \frac{1}{n}\right)}{\frac{1}{n}} \geq 0\,.$$

Set $b_n = a_n - x$, then $b_n > 0$, $a_n = x + b_n$ and $b_n \to 0$ as $n \to \infty$. Note that from $f'(a_n) \geq 0$ it follows that

$$0 \leq \frac{f'(a_n)}{b_n} = \frac{f'(x + b_n) - f'(x)}{b_n} \overset{n\to\infty}{\longrightarrow} f''(x)\,,$$

and therefore $f''(x) \geq 0$ as claimed. Thus, (5) holds for all $x \in I \setminus I_+$ and all $c \in \mathbf{R}$.

Fix $x \in I_+$ arbitrarily and define

$$\Delta(x) = [f'(x)]^2 - f(x)f''(x)\,.$$

Then $\Delta(x)$ is the discriminant of the quadratic equation

$$f(x)c^2 - 2f'(x)c + f''(x) = 0$$

of a variable c. According to (4),

$$\frac{\Delta(x)}{[f(x)]^2} = \left[\frac{f'(x)}{f(x)}\right]^2 - \frac{f''(x)}{f(x)} \leq M_1^2 - M_2\,. \tag{6}$$

To prove (a) observe that if $M_1^2 - M_2 \leq 0$, then $\Delta(x) \leq 0$. Since $\Delta(x)$ is the discriminant, $\Delta(x) \leq 0$ yields that (5) holds for all $c \in \mathbf{R}$.

To prove (b) suppose that $M_1^2 - M_2 > 0$. In this case it is enough to consider the case when $\Delta(x) > 0$. Set

$$c_{\pm}(x) := \frac{f'(x) \pm \sqrt{\Delta(x)}}{f(x)}\,.$$

Then by (4) and (6),

$$-M_1 - \sqrt{M_1^2 - M_2} \leq c_-(x) < c_+(x) \leq M_1 + \sqrt{M_1^2 - M_2}\,,$$

and for each $t \in \mathbf{R} \setminus (c_-(x), c_+(x))$ we have (5). We now set

$$c_1 = \inf\{c_-(x) \in \mathbf{R} : x \in I_+ \text{ with } \Delta(x) > 0\}\,,$$
$$c_2 = \sup\{c_+(x) \in \mathbf{R} : x \in I_+ \text{ with } \Delta(x) > 0\}\,.$$

If we write $T = M_1 + \sqrt{M_1^2 - M_2}$, then $-T \leq c_1 < c_2 \leq T$ so that (5) holds true for each $c \in \mathbf{R} \setminus (c_1, c_2)$ and each $x \in I_+$ with $\Delta(x) > 0$. We thus conclude that (5) holds for all $c \in \mathbf{R} \setminus (c_1, c_2)$.

Consequently, (5) holds for all $x \in I$ and all $c \in \mathbf{R}$ if $M_1^2 - M_2 \leq 0$; (5) holds for all $x \in I$ and all $c \in \mathbf{R} \setminus (c_1, c_2)$ if $M_1^2 - M_2 > 0$. This completes the proof. □

EXAMPLE. Take $I = (0, +\infty)$ and $f(x) = x$. If we put $f_c(x) = f(x)\exp(-cx)$ for each $x \in I$ and each $c \in \mathbf{R}$, then we have $f_c''(x) = (c^2 x - 2c)\exp(-cx)$. Clearly, for $c \leq 0$ the map f_c is convex whereas for $c > 0$ it is neither convex nor concave. Note that since

$$\lim_{x \to 0+} \frac{f'(x)}{f(x)} = +\infty\,,$$

then assumptions of the theorem are not satisfied.

In the next theorem we assume that f satisfies a single inequality of the second degree and we obtain an analogous effect.

THEOREM 3. *If $f \in \mathcal{C}_+^2$ and $M \geq 0$ satisfy*

$$\frac{[f'(x)]^2}{f(x)} \leq f''(x) + M\,,$$

for all $x \in I$ such that $f(x) \neq 0$, then for each $c \in \mathbf{R} \setminus \{0\}$ there exists a convex function $g_c \in \mathcal{C}_+^2$ such that

$$f(x) = -\frac{M}{c^2} + g_c(x)\exp(cx) \quad \textit{for all } x \in I.$$

PROOF. Let $G\colon I \times (\mathbf{R} \setminus \{0\}) \to \mathbf{R}$ be a function defined by

$$G(x,t) = \left(f(x) + \frac{M}{t^2}\right)\exp(-tx).$$

Then we see that $f(x) = -\frac{M}{c^2} + G(x,c)\exp(cx)$ for all $x \in \mathbf{R}$ and $c \in \mathbf{R} \setminus \{0\}$ and

$$\frac{\partial^2 G}{\partial x^2}(x,c)\exp(cx) = f(x)c^2 - 2f'(x)c + f''(x) + M$$

for all $x \in I$ and $c \in \mathbf{R}$. The discriminant of the quadratic equation

$$f(x)c^2 - 2f'(x)c + f''(x) + M = 0$$

of a variable c is

$$[f'(x)]^2 - f(x)[f''(x) + M] = f(x)\left(\frac{[f'(x)]^2}{f(x)} - f''(x) - M\right) \leq 0$$

for every $x \in I$ with $f(x) \neq 0$. This implies that if $x \in I$ and $f(x) \neq 0$, then $\frac{\partial^2 G}{\partial x^2}(x, c) \geq 0$ for all $c \in \mathbf{R}$. For $x \in I$ with $f(x) = 0$, we see by the same arguments as in the proof of Theorem 2 that $f'(x) = 0$ and $f''(x) \geq 0$, and therefore

$$\frac{\partial^2 G}{\partial x^2}(x, c) = f''(x) + M \geq 0$$

for all $c \in \mathbf{R}$. Consequently, $G(x, c)$ is convex for all $c \in \mathbf{R}$. □

We terminate the paper with two corollaries in which we drop the assumption that f is nonnegative.

COROLLARY 4. *Let $f \in \mathcal{C}^2$ and $M_1, M_2 \in \mathbf{R}$ satisfy*

$$|f'(x)| \leq M_1 \quad \textit{and} \quad M_2 \leq f''(x) + [f'(x)]^2$$

for all $x \in I$.

(a) *If $M_1^2 - M_2 \leq 0$, then for each $c \in \mathbf{R}$ there exists a convex map $f_c \in \mathcal{C}_+^2$ such that $f(x) = \log(f_c(x)) + cx$ for all $x \in I$.*

(b) *If $M_1^2 - M_2 > 0$, then there exist constants $c_1, c_2 \in \mathbf{R}$ with $-T \leq c_1 \leq c_2 \leq T$ such that for each $c \in \mathbf{R} \setminus (c_1, c_2)$ there exists a convex map $f_c \in \mathcal{C}_+^2$ satisfying $f(x) = \log(f_c(x)) + cx$ for all $x \in I$, where $T = M_1 + \sqrt{M_1^2 - M_2}$.*

PROOF. Since $\exp \circ f \in \mathcal{C}_+^2$ satisfy

$$\frac{[\exp(f(x))]'}{\exp(f(x))} = f'(x) \quad \text{and} \quad \frac{[\exp(f(x))]''}{\exp(f(x))} = f''(x) + [f'(x)]^2,$$

for each $x \in I$, we see that (4) from Theorem 2 holds for $\exp \circ f$. According to Theorem 2, there exists a convex map $f_c \in \mathcal{C}_+^2$ such that $\exp(f(x)) = f_c(x)\exp(cx)$, and thus $f(x) = \log(f_c(x)) + cx$ for all $x \in I$. □

COROLLARY 5. *Let $f \in \mathcal{C}^2$ and $M_1, M_2 \in \mathbf{R}$ satisfy* (4) *from Theorem 2 for all $x \in I$ with $f(x) \neq 0$.*

(a) *If $2M_1^2 - M_2 \leq 0$, then for each $c \in \mathbf{R}$ there exists a convex map $g_c \in \mathcal{C}_+^2$ such that $|f(x)| = \sqrt{g_c(x)}\exp(\frac{cx}{2})$ for all $x \in I$.*

(b) *If $M_1^2 - M_2 > 0$, then there exist constants $c_1, c_2 \in \mathbf{R}$ with $-T' \leq c_1 \leq c_2 \leq T'$ such that for each $c \in \mathbf{R} \setminus (c_1, c_2)$ there exists a convex map $g_c \in \mathcal{C}_+^2$ satisfying $|f(x)| = \sqrt{g_c(x)}\exp(\frac{cx}{2})$ for all $x \in I$, where $T' = 2M_1 + \sqrt{4M_1^2 - 2M_2}$.*

PROOF. Set $g := f^2$. Then $g \in \mathcal{C}_+^2$ and it satisfies

$$g'(x) = 2f(x)f'(x) \quad \text{and} \quad g''(x) = 2[f'(x)]^2 + 2f(x)f''(x)$$

for all $x \in I$. Therefore, if $x \in I$ with $f(x) \neq 0$, then

$$\left|\frac{g'(x)}{g(x)}\right| = \left|\frac{2f(x)f'(x)}{f^2(x)}\right| = \left|\frac{2f'(x)}{f(x)}\right| \leq 2M_1\,,$$

$$\frac{g''(x)}{g(x)} = \frac{2[f'(x)]^2 + 2f(x)f''(x)}{f^2(x)} = 2\left[\frac{f'(x)}{f(x)}\right]^2 + 2\frac{f''(x)}{f(x)} \geq 2\frac{f''(x)}{f(x)} \geq 2M_2\,.$$

Thus $\left|\frac{g'(x)}{g(x)}\right| < 2M_1$ and $2M_2 \leq \frac{g''(x)}{g(x)}$ for all $x \in I$ with $g(x) \neq 0$. By Theorem 2, if $2M_1^2 - M_2 \leq 0$, then for each $c \in \mathbf{R}$ there exists a convex map $g_c \in \mathcal{C}_+^2$ such that $g(x) = g_c(x)\exp(cx)$ for all $x \in I$. Consequently, $|f(x)| = \sqrt{g_c(x)}\exp(\frac{cx}{2})$ for all $x \in I$. If $2M_1^2 - M_2 > 0$, then there exist constants $c_1, c_2 \in \mathbf{R}$ with $-T' \leq c_1 \leq c_2 \leq T'$ such that for each $c \in \mathbf{R} \setminus (c_1, c_2)$ there exists a convex map $g_c \in \mathcal{C}_+^2$ satisfying $g(x) = g_c(x)\exp(cx)$ for all $x \in I$, where $T' = 2M_1 + \sqrt{4M_1^2 - 2M_2}$. Therefore, $|f(x)| = \sqrt{g_c(x)}\exp(\frac{cx}{2})$ for all $x \in I$. $\square$

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