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A Note on the k-Buchsbaum Property of Symbolic Powers of Stanley-Reisner Ideals

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Abstract. Let I be the Stanley-Reisner ideal of pure simplicial complex Δ of dimension one. We shall give a formula for $S/I^{(r)}$ to be a k-Buchsbaum ring for each $r > 0$, where $I^{(r)}$ is the r-th symbolic power of I. The main result is an improvement of the previous result in [MN] on the k-Buchsbaumness of $S/I^{(r)}$.

1. Introduction

Let Δ be a simplicial complex on a vertex set $[n] = \{1, 2, \dots, n\}$. Let $S = k[x_1, x_2, \dots, x_n]$ be a polynomial ring of n-variables over a field k. Stanley-Reisner ideal I is defined as;

$$I = I_\Delta = \left(\prod_{i \in F} x_i \mid F \notin \Delta \right),$$

which is a square-free monomial ideal of S being associated to Δ. The residue class ring S/I is called the Stanley-Reisner ring. Throughout this article, we assume that Δ is pure and $\dim(\Delta) = 1$, which means that any maximal element of Δ consists of two element. We study the k-Buchsbaum property of $S/I^{(r)}$ for all $r > 0$ and all Δ, where $I^{(r)}$ is the r-th symbolic power of I. In our situation, $S/I^{(r)}$ is a generalized Cohen-Macaulay ring with $\dim S/I^{(r)} = 2$ and depth $S/I^{(r)} > 0$. The condition for $S/I^{(r)}$ to be k-Buchsbaum is equivalent to saying that k is the minimal number satisfying $\mathfrak{m}^k H^1_{\mathfrak{m}}(S/I^{(r)}) = (0)$. We put

$$k(r) = \min\{k \in \mathbb{N} \mid \mathfrak{m}^k H^1_{\mathfrak{m}}(S/I^{(r)}) = (0)\}.$$

Our purpose can be said to determine the value $k(r)$ for any $r > 0$ and Δ.

It is known that S/I is always a Buchsbaum ring, and that S/I is Cohen-Macaulay if and only if Δ is connected (see [BH], [S]). For the case of symbolic powers, the first author and N. V. Trung gave the characterization for $S/I^{(r)}$ to be Cohen-Macaulay in terms of the graphical property of Δ ([MT]). After that, in [MN], we get the characterization of Buchsbaumness of

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$S/I^{(r)}$. In [MN], we also studied the k-Buchsbaum property of $S/I^{(r)}$ when Δ is connected ([MN, Theorem 3.8]). In this paper, we remove the assumption of connectedness of the previous result. Combining the statement of [MN, Theorem 3.8], we have the following result.

THEOREM 1.1. *Let $r > 1$ be an integer. Assume that $S/I^{(r)}$ is not Cohen-Macaulay. Then*

$$k(r) = \mathrm{d}(H^1_{\mathfrak{m}}(S/I^{(r)})) = \begin{cases} r-2 & \text{if } \operatorname{diam}(\Delta) \le 2 \\ r-1 & \text{if } 3 \le \operatorname{diam}(\Delta) < \infty \\ 2r-1 & \text{if } \operatorname{diam}(\Delta) = \infty \end{cases} .$$

Here, we put

$$\mathrm{d}(M) = \max\{n | M_n \neq 0\} - \min\{n | M_n \neq 0\} + 1$$

for the finitely generated $\mathbb{Z}$-graded module M with $M \neq (0)$ and $\mathrm{d}(M) = 0$ if $M = (0)$. It is clear that $k(r) \le \mathrm{d}(H^1_{\mathfrak{m}}(S/I^{(r)}))$. Further, $\operatorname{diam}(\Delta)$, the diameter of simplical complex Δ, is defined as;

$$\operatorname{diam}(\Delta) = \max_{i,j \in [n]} \operatorname{dist}(i, j) ,$$

where $\operatorname{dist}(i, j)$ is the minimal length of the path between nodes i and j. $\operatorname{dist}(i, j)$ is infinite if there is no paths connecting i and j. Thus, $\operatorname{diam}(\Delta) < \infty$ is equivalent to saying that Δ is connected. In [MN], we have determined $k(r)$ in the case that $\operatorname{diam}(\Delta) < \infty$. In order to prove the theorem in disconnected cases, unfortunately the methord used connected cases does not work. We prepare an argument using the concept of *cone* of simplicial complexes.

From Theorem 1.1, we immediately get the characterization of the Buchsbaumness of $S/I^{(r)}$.

COROLLARY 1.2 ([MN, Theorem 3.7]). *Let I be the Stanley-Reisner ideal of a pure simplicial complex Δ of dimension one. Let $r > 0$ be an integer. Then the following statements hold true.*

(1) *$S/I^{(2)}$ is Buchsbaum if and only if Δ is connected.*

(2) *$S/I^{(3)}$ is Buchsbaum if and only if* $\operatorname{diam}(\Delta) \le 2$.

(3) *Let $r > 3$. If $S/I^{(r)}$ is Buchsbaum, then it is Cohen-Macaulay.*

The paper consists of three sections. In Section 2, we set up notations, terminologies. We quote some fundamental results from [MT] and [MN]. In Section 3, we prepare auxiliary arguments with respect to the cone of complexes, and then give the proof of the main result.

2. Preliminaries

We begin with the notation on a simplicial complex. A simplicial complex Δ on a finite set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$. Notice that, for the convenience in the later discussions, we do *not* assume the condition that $\{i\} \in \Delta$ for $i = 1, 2, \ldots, n$. We put $\dim F = |F| - 1$, where $|F|$ means the cardinality of F, and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$, which is called the dimension of Δ. When we assume a linear order on $[n]$, say $<$, Δ is called an oriented simplicial complex. In such a case, we denote $F = \{i_1, \ldots, i_r\}$ for $F \in \Delta$ with the order sequence $i_1 < \cdots < i_r$. Let Δ be an oriented simplicial complex with $\dim \Delta = d$. We denote by $\mathcal{C}(\Delta)_\bullet$ the augmented oriented chain complex of Δ:

$$\mathcal{C}(\Delta)_\bullet : 0 \to C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to C_{-1} \to 0$$

where

$$C_t = \bigoplus_{\substack{F \in \Delta \\ \dim F = t}} \mathbb{Z}F \qquad \text{and} \qquad \partial F = \sum_{j=0}^{t} (-1)^j F_j$$

for all $F \in \Delta$. Here we denote $F_j = \{i_0, \ldots, \hat{i}_j, \ldots, i_t\}$ for $F = \{i_0, \ldots, i_t\}$. For any field k, we define the i-th reduced simplicial homology group $\widetilde{H}_i(\Delta; k)$ of Δ to be the i-th homology group of the complex $\mathcal{C}(\Delta)_\bullet \otimes k$. Further we define the i-th reduced simplicial cohomology group $\widetilde{H}^i(\Delta; k)$ of Δ to be the i-th cohomology group of the dual chain complex $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta)_\bullet, k)$ for all i. Then it follows that

$$\dim_k \widetilde{H}_i(\Delta; k) = \dim_k \widetilde{H}^i(\Delta; k) \quad \text{for all } i \in \mathbb{Z} \quad \text{and}$$

$$\widetilde{H}_{-1}(\Delta; k) \cong \widetilde{H}^{-1}(\Delta; k) \cong \begin{cases} k & \text{if } \Delta = \{\emptyset\} \\ 0 & \text{otherwise} \end{cases} .$$

We also note that $\widetilde{H}_i(\Delta; k) = \widetilde{H}^i(\Delta; k) = 0$ for all $i \in \mathbb{Z}$ if $\Delta = \emptyset$. Moreover, it is known that

$$\dim_k(\widetilde{H}_0(\Delta; k)) = \text{the number of connected components of } \Delta - 1$$

when $\Delta \neq \emptyset$ (see [V, Proposition 5.2.3]). Let $\Gamma \subseteq \Delta$ be a simplicial subcomplex of Δ; then $\mathcal{C}(\Gamma)_\bullet$ is a subcomplex $\mathcal{C}(\Delta)_\bullet$, which yields the quotient complex $\mathcal{C}(\Delta)_\bullet/\mathcal{C}(\Gamma)_\bullet$. The cohomology module

$$\widetilde{H}^i(\Delta, \Gamma; k) = H^i(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta)_\bullet/\mathcal{C}(\Gamma)_\bullet; k))$$

is called the *i-th reduced relative simplicial cohomology* of the pair (Δ, Γ). Let Γ and Δ be simplicial complexes on disjoint vertex sets V and W, respectively. The *join* $\Gamma * \Delta$ is the simplicial complex on the vertex set $V \cup W$ consists of faces $F \cup G$ where $F \in \Gamma$ and $G \in \Delta$.

The *cone*

$$\mathrm{Cone}_x(\Delta) = x * \Delta$$

of Δ is the join of a point $\{x\}$ with Δ.

Let I be a monomial ideal of S. For $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ we put the subset $G_{\mathbf{a}} = \{i | a_i < 0\}$ of $[n]$. Degree complex (see [T]) is denoted by $\Delta_{\mathbf{a}}(I)$ consists of all $F \subseteq [n]$ such that

(1) $F \cap G_{\mathbf{a}} = \emptyset$,
(2) For every minimal generator $x^{\mathbf{b}}$ of I there exists an index $i \notin F \cup G_{\mathbf{a}}$ with $b_i > a_i$.

Here we pick up important results stated in [MT] and [MN], which will be applied several times in our argument.

LEMMA 2.1 ([MN]). *Let I be the Stanley-Reisner ideal of a pure simplicial complex Δ of dimension one. Then, the following assertions hold true for all $0 < r \in \mathbb{N}$.*

(1) *Let $\mathbf{a} \in \mathbb{N}^n$ and $\Delta_{\mathbf{a}}(I^{(r)}) \neq \emptyset$. Then $\Delta_{\mathbf{a}}(I^{(r)})$ is a subcomplex of Δ of pure dimension one.*
(2) *Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. For $i, j \in [n]$, we put $\sigma_{ij} = |\mathbf{a}| - (a_i + a_j)$, where $|\mathbf{a}| = \sum_{k=1}^n a_k$. Then we have the following equivalent conditions:*
   (*a*) $\{i, j\} \in \Delta_{\mathbf{a}}(I^{(r)})$.
   (*b*) $\sigma_{ij} < r$ *and* $\{i, j\} \in \Delta$.

Next is the behaviour of the first local cohomology of $S/I^{(r)}$.

LEMMA 2.2 ([MT], [MN, Section 3]). *Let I be the Stanley-Reisner ideal of a pure simplicial complex Δ of dimension one. Let $r > 0$ be an integer. The following assertions hold true.*

(1) *Let $\mathbf{a} \in \mathbb{Z}^n$. If $G_{\mathbf{a}} \neq \emptyset$ then $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} = 0$.*
(2) $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = 0$ *for all* $j > 2r - 2$.
(3) *Let $0 \leq j < r$. Then $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = 0$ if and only if G is connected.*
(4) *Assume $r > 1$. Then $[H^1_{\mathfrak{m}}(S/I^{(r)})]_r = 0$ if and only if* $\mathrm{diam}(G) \leq 2$.
(5) *Assume $r > 2$ and $r + 1 \leq j \leq 2r - 2$. Then $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = (0)$ if and only if any pair of disjoint edges of G is contained in a cycle of length* 4.

Here, $\mathrm{diam}(G)$, the diameter of simplical comples G, is defined as $\mathrm{diam}(G) = \max_{i,j \in [n]} \mathrm{dist}(i, j)$, where $\mathrm{dist}(i, j)$ is the minimal length of the path between nodes i and j.

At the end of the section, we recall a formula between the local cohomology modules and reduced cohomology modules, due to Takayama.

LEMMA 2.3 ([BH, Lemma 5.3.7], [T, Lemma 2]). *Let I be a monomial ideal of S. For all $t \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^n$, there is an isomorphism of k-vector spaces*

$$H^t_{\mathfrak{m}}(S/I)_{\mathbf{a}} \cong \widetilde{H}^{t-|G_{\mathbf{a}}|-1}(\Delta_{\mathbf{a}}(I); k)\,.$$

The above isomorphism causes more information. Let $\mathbf{b} \in \mathbb{N}^n$ and take the monomial $\mathbf{x}^{\mathbf{b}} = \prod_{j=1}^{n} x_j^{b_j} \in S$. The multiplicative map $S/I \ni f \mapsto \mathbf{x}^{\mathbf{b}} f \in S/I$ induces the homomorphism

$$H_{\mathfrak{m}}^t(S/I)_{\mathbf{a}} \xrightarrow{\mathbf{x}^{\mathbf{b}}} H_{\mathfrak{m}}^t(S/I)_{\mathbf{a}+\mathbf{b}} .$$

LEMMA 2.4 ([MN, Lemma 2.3]). *Let I be a monomial ideal of S and $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. For any integers $j \geq 0$, we have the following commutative diagram:*

$$\begin{array}{ccc} H_{\mathfrak{m}}^j(S/I)_{\mathbf{a}} & \xrightarrow{\mathbf{x}^{\mathbf{b}}} & H_{\mathfrak{m}}^j(S/I)_{\mathbf{a}+\mathbf{b}} \\ \downarrow & & \downarrow \\ \widetilde{H}^{j-1}(\Delta_{\mathbf{a}}(I); k) & \longrightarrow & \widetilde{H}^{j-1}(\Delta_{\mathbf{a}+\mathbf{b}}(I); k) \end{array}$$

where the vertical maps are isomorphisms as in Lemma 2.3 and the bottom map is induced from the natural embedding $\Delta_{\mathbf{a}+\mathbf{b}}(I) \subseteq \Delta_{\mathbf{a}}(I)$ of simplicial complexes.

3. Proof of main result

We begin by establishing the following assertion.

LEMMA 3.1. *Let Δ be an arbitrary simplicial complex over $[n]$ and $\Gamma \subseteq \Delta$ a simplicial subcomplex. Then there is an isomorphism of k-vector spaces*

$$\widetilde{H}^j(\Delta, \Gamma; k) \cong \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma); k) ,$$

for all $j \in \mathbb{Z}$, where $\mathrm{Cone}(\Gamma) = \mathrm{Cone}_x(\Gamma)$ with a new vertex $x \notin [n]$.

PROOF. By definition, there is an isomorphism between the chain complexes

$$\mathcal{C}(\Delta)_\bullet/\mathcal{C}(\Gamma)_\bullet \cong \mathcal{C}(\Delta \cup \mathrm{Cone}(\Gamma))_\bullet/\mathcal{C}(\mathrm{Cone}(\Gamma))_\bullet .$$

Therefore, $\widetilde{H}^j(\Delta, \Gamma; k) \cong \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma), \mathrm{Cone}(\Gamma); k)$ for all $j \in \mathbb{Z}$. On the other hand, the short exact sequence of complexes

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta \cup \mathrm{Cone}(\Gamma))_\bullet/\mathcal{C}(\mathrm{Cone}(\Gamma))_\bullet, k) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta \cup \mathrm{Cone}(\Gamma))_\bullet, k) \\ \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\mathrm{Cone}(\Gamma))_\bullet, k) \longrightarrow 0, \end{aligned}$$

yields the following long exact sequence

$$\begin{aligned} \cdots \longrightarrow \widetilde{H}^{j-1}(\mathrm{Cone}(\Gamma); k) \longrightarrow \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma), \mathrm{Cone}(\Gamma); k) \\ \longrightarrow \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma); k) \longrightarrow \widetilde{H}^j(\mathrm{Cone}(\Gamma)); k) \longrightarrow \cdots \end{aligned}$$

Since $\mathrm{Cone}(\Gamma)$ is acyclic, $\widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma); k) \cong \widetilde{H}^j(\Delta \cup \mathrm{Cone}(\Gamma), \mathrm{Cone}(\Gamma); k)$ for all $j \in \mathbb{Z}$. This implies our assertion.

□

We are now in a position to prove the main theorem.

PROOF OF THEOREM 1.1
By [MN, Theorem 3.8], we may assume that Δ is not connected. Then by Lemma 2.2, $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j \neq (0)$ if and only if $0 \leq j \leq 2r-2$. Thus $\mathrm{d}(H^1_{\mathfrak{m}}(S/I^{(r)})) = 2r-1$. To get the conclusion, it is enough to check that $\mathfrak{m}^{2r-2}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$. Since Δ is not connected, we may assume that $\{1,2\}, \{3,4\}$ belong to different components of Δ. Put $\mathbf{a} = (r-1)\mathbf{e}_1 + (r-1)\mathbf{e}_3$, where $\mathbf{e}_i$ is i-th unit vector in $\mathbb{Z}^n$. Applying Lemma 2.1, one can check

$$\Gamma = \Delta_{\mathbf{a}}(I^{(r)}) = \mathrm{star}_{\Delta}(1) \cup \mathrm{star}_{\Delta}(3)\,.$$

Then $\widetilde{H}^{-1}(\Gamma; k) = 0$ and Γ is not connected. Hence, we have the following long exact sequence of reduced cohomology modules

$$0 \longrightarrow \widetilde{H}^0(\Delta, \Gamma; k) \longrightarrow \widetilde{H}^0(\Delta; k) \longrightarrow \widetilde{H}^0(\Gamma; k) \longrightarrow \widetilde{H}^1(\Delta, \Gamma; k) \longrightarrow \cdots$$

Note that $\widetilde{H}^0(\Delta; k) \longrightarrow \widetilde{H}^0(\Gamma; k)$ in the above is induced from the natural embedding $\Gamma \subseteq \Delta$. On the other hand, by Lemma 3.1,

$$\begin{aligned}
\dim_k(\widetilde{H}^0(\Delta, \Gamma; k)) &= \dim_k(\widetilde{H}^0(\Delta \cup \mathrm{Cone}(\Gamma); k)) \\
&= \text{the number of connected components of } \Delta \cup \mathrm{Cone}(\Gamma) \ - \ 1 \\
&< \text{the number of connected components of } \Delta \ - \ 1 \\
&= \dim_k(\widetilde{H}^0(\Delta; k))\,.
\end{aligned}$$

Hence the natural map $\widetilde{H}^0(\Delta; k) \longrightarrow \widetilde{H}^0(\Gamma; k)$ is never zero map. By Lemma 2.4, we have the following commutative diagram:

$$\begin{array}{ccc}
H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{0}} & \xrightarrow{\ \mathbf{x}^{\mathbf{a}}\ } & H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \\
\downarrow & & \downarrow \\
\widetilde{H}^0(\Delta_{\mathbf{0}}(I^{(r)}); k) & \longrightarrow & \widetilde{H}^0(\Delta_{\mathbf{a}}(I^{(r)}); k)
\end{array}$$

Moreover, since $\Delta_{\mathbf{0}}(I^{(r)}) = \Delta$ then $\mathbf{x}^{\mathbf{a}} H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{0}} \neq (0)$. It implies

$$\mathfrak{m}^{2r-2}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)\,,$$

which is the desired conclusion.

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