TOKYO J. MATH.
VOL. 34, NO. 1, 2011

Surgical Distance between Lens Spaces

Dedicated to Professor Akio Kawauchi on the occasion of his 60th birthday

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(Communicated by K. Ahara)

Abstract. It is well-known that any pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries on knots. Furthermore Kawauchi showed that such knots can be taken to be hyperbolic. In this article, we consider the minimal length of such sequences connecting a pair of 3-manifolds, in particular, a pair of lens spaces.

1. Introduction

As a consequence of the famous Geometrization Conjecture raised by Thurston in [28, section 6, question 1], all closed orientable 3-manifolds are classified as follows: They should be: reducible (i.e., containing essential 2-spheres), toroidal (i.e., containing essential tori), Seifert fibered (i.e., foliated by circles), or hyperbolic manifolds (i.e., admitting a complete Riemannian metric with constant sectional curvature -1). Also see [13, Problem 3.45], and see [26] for a survey.

Now, by the celebrated Perelman's works [17, 18, 19], an affirmative answer to this Geometrization Conjecture could be given. Beyond the classification, one of the next directions in the study of 3-manifolds would be to consider relationships between 3-manifolds. One of the important operations describing such relationships must be *Dehn surgery*. This is an operation to create a new 3-manifold from a given one and a given knot (i.e., an embedded simple closed curve) in the following way: Remove an open tubular neighborhood of the knot, and glue a solid torus back. It gives an interesting subject to study; because, for instance, it is known that any pair of connected closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries on knots, proved by Lickorish [15] and Wallace [29] independently. See also Fact 1 below.

In this article, in terms of Dehn surgery on knots, we introduce a *distance* between pairs of 3-manifolds. Furthermore, by considering the surgery on *hyperbolic knots*, another distance function is also defined, and we report the study of its restriction on the set of lens spaces.

Received October 19, 2009
2000 *Mathematics Subject Classification*: 57M50 (Primary), 57M25 (Secondary)
Key words and phrases: Dehn surgery, hyperbolic knot, lens space

Throughout the article, for convenience, we denote by $\mathcal{M}$ the set of orientation preserving homeomorphism types of connected closed orientable 3-manifolds.

2. Backgrounds

In this section, we will introduce some new definitions about Dehn surgery, and review backgrounds and known results about them. Also we will state a number of open problems which we will consider.

2.1. Surgical distance. First of all, we introduce a function $d : \mathcal{M} \times \mathcal{M} \to \mathbf{Z}_{\geq 0}$ defined as follows; for $[M], [M'] \in \mathcal{M}$, $d([M], [M'])$ is defined as the minimal length of the sequence $[M] = [M_0], [M_1], \ldots, [M_n] = [M'] \in \mathcal{M}$ such that M_{i+1} is obtained from M_i by Dehn surgery on a knot.

It is easy to verify that if the function d is well-defined, then it satisfies the axiom of distance function. Further, as we cited above, the following is known:

FACT 1 (Lickorish [15], Wallace [29]). The function $d : \mathcal{M} \times \mathcal{M} \to \mathbf{Z}_{\geq 0}$ is well-defined. That is, for any pair $[M], [M'] \in \mathcal{M}$, there exists a finite sequence $[M] = [M_0], [M_1], \ldots, [M_n] = [M'] \in \mathcal{M}$ such that M_{i+1} is obtained from M_i by Dehn surgery on a knot.

Remark that, in [1], Auckly defined a similar notion: "surgery number" of $[M] \in \mathcal{M}$. This is equal to $d([S^3], [M])$ in our definition. Also see [13, Problem 3.102].

Also remark that, by a Dehn surgery on a knot, the first betti number β_1 of a 3-manifold can be changed only by ± 1. So $d([M], [M']) \geq |\beta_1(M) - \beta_1(M')|$ holds for $[M], [M'] \in \mathcal{M}$. Thus it would be natural to ask:

PROBLEM 1. *For any given $N > 0$, can we find a pair $[M], [M'] \in \mathcal{M}$ such that $\beta_1(M) = \beta_1(M')$ but $d([M], [M']) \geq N$?*

Here we collect several related known facts:

- All lens spaces have the first betti number at most one. And $d([L], [L']) = 1$ for any lens spaces L, L'. See later for definitions of lens spaces.
- In [11, Theorem 3], Gordon and Luecke showed $d([S^3], [M]) > 1$ if M is non-prime without lens space summands. Thus we can obtain infinitely many 3-manifolds M with $\beta_1(M) = \beta_1(S^3) = 0$ with $d([S^3], [M]) > 1$.
- In [1], Auckly found the first hyperbolic example $[M] \in \mathcal{M}$ with $\beta_1(M) = 0$ such that $d([S^3], [M]) > 1$.

As far as the authors know, there are no explicit examples of pairs of manifolds for which the surgical distance is determined to be three or more.

2.2. Hyperbolic surgical distance. Next we consider Dehn surgery on *hyperbolic knots*, that is, the knots with complements which admit complete hyperbolic metrics of finite volume. In fact, we introduce a function $d_H : \mathcal{M} \times \mathcal{M} \to \mathbf{Z}_{\geq 0}$ defined as follows; for $[M], [M'] \in \mathcal{M}$, $d_H([M], [M'])$ is defined as the minimal length of the sequence $[M] = [M_0], [M_1], \ldots, [M_n] = [M'] \in \mathcal{M}$ such that M_{i+1} is obtained from M_i by Dehn surgery on a hyperbolic knot.

One reason why we choose to consider hyperbolic knots is as follows: Following the classification of 3-manifolds, all knots are also classified into several types. When one considers only knots in types of hyperbolic, the next was established by Kawauchi using his "Imitation Theory". See [12, Theorem 3.1] for example.

FACT 2 (Kawauchi). $d_H : \mathcal{M} \times \mathcal{M} \to \mathbf{Z}_{\geq 0}$ is well-defined. That is, for any pair $[M], [M'] \in \mathcal{M}$, there exists a finite sequence $[M] = [M_0], [M_1], \ldots, [M_n] = [M'] \in \mathcal{M}$ such that M_{i+1} is obtained from M_i by Dehn surgery on a hyperbolic knot.

It is then easy to verify that this function also satisfies the axiom of distance function. Furthermore, Kawauchi showed the following:

FACT 3 (Kawauchi). For $[M], [M'] \in \mathcal{M}$,

$$d_H([M], [M']) = \begin{cases} 1 \quad \text{or} \quad 2 & \text{if} \quad d([M], [M']) = 1 \\ d([M], [M']) & \text{otherwise} \end{cases}$$

Then it seems to be natural to ask:

PROBLEM 2. *When can* $d([M], [M']) \neq d_H([M], [M'])$ *occur?*

Concerning this question, there are several known facts. We collect them in the following.

- $d([S^3], [L(p, q)]) = 1$ and $d_H([S^3], [L(p, q)]) = 2$ if q is not a quadratic residue modulo p; i.e., $x^2 \not\equiv \pm q \mod p$ for any x. (Fintushel-Stern [6, Proposition 1])
- $d_H([S^3], [S^2 \times S^1]) = 2$, while $d([S^3], [S^2 \times S^1]) = 1$. (Gabai [7])
- There is a pair of lens spaces L, L' such that $d_H([L], [L']) = 1$ and L and L' are orientation-reversingly homeomorphic. (Bleiler-Hodgson-Weeks [4]) There is only one known example with such a property. See [13, Problem 1.81] for related conjectures.
- $d([S^3], [L(p, q)]) = 1$ and $d_H([S^3], [L(p, q)]) = 2$ if $|p| < 9$. In particular, $d([S^3], [RP^3]) = 1$ and $d_H([S^3], [RP^3]) = 2$. (Kronheimer-Mrowka-Ozsváth-Z. Szabó [14, Theorem 1.1.], Ozsváth-Szabó [20])
- For the Poincaré homology sphere P, $d([S^3], [P]) = 1$ and $d_H([S^3], [P]) = 2$. (Ghiggini [9])
- There is a sufficient condition to be $d_H([S^2 \times S^1], [L]) = 2$ for a lens space L. (Lisca [16])

We remark that the facts above could be obtained mainly from the results in the references.

3. On the set of lens spaces

In the rest of the article, we will concentrate on the set of lens spaces. We here call a 3-manifold L with Heegaard genus at most one (i.e., constructed by gluing two solid tori) a *lens space*. Thus, in this article, we say that S^3, $S^2 \times S^1$ and RP^3 are all lens spaces. Denote by $\mathcal{L}$ the set of orientation preserving homeomorphism types of lens spaces. Then note that $d([L], [L']) = 1$ and $d_H([L], [L']) \leq 2$ for any $[L], [L'] \in \mathcal{L}$

By regarding $\mathcal{L}$ as a subset of $\mathcal{M}$, we can consider the following definition naturally.

DEFINITION 1. For $[L], [L'] \in \mathcal{L}$, suppose that there exists a sequence $[L] = [L_0], [L_1], \ldots, [L_n] = [L'] \in \mathcal{L}$ such that L_{i+1} is obtained from L_i by Dehn surgery on a hyperbolic knot. Then we define $d_H([L], [L'])_{\mathcal{L}}$ as the minimal length of such sequence.

However, as far as the authors know, it is unknown whether the definition above could give a distance function on $\mathcal{L}$:

PROBLEM 3. *Can $d_H([L], [L'])_{\mathcal{L}}$ be well-defined for any $[L], [L'] \in \mathcal{L}$? Equivalently, for any pair $[L], [L'] \in \mathcal{L}$, does there exist a finite sequence $[L] = [L_0], [L_1], \ldots, [L_n] = [L'] \in \mathcal{L}$ such that L_{i+1} is obtained from L_i by Dehn surgery on a hyperbolic knot*?

Recall that: If $d_H([L], [L']) = 1$, then $d_H([L], [L'])_{\mathcal{L}} = 1$ by definition. However $d_H([L], [L'])_{\mathcal{L}} \geq d_H([L], [L']) = 2$ in general. Thus we can consider the following problem, which gives one of the motivations for our study:

PROBLEM 4. *Are there $[L], [L'] \in \mathcal{L}$ such that $d_H([L], [L'])_{\mathcal{L}} > d_H([L], [L'])$? Equivalently, are there $[L], [L'] \in \mathcal{L}$ such that $d_H([L], [L'])_{\mathcal{L}} > 2$?*

Here we include basic facts on lens spaces. Usually, lens spaces are parametrized by a pair of coprime integers as follows. Let V_1 be a regular neighborhood of a trivial knot in S^3, m a meridian of V_1 and ℓ a longitude of V_1 such that ℓ bounds a disk in $\mathrm{cl}(S^3 \setminus V_1)$. We fix an orientation of m and ℓ as illustrated in Figure 3. By attaching a solid torus V_2 to V_1 so that $\bar{m}$ is isotopic to a representative of $p[\ell] + q[m]$ in ∂V_1, we obtain a lens space, which is denoted by $L(p, q)$, where p and q are integers with $p > 0$ and $(p, q) = 1$, and $\bar{m}$ is a meridian of V_2. It is known that two lens spaces $L(p, q)$ and $L(p', q')$ are (possibly orientation reversing) homeomorphic, i.e., $L(p, q) \cong L(p', q')$ if and only if $|p| = |p'|$, and $q \equiv \pm q' \pmod{p}$ or $qq' \equiv \pm 1 \pmod{p}$. See [22] for example.

4. Results

In this section, we will give our results concerning to Problems 2 and 4.

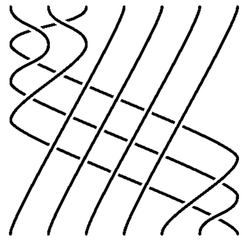

FIGURE 1. $\beta = W_3^{-1} W_7^3$.

Recall that $d([L],[L']) = 1$ and $d_H([L],[L']) \leq 2$ for any $[L],[L'] \in \mathcal{L}$. So consider the question: For which $[L],[L'] \in \mathcal{L}$, $d_H([L],[L']) = 1$?

We here recall that basic terminology about Dehn surgery on knots in the 3-sphere. See [22] in details for example. As usual, by a *slope*, we call an isotopy class of a non-trivial unoriented simple closed curve on a torus. Then Dehn surgery on a knot K is characterized by the slope on the peripheral torus of K which is represented by the simple closed curve identified with the meridian of the attached solid torus via the surgery. When K is a knot in S^3, by using the standard meridian-longitude system, slopes on the peripheral torus are parametrized by rational numbers with $1/0$. For example, the meridian of K corresponds to $1/0$ and the longitude to 0. We say that a Dehn surgery on K in S^3 is p/q-surgery if it is along the slope p/q. This means that the curve representing the slope runs meridionally p times and longitudinally q times.

Let V be a solid torus standardly embedded in S^3, K_1 the closure of an n-string braid in $V \subset S^3$, and K_0 a core loop of the solid torus which is the exterior of V in S^3. Set $K := K_0 \cup K_1$, and let $K(p/q, r/s)$ denotes the 3-manifold obtained by the p/q-surgery on K_0 and the r/s-surgery on K_1. In this paper, $K(p/q, -)$ (resp. $K(-, r/s)$) denotes the 3-manifold obtained by the p/q-surgery on K_0 (resp. the r/s-surgery on K_1) and removing an open tubular neighborhood of K_1 (resp. K_0).

PROPOSITION 1. *Suppose that* $K(-, r/s) \cong D^2 \times S^1$. *Then* $K(p/q, r/s) \cong L(pr - (n^2 s)q, xq - yp)$, *where x and y are coprime integers satisfying* $y(n^2 s) - xr = 1$.

PROOF. Since we suppose that $K(-, r/s) \cong D^2 \times S^1$, it follows from [10, Lemma 3.3(ii)] that the meridian of the new solid torus is given by the slope $r/(n^2 s)$. Hence the conclusion immediately follows from [4, Lemma 3]. □

In the following of this section, let β be the 7-string braid and K_1 its closure in the solid torus V illustrated in Figure 1. We note that K_1 is denoted by $W_3^{-1} W_7^3$ in [3]. It follows from [3] and [4] that $K(-, 18/1) \cong D^2 \times S^1$. Since K_1 is a 7-string braid, we see that $K(p/q, 18/1)$ is orientation preservingly homeomorphic to $L(18p - 49q, 19q - 7p)$, denote by $K(p/q, 18/1) \equiv L(18p - 49q, 19q - 7p)$.

PROPOSITION 2. *Let K_1' be the image of K_1 in the lens space obtained by the p/q-surgery on K_0 with $p > 0$. Then there exists an integer $c > 0$ such that K_1' is hyperbolic in the lens space for any integer q with $(p, q) = 1$ and $|q| \geq c$.*

PROOF. Since the number of strands of the braid β is 7, which is a prime, and the exponent sum of β is 16, which is not a multiple of $7 - 1 = 6$, it follows from [5, Proposition 9.4] that the braid β is pseudo-Anosov. Let φ_β be the pseudo-Anosov homeomorphism of a punctured disk obtained from the braid β. See [5] for the definition of pseudo-Anosov homeomorphisms for example. By the definition, there exits a measured foliation τ_β on the punctured disk, which is invariant for φ_β. Let $E(K_1)$ be the exterior of K_1 in V, that is, $E(K_1) = \mathrm{cl}(V - N(K_1))$, where $N(K_1)$ denotes a tubular neighborhood of K_1. By regarding this $E(K_1)$ as the surface bundle over the circle with monodromy φ_β, we find an essential lamination $\mathcal{L}_\beta$ in the exterior as a suspension of τ_β. See [8] for example.

In the complement of $\mathcal{L}_\beta$, we have an annulus A connecting from a leaf of $\mathcal{L}_\beta$ to the boundary $\partial N(K_0)$, which comes from the suspension of the arc on the punctured disk connecting a leaf of τ_β to a boundary circle. The boundary component of A on $\partial N(K_0)$ determines a slope, which is so-called *degeneracy slope* for $\mathcal{L}_\beta$. Denote it by $\gamma = u/v$ with $u > 0$.

It then follows from [31, Theorem 2.5] that K_1' is hyperbolic if $\Delta(p/q, \gamma) = |pv - qu| \geq 3$.

Since p, u and v are constant and $u > 0$, if we take an integer q with $q \leq pv + 3$, we see that $pv - qu \leq pv - (pv + 3)u = -3u \leq -3$ and hence K_1' is hyperbolic. □

For a given lens space $L(p, q)$ and an integer $c > 0$, we can always find an integer q_0' with $q_0' > c$ and $L(p, q_0') \equiv L(p, q)$, because we, if necessary, can replace q by $q_0' := q + pn$ ($n \in \mathbf{Z}$). Moreover, there is an infinite set of integers $Q = \{q' \mid q' > c,\ q' = q + pn (n \in \mathbf{Z})\}$. Then it follows from Proposition 2 that K_1' is hyperbolic in the lens space $L(p, q') \equiv L(p, q)$ for any $q' \in Q$. Since $K(p/q', 18/1) \equiv L(18p - 49q', 19q' - 7p)$ and $L(p, q') \equiv L(p, q)$, we have:

THEOREM 1. *For every $[L] \in \mathcal{L}$, there exists an infinite family $[L_i] \in \mathcal{L}$ such that $d_H([L], [L_i]) = 1$ for any $i \in \mathbf{N}$.*

Using arguments similar to the above, we also have:

THEOREM 2. *For every $p \in \mathbf{N}$, there exist two pairs of coprime integers (r, s) and (r', s') such that $d_H([L(r, s)], [L(r', s')]) = 1$ and $|r - r'| = p$.*

PROOF. We use the same link $K = K_0 \cup K_1$ as above, and let K_1' be again the image of K_1 in the lens space obtained by the p/q-surgery on K_0 with $p > 0$. Then it follows from [3] that $K(-, 19/1) \cong D^2 \times S^1$. Also see [4]. Hence we see that $K(p/q, 19/1) \equiv L(19p - 49q, 18q - 7p)$. As mentioned above, $K(p/q, 18/1) \equiv L(18p - 49q, 19q - 7p)$. This implies that the dual knot of K_1' in $L(19p - 49q, 18q - 7p)$ admits a Dehn surgery

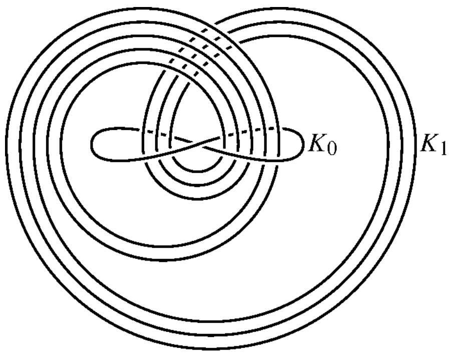

FIGURE 2. $K := K_0 \cup K_1$.

yielding $L(18p - 49q, 19q - 7p)$. Hence we have

$$d([L(18p - 49q, 19q - 7p)], [L(19p - 49q, 18q - 7p)]) = 1 .$$

Moreover, retaking q with an appropriate integer q' with $q' > c$, we see that

$$d_H([L(18p - 49q', 19q' - 7p)], [L(19p - 49q', 18q' - 7p)]) = 1 .$$

Setting $(r, s) = (18p - 49q', 19q' - 7p)$ and $(r', s') = (19p - 49q', 18q' - 7p)$, we obtain the desired conclusion. □

5. Sample calculations

Recall that if $d_H([L], [L']) = 1$, then $d_H([L], [L'])_{\mathcal{L}} = 1$ by definition; however, if $d_H([L], [L']) = 2$, then $d_H([L], [L'])_{\mathcal{L}} \geq 2$ in general. Consider the question: For which $[L], [L'] \in \mathcal{L}$, $d_H([L], [L']) = d_H([L], [L'])_{\mathcal{L}} = 2$. In this section, we give some examples concerning this question.

In the following, the link $K := K_0 \cup K_1$ illustrated in Figure 2 plays an important role. We note that K is introduced by Yamada and is denoted by $k(3, 5) \cup u$ in [27].

5.1. $d_H([S^3], [S^2 \times S^1]) = d_H([S^3], [S^2 \times S^1])_{\mathcal{L}} = 2$. By an argument similar to that in [27], we have $K(r/1, 15/1) \equiv L(64-15r, 23-5r)$. This implies that $K(0/1, 15/1) \equiv L(64, 23)$ and hence $d([S^2 \times S^1], [L(64, 23)]) = 1$. Let K_1' be the image of K_1 in $S^2 \times S^1$ which is obtained by the $0/1$-surgery on K_0. Then it is verified by using computer program SnapPea [30] that K_1' is hyperbolic in $S^2 \times S^1$. Hence $d_H([S^2 \times S^1], [L(64, 23)]) = 1$.

On the other hand, we have $d_H([S^3], [L(64, 23)]) = 1$ as follows. Let K'' be the knot in $L(64, 23)$ denoted by $K(L(64, 23); 19)$ (see the Appendix for the definition). Then we see that K'' admits a Dehn surgery yielding S^3. Moreover, it follows from [25, Theorem 1.3] that K'' is hyperbolic in $L(64, 23)$ (see the appendix for detail).

It is remarked by the referee that the knot $K(L(64, 23); 19)$ coming from the closure of the braid $W_5^2 W_7^8$ described in [3], which corresponds to Sporadic (b) ($P = 22j^2 + 13j + 2$ with $j = -2$) in Berge's list [2].

5.2. $d_H([S^3], [RP^3]) = d_H([S^3], [RP^3])_{\mathcal{L}} = 2$. In the same way as above, we have that $K(2/1, 15/1) \equiv L(34, 13)$ and hence $d([RP^3], [L(34, 13)]) = 1$. Let K_1' be the image of K_1 in RP^3 which is obtained by the $2/1$-surgery on K_0.

Again it is verified by using computer program SnapPea [30] that K_1' is hyperbolic in RP^3. Hence $d_H([RP^3], [L(34, 13)]) = 1$.

On the other hand, we see $d_H([S^3], [L(34, 13)]) = 1$ as follows. Let K'' be the knot in $L(34, 13)$ denoted by $K(L(34, 13); 9)$. Then we see that K'' admits a Dehn surgery yielding S^3. Moreover, it follows from [25, Theorem 1.3] that K'' is hyperbolic in $L(34, 13)$.

It is also remarked by the referee that the knot $K(L(34, 13); 9)$ coming from the mirror image of the closure of the braid $W_7 W_9^3$ described in [3], which corresponds to Type III in Berge's list [2].

These examples would be of interest independently, for the surgeries yielding lens spaces are different "type" from that in Section 4. In fact, for example, the formula $K(r/1, 15/1) = L(64 - 15r, 23 - 5r)$ is not derived from Proposition 1.

ACKNOWLEDGMENT. The first author is partially supported by Grant-in-Aid for Young Scientists (B), No.20740039, Ministry of Education, Culture, Sports, Science and Technology, Japan. The second author is supported by JSPS Postdoctoral Fellowships for Research Abroad.

The authors would like to thank Akio Kawauchi for teaching them about his imitation theory, for example, in [12]. They also thank Yukihiro Tsutsumi, Yuichi Yamada, Eiko Kin for useful discussion, and the referee for his/her careful reading of our paper.

A. Definition and properties of $K(L(p, q); u)$

Recall the definition and the parametrization of lens spaces as follows. Let V_1 be a regular neighborhood of a trivial knot in S^3, m a meridian of V_1 and ℓ a longitude of V_1 such that ℓ bounds a disk in $\mathrm{cl}(S^3 \setminus V_1)$. We fix an orientation of m and ℓ as illustrated in Figure 3. By attaching a solid torus V_2 to V_1 so that $\bar{m}$ is isotopic to a representative of $p[\ell] + q[m]$ in ∂V_1, we obtain a lens space $L(p, q)$, where p and q are integers with $p > 0$ and $(p, q) = 1$, and $\bar{m}$ is a meridian of V_2. Then the intersection points of m and $\bar{m}$ are labeled $P_0, \ldots, P_{p-1}$ successively along the positive direction of m. Let t_i^u $(i = 1, 2)$ be a simple arc in D_i joining P_0 to P_u $(u = 1, 2, \ldots, p-1)$. Then the notation $K(L(p, q); u)$ denotes the knot $t_1^u \cup t_2^u$ in $L(p, q)$ (*cf.* Figure 3).

We then prepare the following notations. Let p and q be integers with $p > 0$ and $(p, q) = 1$. Let $\{s_j\}_{1 \le j \le p}$ be the finite sequence, which we call the *basic sequence*, such that $0 \le s_j < p$ and $s_j \equiv jq \pmod{p}$. For an integer u with $0 < u < p$, $\Psi_{p,q}(u)$ denotes the

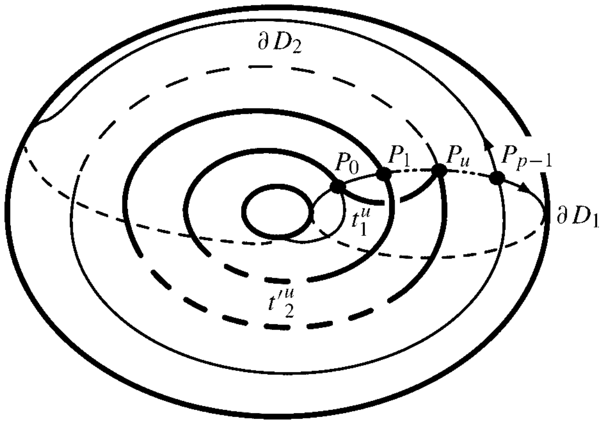

FIGURE 3. Here, t'^u_2 is a projection of t_2^u on ∂V_1.

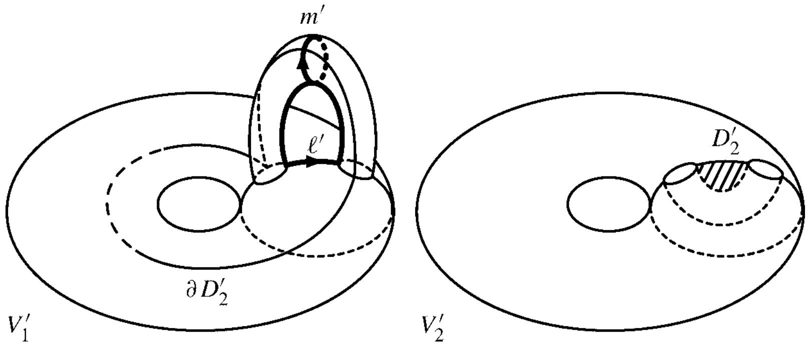

FIGURE 4

integer satisfying $\Psi_{p,q}(u) \cdot q \equiv u \pmod p$ and $\Phi_{p,q}(u)$ denotes the number of elements of the following set (possibly, the empty set):

$$\{s_j \mid 1 \leq j < \Psi_{p,q}(u),\ s_j < u\}.$$

Also, $\widetilde{\Phi}_{p,q}(u)$ denotes the following:

$$\widetilde{\Phi}_{p,q}(u) = \min \left\{ \begin{array}{l} \Phi_{p,q}(u),\ \Phi_{p,q}(u) - \Psi_{p,q}(u) + p - u, \\ \Psi_{p,q}(u) - \Phi_{p,q}(u) - 1,\ u - \Phi_{p,q}(u) - 1 \end{array} \right\}.$$

Set $V_1' = V_1 \cup \eta(t_2^u; V_2)$, $V_2' = \mathrm{cl}(V_2 \setminus \eta(t_2^u; V_2))$ and $S' = \partial V_1' = \partial V_2'$. Then $(V_1', V_2'; S')$ is a genus two Heegaard splitting of $L(p, q)$. Let $D_2' \subset (D_2 \cap V_2')$ be a meridian disk of V_2' with $\partial D_2' \supset (t'^u_2 \cap S')$. Let m' be a meridian of $K = t_1^u \cup t_2^u$ in the annulus $S' \cap \partial\eta(t_2^u; V_2)$.

Let ℓ' be an essential loop in S' which is a union of $t'^{u}_1 \cap S'$ and an essential arc in the annulus $S' \cap \partial\eta(t^u_2; V_2)$ disjoint from $\partial D'_2$

Let m^* be a meridian of K in $\partial\eta(K; V'_1)$ and ℓ^* a longitude of $\partial\eta(K; V'_1)$ such that $\ell' \cup \ell^*$ bounds an annulus in $\mathrm{cl}(V'_1 \setminus \eta(K; V'_1))$. The loops m^* and ℓ^* are oriented as illustrated in Figure 4. Then $\{[m^*], [\ell^*]\}$ is a basis of $H_1(\partial\eta(K; V'_1); \mathbf{Z})$. Let V''_1 be a genus two handlebody obtained from $\mathrm{cl}(V'_1 \setminus \eta(K; V'_1))$ by attaching a solid torus $\bar{V}$ so that the boundary of a meridian disk $\bar{D}$ of $\bar{V}$ is identified with a loop represented by $r[m^*] + s[\ell^*]$. Set $M' = V''_1 \cup_{S'} V'_2$. Then we say that M' is obtained by $(r/s)^*$-*surgery* on K. We note that $(r/s)^*$-surgery is longitudinal if and only if r/s is an integer.

A.1. The fundamental group. Since $K(L(p,q); u)$ is a $(1,1)$-knot in $L(p,q)$, particularly is a so-called 1-bridge braid, we can easily obtain a presentation of the fundamental group of a surgered manifold as follows.

PROPOSITION 3 ([24, Theorem 5.1]). *Set $K = K(L(p,q); u)$ and let $\{s_j\}_{1 \le j \le p}$ be the basic sequence for (p,q). Let N' be the 3-manifold obtained by r^*-surgery on K, where r be an integer. Then we have:*

$$\pi_1(N') \cong \left\langle a, b \;\middle|\; \prod_{j=1}^{\Psi_{p,q}(u)} W_1(j) = 1,\ \prod_{j=1}^{p} W_2(j) = 1 \right\rangle,$$

where

$$W_1(j) = \begin{cases} a & \text{if } s_j > u \\ ab^r & \text{if } s_j = u \\ ab & \text{otherwise} \end{cases} \quad \text{and} \quad W_2(j) = \begin{cases} a & \text{if } s_j \ge u \\ ab & \text{otherwise} \end{cases}.$$

A.2. Hyperbolicity. Though $K = K(L(p,q); u)$ admits several representation (*cf.* [25, Proposition 4.5]), it is proven that $\widetilde{\Phi}_{p,q}(u)$ is an invariant for K if K admits a longitudinal surgery yielding S^3 (*cf.* [25, Corollary 4.6]). Hence when K admits a longitudinal surgery yielding S^3, $\widetilde{\Phi}_{p,q}(u)$ is denoted by $\Phi(K)$. Moreover, we have a necessary and sufficient condition for such knots to be hyperbolic.

PROPOSITION 4 ([25, Theorem 1.3]). *Set $K = K(L(p,q); u)$. Suppose that K admits a longitudinal surgery yielding S^3. Then we have the following:*

1. *$\Phi(K) = 0$ if and only if K is a torus knot.*
2. *$\Phi(K) = 1$ if and only if K contains an essential torus in its exterior.*
3. *$\Phi(K) \ge 2$ if and only if K is a hyperbolic knot.*

A.3. Example. Set $K = K(L(64, 23); 19)$. The basic sequence for $(64, 23)$ is:
$\{s_j\}_{1 \le j \le 64}$: 23, 46, 5, 28, 51, 10, 33, 56, 15, 38, 61, 20, 43, 2, 25, 48, 7, 30, 53, 12, 35, 58, 17, 40, 63, 22, 45, 4, 27, 50, 9, 32, 55, 14, 37, 60, 19, 42, 1, 24, 47, 6, 29, 52, 11, 34, 57, 16, 39, 62, 21, 44, 3, 26, 49, 8, 31, 54, 13, 36, 59, 18, 41, 0.

Let N' be the 3-manifold obtained by 1^*-surgery on K. Then we have:

$$\pi_1(N') \cong \left\langle a, b \;\middle|\; \begin{array}{l} (a^3b)^3a^5b(a^3b)^3a^5b(a^3b)^3, \\ (a^3b)^3a^5b(a^3b)^3a^5b(a^3b)^2a^5b(a^3b)^3a^5b(a^3b)^3a^2b \end{array} \right\rangle .$$

Repeating word reduction, we see that $\pi_1(N')$ is trivial. This implies that $N' \cong S^3$ since Geometrization Conjecture is true [17, 18, 19]. Moreover, K is hyperbolic since $\Phi(K) = 8$.

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