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On the Limit of the Colored Jones Polynomial of a Non-simple Link

Dedicated to Professor Akio Kawauchi for his 60th birthday

Mayuko YAMAZAKI and Yoshiyuki YOKOTA

Tokyo Metropolitan University

Abstract. We compute the limit of the colored Jones invariant of a prime link, which gives the first evidence for Volume Conjecture of a link whose complement decomposes into two hyperbolic pieces

Introduction

In [3], Kashaev defined an invariant $\langle K\rangle_N \in \mathbf{C}$ of a link K by using quantum dilogarithm functions, and conjectured in [4] that

HYPERBOLIC VOLUME CONJECTURE. *If K is a hyperbolic knot in S^3,*

$$\lim_{N\to\infty} \frac{2\pi}{N} \log|\langle K\rangle_N| = \mathrm{vol}(S^3 \setminus K)\,.$$

In [10], H. Murakami and J. Murakami proved that Kashaev's invariant $\langle K\rangle_N$ is nothing but the N-colored Jones polynomial of K evaluated at $\omega = \exp 2\pi\sqrt{-1}/N$, and generalized Kashaev's conjecture to

VOLUME CONJECTURE. *Let K be a link in S^3 and $J_K(N;q)$ the N-colored Jones polynomial of K. Then,*

$$\lim_{N\to\infty} \frac{2\pi}{N} \log|J_K(N;\omega)| = v_3\,\|S^3 \setminus K\|\,,$$

where $\|S^3 \setminus K\|$ denotes the Gromov norm of $S^3 \setminus K$ and v_3 is the volume of the ideal regular tetrahedron in the 3-dimensional hyperbolic space.

Since the Gromov norm is equal to the sum of the volumes of the hyperbolic pieces divided by v_3 (see [2]), Volume Conjecture is a natural generalization of Hyperbolic Volume Conjecture. The purpose of this paper is to show that Volume Conjecture holds for the 2-component link L depicted in Figure 1, that is,

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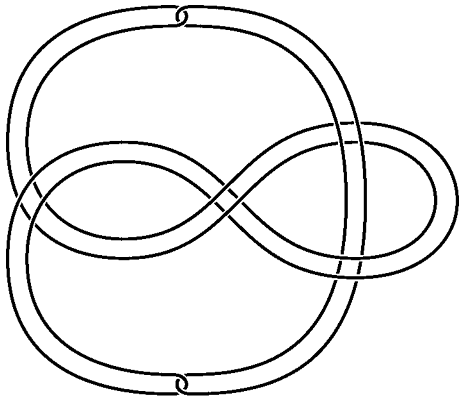

FIGURE 1

MAIN THEOREM

$$\lim_{N\to\infty}\frac{2\pi}{N}\log J_L(N;\omega) = 6\Lambda(\pi/3) + 16\Lambda(\pi/4)\,,$$

where $\Lambda(\theta)$ is the Lobachevsky function defined by

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin x|dx\,.$$

This is an important evidence for Volume Conjecture because L is prime and the complement of L decomposes into the figure-eight knot complement, whose volume is $6\Lambda(\pi/3)$, and the Borromean ring complement, whose volume is $16\Lambda(\pi/4)$ (see [11]). It should be noted that Volume conjecture is proved for the figure-eight knot by Ekholm, for torus knots by Kashaev and O. Tirkkonen [5], for Whitehead doubles of torus knots by H. Zheng [13], for Whitehead chains by R. van der Veen [12], and for some cabled links by T. Le and A. Tran [7]. For the other results, see [9].

This paper is organized as follows. In Section 1, we compute the colored Jones polynomial of L by using Masbaum's method. Then, we investigate the asymptotic behavior of the main part of $J_L(N;\omega)$ in Sections 2 and prove Main Theorem in Section 3 under the assumption $N \equiv 1 \bmod 4$ because the proofs for the other cases are quite similar.

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1. The colored Jones polynomial of L

We first review the results in [8]. Recall that the Kauffman bracket skein module of $S^1 \times [-1,1]$, denoted by $\mathcal{B}$, is the ring $\mathbf{Z}[q^{\pm 1/4}][z]$, where z represents $S^1 \times \{0\}$ and z^n

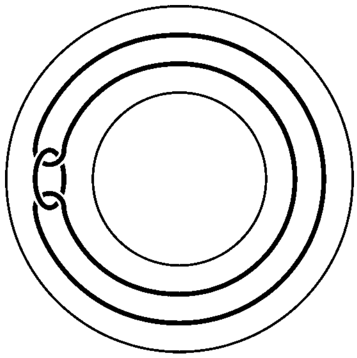

FIGURE 2

represents the n parallel copies of z. If we put

$$R_n = \prod_{i=0}^{n-1} (z + q^{(2i+1)/2} + q^{-(2i+1)/2}) , \tag{1}$$

then $\{R_n\}$ form a basis of $\mathcal{B}$. On the other hand, there is another basis $\{e_n\}$ of $\mathcal{B}$ satisfying $e_0 = 1, e_1 = z$ and $e_n = ze_{n-1} - e_{n-2}$ for $n \geq 2$. In fact, e_n is the closure of the Jones-Wenzl idempotent f_n of the n-th Temperley-Lieb algebra T_n (see [6] for their definitions) and $J_L(N; q) \cdot \{N\}/\{1\}$ is obtained by cabling each component of L with e_{N-1} and by taking the Kauffman bracket. Furthermore, we have

$$e_{N-1} = \sum_{n=0}^{N-1} (-1)^{N-1-n} \begin{Bmatrix} N+n \\ N-1-n \end{Bmatrix} R_n , \tag{2}$$

by [8, (47)], where $\{n\} = q^{n/2} - q^{-n/2}$, $\{n\}! = \{n\}\{n-1\} \cdots \{2\}\{1\}$ and

$$\begin{Bmatrix} m \\ n \end{Bmatrix} = \frac{\{m\}!}{\{n\}!\{m-n\}!} .$$

LEMMA 1.1. *For $x, y \in \mathcal{B}$, we define $\phi(x, y) \in \mathcal{B}$ by cabling the 2-component link diagram in $S^1 \times [-1, 1]$ depicted in Figure 2 with x, y. Then, we have*

$$\phi(e_{N-1}, e_{N-1}) = \sum_{m=0}^{N-1} \sum_{n=m}^{N-1} (-1)^m \left(\{n\}!\right)^2 \begin{Bmatrix} N+n \\ 2n+1 \end{Bmatrix}^2 \begin{Bmatrix} 2n+1 \\ n-m \end{Bmatrix} e_{2m} .$$

PROOF. First of all, we have

$$\phi(e_{N-1}, e_{N-1}) = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} (-1)^{n+k} \begin{Bmatrix} N+n \\ 2n+1 \end{Bmatrix} \begin{Bmatrix} N+k \\ 2k+1 \end{Bmatrix} \phi(R_n, R_k)$$

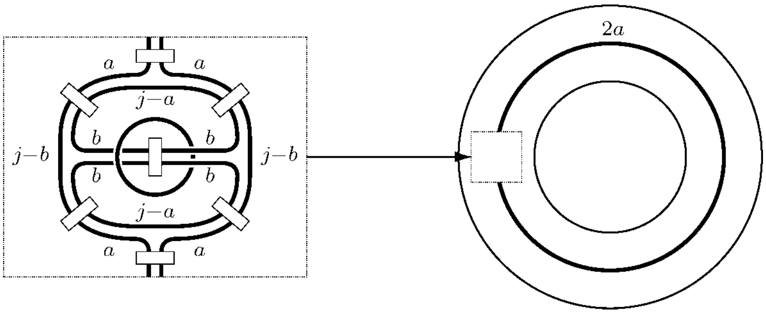

FIGURE 3

by (2), and $\phi(R_n, R_k)$ is a linear sum of $\phi(R_n, e_j)$'s with $j \le k$. Then, by [8, (26)],

$$\phi(R_n, e_j) = \sum_{a=0}^{j} \frac{\langle 2a \rangle}{\langle j, j, 2a \rangle} \sum_{b=0}^{j} \frac{\langle 2b \rangle}{\langle j, j, 2b \rangle} \Gamma^j_{a,b}(R_n)\,, \tag{3}$$

where

$$\langle 2i \rangle = \frac{\{2i+1\}}{\{1\}}\,, \quad \langle j, j, 2i \rangle = (-1)^{j+i} \frac{\{j+i+1\}!\{j-i\}!(\{i\}!)^2}{(\{j\}!)^2\{2i\}!}$$

and $\Gamma^j_{a,b}(x) \in \mathcal{B}$ is defined by Figure 3, where the center circle in the dotted square is cabled with $x \in \mathcal{B}$, an edge colored i stands for i parallel strands, and a white box represents a Jones-Wenzl idempotent. Since

$$\Gamma^j_{a,b}(z) = (-q^{(2b+1)/2} - q^{-(2b+1)/2})\Gamma^j_{a,b}(1)$$

(see [6, 9.6] for example), we have $\Gamma^j_{a,b}(R_n) = 0$ when $b < n$ by (1), which implies $\phi(R_n, R_k) = 0$ if $k < n$. Since ϕ is symmetric, we have

$$\phi(e_{N-1}, e_{N-1}) = \sum_{n=0}^{N-1} \begin{Bmatrix} N+n \\ 2n+1 \end{Bmatrix}^2 \phi(R_n, R_n)\,,$$

where $\phi(R_n, R_n) = \phi(R_n, e_n)$ because $R_n - e_n$ is the sum of R_k's with $k < n$ by (2), and it suffices to show

$$\phi(R_n, e_n) = \sum_{m=0}^{n} (-1)^m (\{n\}!)^2 \begin{Bmatrix} 2n+1 \\ n-m \end{Bmatrix} e_{2m}\,.$$

In fact, by (3) and by the observation above, we have

$$\phi(R_n, e_n) = \sum_{m=0}^{n}\sum_{k=0}^{n} \frac{\langle 2m\rangle}{\langle n, n, 2m\rangle}\frac{\langle 2k\rangle}{\langle n, n, 2k\rangle}\Gamma^n_{m,k}(R_n) = \sum_{m=0}^{n} \frac{\langle 2m\rangle}{\langle n, n, 2m\rangle}\Gamma^n_{m,n}(R_n)\,,$$

where

$$\Gamma^n_{m,n}(R_n) = \frac{\langle e_{2n}, R_n\rangle}{\langle e_{2n}, 1\rangle}\,\Gamma^n_{m,n}(1) = (-1)^n\{2n\}!\,\Gamma^n_{m,n}(1)$$

by [8, (5)] and

$$\Gamma^n_{m,n}(1) = \frac{\langle 2n\rangle}{\langle 2m\rangle}\Gamma^m_{m,m}(1) = \frac{\{2n+1\}}{\{2m+1\}}\frac{(\{m\}!)^2}{\{2m\}!}e_{2m}$$

by [6, 3.3] and [8, Lemma 3.1]. Consequently, $\phi(R_n, e_n)$ is equal to

$$\sum_{m=0}^{n} \frac{(-1)^{n+m}\{2m+1\}(\{n\}!)^2\{2m\}!}{\{n-m\}!\{n+m\}!(\{m\}!)^2}\cdot(-1)^n\{2n\}!\cdot\frac{\{2n+1\}}{\{2m+1\}}\frac{(\{m\}!)^2}{\{2m\}!}\cdot e_{2m}$$

$$= \sum_{m=0}^{n} \frac{(-1)^m(\{n\}!)^2\{2n+1\}!}{\{n+m+1\}!\{n-m\}!}\cdot e_{2m} = \sum_{m=0}^{n}(-1)^m\,(\{n\}!)^2\begin{Bmatrix}2n+1\\ n-m\end{Bmatrix}e_{2m}\,. \qquad \square$$

Since the $(2m+1)$-colored Jones polynomial of the figure-eight knot is given by

$$\sum_{l=0}^{2m}\frac{\{2m+1+l\}!}{\{2m-l\}!\{2m+1\}}$$

(see [8, (49)] for example), the colored Jones polynomial $J_L(N; q)$ of L is equal to

$$\frac{\{1\}}{\{N\}}\sum_{n=0}^{N-1}\sum_{m=0}^{n}\sum_{l=0}^{2m}A_q(n,m)B_q(m,l) = \frac{\{1\}}{\{N\}}\sum_{m=0}^{N-1}\sum_{n=m}^{N-1}\sum_{l=0}^{2m}A_q(n,m)B_q(m,l)$$

by Lemma 1.1, where we put

$$A_q(n,m) = (-1)^m\,(\{n\}!)^2\begin{Bmatrix}N+n\\ 2n+1\end{Bmatrix}^2\begin{Bmatrix}2n+1\\ n-m\end{Bmatrix},\ \ B_q(m,l) = \frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}}\,.$$

From now on, we suppose $N \equiv 1 \mod 4$ for simplicity, and put

$$T = \frac{N-1}{4}\,,\quad m' = 4T - m\,.$$

Note that $A_q(n,m) \equiv 0 \mod \{N\}^2$ if $n < 2T$ and $B_q(m,l) \equiv 0 \mod \{N\}^2$ if $l > 2m'$ and $l > m - m'$, and $J_L(N; q)$ is equal to

$$\frac{\{1\}}{\{N\}}\left(\sum_{m=0}^{2T-1}\sum_{n=2T}^{4T}\sum_{l=0}^{2m} + \sum_{m=2T}^{3T}\sum_{n=m}^{4T}\sum_{l=0}^{2m'} + \sum_{m=3T+1}^{4T}\sum_{n=m}^{4T}\sum_{l=0}^{m-m'}\right)A_q(n,m)B_q(m,l)$$

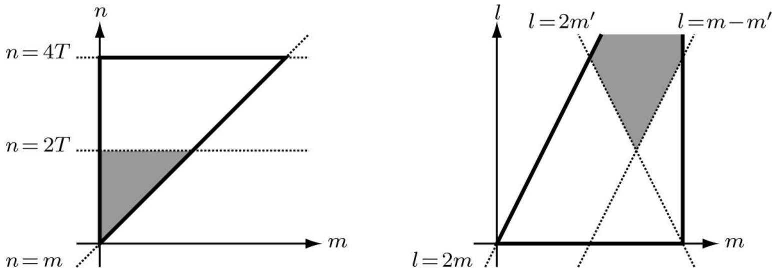

FIGURE 4

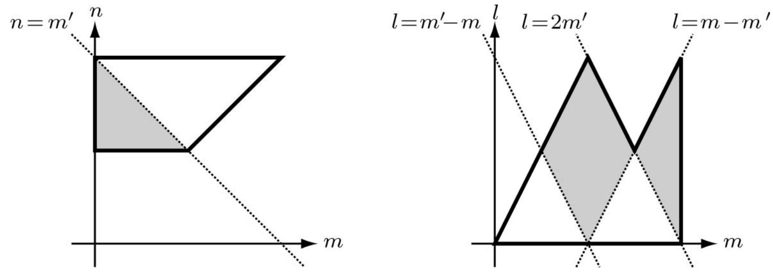

FIGURE 5

modulo $\{N\}/\{1\}$. See the dark-gray regions in Figure 4.

Furthermore, there exists $\tilde{A}_q(n,m) \in \mathbf{Z}[q^{\pm 1/2}]$ such that

$$A_q(n,m) = \tilde{A}_q(n,m)\{N\}/\{1\}$$

if $n < m'$, and there exists $\tilde{B}_q(m,l) \in \mathbf{Z}[q^{\pm 1/2}]$ such that

$$B_q(m,l) = \tilde{B}_q(m,l)\{N\}/\{1\}$$

if $l \geq |m' - m|$ or $l > 2m'$. See the gray regions in Figure 5. Therefore,

$$\frac{\{1\}}{\{N\}} \sum_{m=T}^{2T-1} \sum_{n=2T}^{m'-1} \sum_{l=m'-m}^{2m} A_q(n,m) B_q(m,l) \equiv 0 \mod \{N\}/\{1\},$$

and $J_K(N;q)$ is equal to

$$\frac{\{1\}}{\{N\}} \left(\sum_{m=0}^{T-1} \sum_{n=2T}^{m'-1} \sum_{l=0}^{2m} + \sum_{m=0}^{T-1} \sum_{n=m'}^{4T} \sum_{l=0}^{2m} \right.$$

$$+\sum_{m=T}^{2T-1}\sum_{n=2T}^{m'-1}\sum_{l=0}^{m'-m-1}+\sum_{m=T}^{2T-1}\sum_{n=m'}^{4T}\sum_{l=0}^{m'-m-1}+\sum_{m=T}^{2T-1}\sum_{n=m'}^{4T}\sum_{l=m'-m}^{2m}$$
$$+\sum_{m=2T+1}^{3T}\sum_{n=m}^{4T}\sum_{l=0}^{m-m'-1}+\sum_{m=2T}^{3T}\sum_{n=m}^{4T}\sum_{l=m-m'}^{2m'}$$
$$\left.+\sum_{m=3T+1}^{4T}\sum_{n=m}^{4T}\sum_{l=0}^{2m'}+\sum_{m=3T+1}^{4T}\sum_{n=m}^{4T}\sum_{l=2m'+1}^{m-m'}\right)A_q(n,m)B_q(m,l)$$
$$=P(N;q)+Q(N;q)+R(N;q)$$
$$+\frac{\{1\}}{\{N\}}\left(\sum_{m=0}^{T-1}\sum_{n=m'}^{4T}\sum_{l=0}^{2m}+\sum_{m=T}^{2T-1}\sum_{n=m'}^{4T}\sum_{l=0}^{m'-m-1}\right)A_q(n,m)B_q(m,l)$$
$$+\frac{\{1\}}{\{N\}}\left(\sum_{m'=0}^{T-1}\sum_{n=(m')'}^{4T}\sum_{l=0}^{2m'}+\sum_{m'=T}^{2T-1}\sum_{n=(m')'}^{4T}\sum_{l=0}^{(m')'-m'-1}\right)A_q(n,(m')')B_q((m')',l)$$

modulo $\{N\}/\{1\}$, where

$$P(N;q)=\left(\sum_{m=0}^{T-1}\sum_{n=2T}^{m'-1}\sum_{l=0}^{2m}+\sum_{m=T}^{2T-1}\sum_{n=2T}^{m'-1}\sum_{l=0}^{m'-m-1}\right)\tilde{A}_q(n,m)B_q(m,l)\,,$$
$$Q(N;q)=\left(\sum_{m=T}^{2T-1}\sum_{n=m'}^{4T}\sum_{l=m'-m}^{2m}+\sum_{m=2T}^{3T}\sum_{n=m}^{4T}\sum_{l=m-m'}^{2m'}\right)A_q(n,m)\tilde{B}_q(m,l)\,,$$
$$R(N;q)=\sum_{m=3T+1}^{4T}\sum_{n=m}^{4T}\sum_{l=2m'+1}^{m-m'}A_q(n,m)\tilde{B}_q(m,l)\,.$$

If we put

$$C_q(n,m,l)=A_q(n,m)B_q(m,l)+A_q(n,m')B_q(m',l)$$

and

$$S(N;q)=\frac{\{1\}}{\{N\}}\left(\sum_{m=0}^{T-1}\sum_{n=m'}^{4T}\sum_{l=0}^{2m}+\sum_{m=T}^{2T-1}\sum_{n=m'}^{4T}\sum_{l=0}^{m'-m-1}\right)C_q(n,m,l)\,,$$

we have

$$J_L(N;q)\equiv P(N;q)+Q(N;q)+R(N;q)+S(N;q)\ \mathrm{mod}\ \{N\}/\{1\}\,. \tag{4}$$

2. Asymptotics of $Q(N;\omega)$

From now on, we suppose N is sufficiently large.

LEMMA 2.1. *Suppose n is greater than or equal to m and m'. Then,*

$$0 \le A_\omega(n,m) \le A_\omega(3T,2T) = \frac{\left(\prod_{k=1}^{3T} 2\sin\frac{k}{N}\pi\right)^6}{N^2\left(\prod_{k=1}^{2T} 2\sin\frac{k}{N}\pi\right)\left(\prod_{k=1}^{T} 2\sin\frac{k}{N}\pi\right)^2}.$$

PROOF. If $q=\omega$, we have

$$\begin{Bmatrix} N+n \\ 2n+1 \end{Bmatrix} = \frac{\prod_{k=1}^{n}\{N+k\}\{N-k\}}{\{N-1\}!\prod_{k=1}^{2n-4T}\{N+k\}} = \frac{(-1)^n(\{n\}!)^2}{\{N-1\}!\{2n-4T\}!},$$

$$\begin{Bmatrix} 2n+1 \\ n-m \end{Bmatrix} = \frac{\prod_{k=1}^{2n-4T}\{N+k\}}{\{n-m\}!\prod_{k=1}^{n-m'}\{N+k\}} = \frac{(-1)^{n+m}\{2n-4T\}!}{\{n-m\}!\{n-m'\}!},$$

and $\{N-1\}! = \omega^{-N(N-1)/4}(\omega-1)(\omega^2-1)\cdots(\omega^{N-1}-1) = N$, and so

$$\begin{aligned} A_\omega(n,m) &= \frac{(-1)^n(\{n\}!)^6}{(\{N-1\}!)^2\{2n-4T\}!\{n-m\}!\{n-m'\}!} \\ &= \frac{\left(\prod_{k=1}^{n} 2\sin\frac{k}{N}\pi\right)^6}{N^2\left(\prod_{k=1}^{2n-4T} 2\sin\frac{k}{N}\pi\right)\left(\prod_{k=1}^{n-m} 2\sin\frac{k}{N}\pi\right)\left(\prod_{k=1}^{n-m'} 2\sin\frac{k}{N}\pi\right)} \end{aligned}$$

is positive because n , $2n-4T$, $n-m$, $n-m' \in [0,N)$. On the other hand,

$$\frac{A_\omega(n,m)}{A_\omega(n,m-1)} = \frac{\{n-m+1\}}{\{n-m'\}} = -\frac{\sin\frac{n-m+1}{N}\pi}{\sin\frac{n+m+1}{N}\pi}$$

is equal to 1 if $n=4T$, greater than 1 if $n<4T$ and $m\le 2T$, and less than 1 if $n<4T$ and $m>2T$, and so we have $A_\omega(n,m)\le A_\omega(n,2T)$. Similarly,

$$\frac{A_\omega(n,2T)}{A_\omega(n-1,2T)} = \frac{4\sin^6\frac{n}{N}\pi}{\sin\frac{2n}{N}\pi\ \sin\frac{2n+1}{N}\pi\cos^2\frac{2n+1}{2N}\pi}$$

is greater than

$$\frac{4\sin^6\frac{2n+1}{2N}\pi}{\sin^2\frac{2n+1}{N}\pi\cos^2\frac{2n+1}{2N}\pi} = \tan^4\frac{(2n+1)\pi}{2N} > 1$$

if $2T<n\le 3T$, and is less than

$$\frac{4\sin^4\frac{n}{N}\pi}{\sin^2\frac{2n+1}{N}\pi}\cdot\frac{\sin^2\frac{n}{N}\pi}{\cos^2\frac{2n+1}{2N}\pi} \le \frac{4\sin^6\frac{3}{4}\pi}{\sin^2(\frac{3}{2}\pi+\frac{\pi}{N})\cos^2(\frac{3}{4}\pi+\frac{\pi}{2N})} = \frac{1}{\cos^2\frac{\pi}{N}(1+\sin\frac{\pi}{N})} < 1$$

if $n > 3T$ because

$$\frac{d}{dx}\left\{\frac{-2\sin^2 x}{\sin(2x+\frac{\pi}{N})}\right\} = \frac{4\sin\frac{\pi}{N}\sin(x+\frac{\pi}{N})}{-\sin^2(2x+\frac{\pi}{N})}, \quad \frac{d}{dx}\left\{\frac{-\sin x}{\cos(x+\frac{\pi}{2N})}\right\} = \frac{-\cos\frac{\pi}{2N}}{\cos^2(x+\frac{\pi}{2N})}$$

are negative if $3\pi/4 < x < 4T\pi/N$. Consequently, we have

$$0 \le A_\omega(n, m) \le A_\omega(n, 2T) \le A_\omega(3T, 2T). \qquad \square$$

In what follows, for $x \in \mathbf{R}$, $\lfloor x \rfloor$ denotes the integer part of x.

LEMMA 2.2. *Suppose $m - m' \le l \le 2m$ and $m' - m \le l \le 2m'$. Then,*

$$0 \le \tilde{B}_\omega(m, l) \le \tilde{B}_\omega(2T, \lfloor 5N/6 \rfloor) = \prod_{k=1}^{\lfloor 5N/6 \rfloor}\left(2\sin\frac{k\pi}{N}\right)^2.$$

PROOF. By definition, $\tilde{B}_\omega(m, l)$ is equal to

$$\prod_{k=1}^{l+m-m'}\{N+k\}\prod_{k=1}^{l+m'-m}\{N-k\} = (-1)^{l+m-m'}\{l+m-m'\}!\{l+m'-m\}!$$
$$= \prod_{k=1}^{l+m-m'} 2\sin\frac{k\pi}{N} \cdot \prod_{k=1}^{l+m'-m} 2\sin\frac{k\pi}{N}$$

and so $\tilde{B}_\omega(m, l) \ge 0$ because $l + m - m',\ l + m' - m \in [0, N)$. Then, we consider

$$\frac{\tilde{B}_\omega(m, l)}{\tilde{B}_\omega(m-1, l)} = \frac{\{2m+l\}\{2m+1+l\}}{\{2m-l\}\{2m-1-l\}} = \frac{\sin\frac{2m+l}{N}\pi \cdot \sin\frac{2m+1+l}{N}\pi}{\sin\frac{2m-l}{N}\pi \cdot \sin\frac{2m-1-l}{N}\pi} \tag{5}$$

which is equal to 1 if $l = 2T$.

If $l > 2T$, (5) is greater than 1 when $m \le 2T$ and less than 1 when $m > 2T$. Therefore we have

$$\tilde{B}_\omega(m, l) \le \tilde{B}_\omega(2T, l) = (-1)^l(\{l\}!)^2,$$

and Lemma 2.2 is true in this case because

$$\frac{\tilde{B}_\omega(2T, l)}{\tilde{B}_\omega(2T, l-1)} = -\{l\}^2 = \left(2\sin\frac{l\pi}{N}\right)^2$$

is greater than 1 if $2T < l \le \lfloor 5N/6 \rfloor$ and less than 1 if $\lfloor 5N/6 \rfloor < l \le 4T$.

If $l < 2T$, (5) is less than 1 when $m \le 2T$ and greater than 1 when $m > 2T$. Therefore we have

$$\tilde{B}_\omega(m, l) \le \tilde{B}_\omega(2T \pm \lfloor l/2 \rfloor, l) = (-1)^{l \pm 2\lfloor l/2 \rfloor}\{l + 2\lfloor l/2 \rfloor\}!\{l - 2\lfloor l/2 \rfloor\}!.$$

Since the right hand side is bounded by $(-1)^l\{2l\}!$ and

$$\frac{(-1)^l\{2l\}!}{(-1)^{l-1}\{2(l-1)\}!} = -\{2l\}\{2l-1\} = 2\sin\frac{2l\pi}{N}\cdot 2\sin\frac{(2l-1)\pi}{N}$$

is greater than 1 if $\lfloor N/12\rfloor < l \le \lfloor 5N/12\rfloor$ and less than 1 otherwise, we have

$$\tilde{B}_\omega(m,l) \le (-1)^{\lfloor 5N/12\rfloor}\{2\lfloor 5N/12\rfloor\}! \le |\{\lfloor 5N/6\rfloor\}!|\,.$$

Since

$$|\{\lfloor 5N/6\rfloor\}!| = \{2T\}!\prod_{k=2T+1}^{\lfloor 5N/6\rfloor} 2\sin\frac{k\pi}{N} = \sqrt{N}\prod_{k=2T+1}^{\lfloor 5N/6\rfloor} 2\sin\frac{k\pi}{N} > 1\,,$$

we have

$$\tilde{B}_\omega(m,l) \le |\{\lfloor 5N/6\rfloor\}!| < (-1)^{\lfloor 5N/6\rfloor}(\{\lfloor 5N/6\rfloor\}!)^2 = \tilde{B}_\omega(2T, \lfloor 5N/6\rfloor),$$

and Lemma 2.2 is true in this case. This completes the proof. □

The following is the main result of this section.

PROPOSITION 2.3.

$$\log Q(N;\omega) = \frac{N}{2\pi}\left(6\Lambda(\pi/3) + 16\Lambda(\pi/4)\right) + O(\log N)\,,$$

where $O(\log N)$ *stands for a term bounded by a constant times* $\log N$.

PROOF. By Lemmas 2.1 and 2.2, $Q(N,\omega)$ consists of at most N^3 positive terms and the largest term is $A_\omega(3T,2T)\tilde{B}_\omega(2T,\lfloor 5N/6\rfloor)$. Therefore we have

$$A_\omega(3T,2T)\tilde{B}_\omega(2T,\lfloor 5N/6\rfloor) \le Q(N;\omega) \le N^3 A_\omega(3T,2T)\tilde{B}_\omega(2T,\lfloor 5N/6\rfloor)\,,$$

and so

$$\begin{aligned}\log Q(N;\omega) &= \log A_\omega(3T,2T) + \log \tilde{B}_\omega(2T,\lfloor 5N/6\rfloor) + O(\log N)\\ &= \left(6\sum_{k=1}^{3T} - \sum_{k=1}^{2T} - 2\sum_{k=1}^{T} + 2\sum_{k=1}^{\lfloor 5N/6\rfloor}\right)\log\left(2\sin\frac{k\pi}{N}\right) + O(\log N)\,.\end{aligned}$$

Since

$$\sum_{k=1}^{n}\log\left|2\sin\frac{k\pi}{N}\right| = -\frac{N}{2\pi}\cdot 2\Lambda(n\pi/N) + O(\log N) \tag{6}$$

(see [1, Lemma 4.1] for example), $\log Q(N;\omega)$ is equal to

$$\frac{N}{2\pi}\left(-12\Lambda(3\pi/4) + 2\Lambda(\pi/2) + 4\Lambda(\pi/4) - 4\Lambda(5\pi/6)\right) + O(\log N)\,.$$

Then, by using the famous identities

$$\Lambda(-\theta) = -\Lambda(\theta),\ \Lambda(\pi+\theta) = \Lambda(\theta),\ \Lambda(2\theta) = 2\Lambda(\theta) + 2\Lambda(\theta+\pi/2)\,,$$

we can observe $-12\Lambda(3\pi/4) + 2\Lambda(\pi/2) + 4\Lambda(\pi/4) = 16\Lambda(\pi/4)$ and

$$-4\Lambda(5\pi/6) = 4\Lambda(\pi/6) = 2(\Lambda(\pi/3) - 2\Lambda(2\pi/3)) = 6\Lambda(\pi/3)\,. \qquad \square$$

3. Proof of Main Theorem

In this section, we show the absolute values of $P(N;\omega)$, $R(N;\omega)$ and $S(N;\omega)$ are much smaller than $Q(N;\omega)$, and complete the proof of Main Theorem.

3.1. Asymptotics of $S(N;\omega)$

LEMMA 3.1. *Suppose $m < 2T$, $n \geq m'$, $l \leq 2m$ and $l < m' - m$. Then,*

$$C_q(n,m,l) \equiv A_q(n,m)B_q(m,l)\cdot f_q(n,m,l)\cdot \frac{\{N\}}{\{N/2\}} \mod \frac{\{N\}^2}{\{N/2\}^2}\,,$$

where

$$f_q(n,m,l) = -\sum_{k=n-m'+1}^{n-m}\frac{\{k+N/2\}}{\{k\}} + \{N/2\}\sum_{k=2m+1-l}^{2m+1+l}\frac{\{2(N-k)\}}{\{N-k\}\{k\}}\,.$$

PROOF. It suffices to show that

$$\begin{Bmatrix}2n+1\\ n-m\end{Bmatrix}\frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}} + \begin{Bmatrix}2n+1\\ n-m'\end{Bmatrix}\frac{\{2m'+1+l\}!}{\{2m'-l\}!\{1\}}$$
$$\equiv \begin{Bmatrix}2n+1\\ n-m\end{Bmatrix}\frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}}\cdot f_q(n,m,l)\cdot\frac{\{N\}}{\{N/2\}} \mod \frac{\{N\}^2}{\{N/2\}^2}\,.$$

In fact, by using the identities

$$\{N+k\} = -\{k\} + \{k+N/2\}\cdot\frac{\{N\}}{\{N/2\}},\ \{2N-k\} = -\{k\} + \frac{\{2(N-k)\}}{\{N-k\}}\cdot\{N\}\,,$$

we can observe that

$$\begin{Bmatrix}2n+1\\ n-m\end{Bmatrix} - (-1)^{n-m}\begin{Bmatrix}2n-4T\\ n-m\end{Bmatrix}\left(1 - \frac{\{N\}}{\{N/2\}}\sum_{k=n-m'+1}^{2n-4T}\frac{\{k+N/2\}}{\{k\}}\right),$$

$$\begin{Bmatrix}2n+1\\ n-m'\end{Bmatrix} - (-1)^{n-m'}\begin{Bmatrix}2n-4T\\ n-m'\end{Bmatrix}\left(1 - \frac{\{N\}}{\{N/2\}}\sum_{k=n-m+1}^{2n-4T}\frac{\{k+N/2\}}{\{k\}}\right),$$

$$\frac{\{2m'+1+l\}!}{\{2m'-l\}!\{1\}} - (-1)^{2l+1}\frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}}\left(1 - \{N\}\sum_{k=2m+1-l}^{2m+1+l}\frac{\{2(N-k)\}}{\{N-k\}\{k\}}\right)$$

are divisible by $\{N\}^2/\{N/2\}^2$ and that

$$\begin{aligned}
&\begin{Bmatrix} 2n+1 \\ n-m \end{Bmatrix} \frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}} + \begin{Bmatrix} 2n+1 \\ n-m' \end{Bmatrix} \frac{\{2m'+1+l\}!}{\{2m'-l\}!\{1\}} \\
&\quad \equiv \begin{Bmatrix} 2n-4T \\ n-m \end{Bmatrix} \frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}} \cdot (-1)^{n-m} f_q(n,m,l) \cdot \frac{\{N\}}{\{N/2\}} \mod \frac{\{N\}^2}{\{N/2\}^2} \\
&\quad \equiv \begin{Bmatrix} 2n+1 \\ n-m \end{Bmatrix} \frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}} \cdot f_q(n,m,l) \cdot \frac{\{N\}}{\{N/2\}} \mod \frac{\{N\}^2}{\{N/2\}^2} . \qquad \square
\end{aligned}$$

LEMMA 3.2. *Suppose*

$$|n-3T| + 2|m-2T| \geq T .$$

Then, there exists $\alpha > 0$, *which is independent of* n, m *and* N, *such that*

$$\log A_\omega(n,m) \leq \frac{N}{2\pi}\,(U(3\pi/4, \pi/2) - \alpha) + O(\log N) ,$$

where $U(\nu,\mu) = 2\Lambda(2\nu) + 2\Lambda(\nu-\mu) + 2\Lambda(\nu+\mu) - 12\Lambda(\nu)$.

PROOF. From the proof of Lemma 2.1, it suffices to show when

$$|n-3T| + 2|m-2T| = T .$$

By (6), this is enough to show $U(3\pi/4, \pi/2) - U(\nu,\mu) > 0$ if

$$|\nu - 3\pi/4| + 2|\mu - \pi/2| = \pi/4$$

because

$$\log A_\omega(n,m) = \frac{N}{2\pi} \cdot U(n\pi/N, m\pi/N) + O(\log N) .$$

In fact, as in Lemma 2.1,

$$\frac{\partial U(\nu,\mu)}{\partial\mu} = \log \frac{\sin(\mu-\nu)}{\sin(\mu+\nu)}$$

is positive if $\mu < \pi/2$ and negative if $\mu > \pi/2$, and

$$\frac{\partial U(\nu,\pi/2)}{\partial\nu} = \log \frac{4\sin^6\nu}{\sin^2 2\nu \cdot \sin^2(\nu-\pi/2)}$$

is positive if $\nu < 3\pi/4$ and negative if $\nu > 3\pi/4$. Therefore,

$$U(3\pi/4,\pi/2) - U(\nu,\mu) = \int_\mu^{\pi/2} \frac{\partial U(\nu,x)}{\partial x} dx + \int_\nu^{3\pi/4} \frac{\partial U(y,\pi/2)}{\partial y} dy > 0 . \qquad \square$$

LEMMA 3.3. *Suppose* $l \leq 2m$ *and* $l < m' - m$. *Then*, $(-1)^l B_\omega(m,l) > 0$ *and*

$$\log\{(-1)^l B_\omega(m,l)\} \leq \frac{N}{2\pi} \cdot V(\pi/4, \pi/3) + O(\log N) ,$$

where $V(\mu, \lambda) = 2\Lambda(2\mu - \lambda) - 2\Lambda(2\mu + \lambda)$. *In particular, if*

$$2|m - T| + |l - 2T| \geq T\ ,$$

there exists $\beta > 0$, *which is independent of* m, l *and* N, *such that*

$$\log\{(-1)^l B_\omega(m, l)\} \leq \frac{N}{2\pi}\,(V(\pi/4, \pi/3) - \beta) + O(\log N)\,.$$

PROOF. If $l \leq 2m$ and $l < m' - m$,

$$(-1)^l B_\omega(m, l) = (-1)^l \frac{\{2m+1+l\}!}{\{2m-l\}!\{1\}} = \frac{\prod_{k=1}^{2m+1+l} 2\sin\frac{k}{N}\pi}{2\sin\frac{\pi}{N}\prod_{k=1}^{2m-l} 2\sin\frac{k}{N}\pi}$$

is positive and

$$\log\{(-1)^l B_\omega(m, l)\} = \frac{N}{2\pi}\cdot V(m\pi/N, l\pi/N) + O(\log N)$$

by (6). Then, the proof of Lemma 3.3 is similar to that of Lemma 3.2 because

$$\frac{\partial V(\mu, \lambda)}{\partial \mu} = 2\log\frac{\sin(2\mu + \lambda)}{\sin(2\mu - \lambda)}$$

is positive if $\mu < \pi/4$ and negative if $\mu > \pi/4$, and

$$\frac{\partial V(\pi/4, \lambda)}{\partial \lambda} = 2\log\{4\sin(\pi/2 - \lambda)\sin(\pi/2 + \lambda)\}$$

is positive if $\lambda < \pi/3$ and negative if $\lambda > \pi/3$. □

PROPOSITION 3.4. *There exists* $\varepsilon > 0$, *which is independent of* N, *such that*

$$\log|S(N;\omega)| \leq \frac{N}{2\pi}\,(U(3\pi/4, \pi/2) + V(\pi/4, \pi/3) - \varepsilon) + O(\log N)$$

PROOF. By Lemma 3.1, $S(N;\omega)$ is equal to

$$\left(\sum_{m=0}^{T-1}\sum_{n=m'}^{4T}\sum_{l=0}^{2m} + \sum_{m=T}^{2T-1}\sum_{n=m'}^{4T}\sum_{l=0}^{m'-m-1}\right) A_\omega(n, m) B_\omega(m, l) e^{O(\log N)}\,.$$

On the other hand, by Lemmas 3.2 and 3.3,

$$\log\{A_\omega(n, m)|B_\omega(m, l)|\} \leq \frac{N}{2\pi}\,(U(3\pi/4, \pi/2) + V(\pi/4, \pi/3) - \alpha) + O(\log N)$$

if $m < 3T/2$ and

$$\log\{A_\omega(n, m)|B_\omega(m, l)|\} \leq \frac{N}{2\pi}\,(U(3\pi/4, \pi/2) + V(\pi/4, \pi/3) - \beta) + O(\log N)$$

if $m \geq 3T/2$. This completes the proof. □

3.2. Asymptotics of $P(N;\omega)$. Suppose $2T \le n < m'$. Then,

$$\frac{1}{\{N\}}\begin{Bmatrix}2n+1\\ n-m\end{Bmatrix} = \frac{\{N-1\}!\prod_{k=1}^{2n-4T}\{N+k\}}{\{n-m\}!\{n+m+1\}!} = \frac{\{N-1\}!\{2n-4T\}!}{\{n-m\}!\{n+m+1\}!}\,,$$

and so

$$\begin{aligned}\frac{(-1)^{n+m}\tilde{A}_\omega(n,m)}{\{1\}} &= \frac{(-1)^n(\{n\}!)^6}{\{N-1\}!\{2n-4T\}!\{n-m\}!\{n+m+1\}!}\\ &= \frac{\left(\prod_{k=1}^{n}2\sin\frac{k}{N}\pi\right)^6}{N\left(\prod_{k=1}^{2n-4T}2\sin\frac{k}{N}\pi\right)\left(\prod_{k=1}^{n-m}2\sin\frac{k}{N}\pi\right)\left(\prod_{k=1}^{n+m+1}2\sin\frac{k}{N}\pi\right)}\end{aligned}$$

is positive. On the other hand, as in Lemma 2.1, we can show

$$\frac{(-1)^{n+m}\tilde{A}_\omega(n,m)}{\{1\}} \le \frac{-\tilde{A}_\omega(n,4T-n-1)}{\{1\}} = 2\sin\frac{(2n+2)\pi}{N}\cdot\frac{A_\omega(n,4T-n)}{\{1\}}\,.$$

Therefore, by Lemmas 3.2 and 3.3, we have

PROPOSITION 3.5

$$\log|P(N;\omega)| \le \frac{N}{2\pi}\left(U(3\pi/4,\pi/2)+V(\pi/4,\pi/3)-\alpha\right)+O(\log N)\,.$$

3.3. Asymptotics of $R(N;\omega)$. Suppose $2m' < l \le m-m'$. Then, $\tilde{B}_\omega(m,l)$ is equal to

$$-2\prod_{k=1}^{l-2m'-1}\{2N+k\}\prod_{k=1}^{l+2m'+1}\{2N-k\} = (-1)^{l+1}2\prod_{k=1}^{l-2m'-1}2\sin\frac{k\pi}{N}\cdot\prod_{k=1}^{l+2m'+1}2\sin\frac{k\pi}{N}\,,$$

and so $(-1)^{l+1}\tilde{B}_\omega(m,l)$ is positive. Furthermore, as in Lemmas 2.2 and 3.3, we can show

$$\log\{(-1)^{l+1}\tilde{B}_\omega(m,l)\} \le \frac{N}{2\pi}\cdot V(\pi,5\pi/6)+O(\log N)\,.$$

Therefore, by Lemma 3.2, we have

PROPOSITION 3.6

$$\log|R(N;\omega)| \le \frac{N}{2\pi}\left(U(3\pi/4,\pi/2)+V(\pi,5\pi/6)-\alpha\right)+O(\log N)\,.$$

3.4. Proof of Main Theorem. First of all, by (4),

$$\log J_L(N;\omega) = \log Q(N;\omega)+\log\left\{1+\frac{P(N;\omega)}{Q(N;\omega)}+\frac{R(N;\omega)}{Q(N;\omega)}+\frac{S(N;\omega)}{Q(N;\omega)}\right\}\,.$$

On the other hand,

$$\lim_{N\to\infty}\frac{P(N;\omega)}{Q(N;\omega)} = \lim_{N\to\infty}\frac{R(N;\omega)}{Q(N;\omega)} = \lim_{N\to\infty}\frac{S(N;\omega)}{Q(N;\omega)} = 0$$

by Propositions 2.3, 3.4, 3.5 and 3.6 because

$$V(\pi/4, \pi/3) = 6\Lambda(\pi/3) = V(\pi, 5\pi/6)\,.$$

Consequently, by Proposition 2.3 again,

$$\lim_{N\to\infty} \frac{2\pi}{N} \log J_L(N;\omega) = \lim_{N\to\infty} \frac{2\pi}{N} \log Q(N;\omega) = 6\Lambda(\pi/3) + 16\Lambda(\pi/4)\,. \qquad \square$$

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Present Addresses:
MAYUKO YAMAZAKI
DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES,
TOKYO METROPOLITAN UNIVERSITY,
MINAMI-OSAWA, HACHIOJI, TOKYO, 192–0397 JAPAN.

YOSHIYUKI YOKOTA
DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES,
TOKYO METROPOLITAN UNIVERSITY,
MINAMI-OSAWA, HACHIOJI, TOKYO, 192–0397 JAPAN.
e-mail: jojo@tmu.ac.jp