TOKYO J. MATH.
VOL. 33, NO. 2, 2010

The IH-complex of Spatial Trivalent Graphs

Atsushi ISHII and Kengo KISHIMOTO

University of Tsukuba and Osaka City University

(Communicated by J. Murakami)

Abstract. We define the IH-complex on the set of spatial trivalent graphs by using the IH-move, which is a local spatial move appeared in a study of knotted handlebodies. The IH-distance between two spatial trivalent graphs is defined by the minimal number of IH-moves needed to transform one into the other. It gives a distance function on the IH-complex. We give a lower bound for the IH-distance, and evaluate it.

1. Introduction

A spatial graph is a finite graph embedded in the 3-sphere S^3. Two spatial graphs are assumed to be the same if they can be transformed into each other by an isotopy of S^3. We introduce the IH-complex, whose vertex set consists of all spatial trivalent graphs, and investigate it in the IH-complex. Some properties of a spatial trivalent graph come out in this complex. For example, the trivial spatial θ-curve K_θ and the trivial spatial handcuff graph K_ϕ, which are the two simplest spatial graphs, are entirely different vertices in the IH-complex: Although the vertex-degree of K_θ is infinite, that of K_ϕ is just one.

For a set of topological objects and a local move for the objects, we can construct a simplicial complex whose vertex set consists of the topological objects. In the simplicial complex, a family of $n+1$ vertices spans an n-simplex if and only if any two of them are related by the single local move. Hirasawa and Uchida [2] introduced the Gordian complex which is defined on the set of knots with the crossing change operation. The unknotting number of a knot is the distance from the trivial knot in this complex. Such simplicial complexes are studied in [8], [9] and [10]. We also note that the tunnel complex defined in [6] is closely related to the IH-move.

The IH-complex is defined on the set of spatial trivalent graphs with the IH-move, which is a local spatial move appeared in a study of knotted handlebodies. The first author [3] showed that two spatial trivalent graphs are neighborhood equivalent [11] if and only if they are related by IH-moves. We show some properties of the IH-complex, and give a lower bound for the IH-distance, which gives a distance function on the IH-complex. We construct a family which gives all IH-distances as a corollary of the evaluation.

Received August 3, 2009; revised May 25, 2010
Mathematics Subject Classification: 57M25
Key words and phrases: Spatial graph, simplicial complex, IH-move

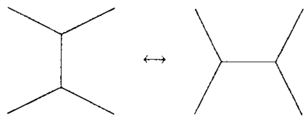


FIGURE 1

It is not easy to determine the IH-distance between two spatial trivalent graphs, like the unknotting number for knots. The key to the evaluation of the IH-distance is the flow expansion of the IH-complex. The flow expansion is a simplicial complex whose vertex set consists of pairs of spatial trivalent graphs and their flows. We have a natural projection from the flow expansion onto the IH-complex. We can lift a path in the IH-complex to its flow expansion. The lower bound for the IH-distance is obtained by reducing it to the length of a path in the flow expansion.

A connected component of the IH-complex represents a neighborhood equivalence class of a spatial trivalent graph. We show that the greatest common divisor plays a large role in the study of the connectivity of the flow expansion. Furthermore, we determine the connected components containing the preimage of the trivial spatial handcuff graph with respect to the natural projection.

In Section 2, we introduce the IH-complex, and show its properties. In Section 3, we define the flow expansion of the IH-complex, and we see relationships between paths in the IH-complex and its flow expansion. In Section 4, we investigate the connectivity of the flow expansion. In Section 5, we define a map whose Lipschitz constant is equal to one, and give a lower bound for the IH-distance. In Section 6, we construct a family which gives all IH-distances.

2. The IH-complex of spatial trivalent graphs

We denote by $\mathcal{K}$ the set of spatial trivalent graphs. An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1. For $K, K' \in \mathcal{K}$, we define the *IH-distance* $d_{\mathrm{IH}}(K, K')$ by the minimal number of IH-moves needed to transform K into K'. We set $d_{\mathrm{IH}}(K, K') := \infty$ if we do not have such a number. We remark that d_{IH} is a distance function on $\mathcal{K}$.

The *IH-complex* $\mathcal{C}_{\mathrm{IH}}$ of spatial trivalent graphs is the simplicial complex defined by the following condition:

- The vertex set of $\mathcal{C}_{\mathrm{IH}}$ consists of all spatial trivalent graphs.
- A family of $n+1$ vertices $\{K_0, K_1, \ldots, K_n\}$ spans an n-simplex if and only if $d_{\mathrm{IH}}(K_i, K_j) = 1$ for any $i, j \in \{0, 1, \ldots, n\}$ such that $i \neq j$.

Two spatial graphs are *neighborhood equivalent* [11] if their regular neighborhoods can be transformed into each other by an isotopy of S^3. We denote by $\deg(v)$ the vertex-degree of a vertex v. A simplicial complex is said to be *locally finite* if each vertex belongs only to

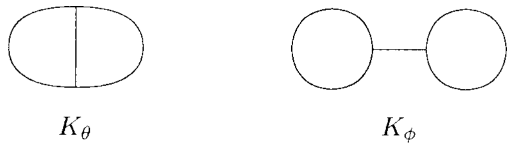


FIGURE 2

finitely many simplices. The trivial spatial θ-curve K_θ and the trivial spatial handcuff graph K_ϕ are the spatial trivalent graphs depicted in Figure 2.

PROPOSITION 1. *The IH-complex $\mathcal{C}_{\mathrm{IH}}$ satisfies the following properties.*

- *Two spatial trivalent graphs belong to the same connected component of $\mathcal{C}_{\mathrm{IH}}$ if and only if they are neighborhood equivalent.*
- *The trivial spatial θ-curve K_θ is the only spatial trivalent graph K such that $d_{\mathrm{IH}}(K, K_\phi) = 1$. Thus* $\deg(K_\phi) = 1$.
- *We have* $\deg(K_\theta) = \infty$. *Hence the IH-complex $\mathcal{C}_{\mathrm{IH}}$ is not locally finite.*
- *The dimension of $\mathcal{C}_{\mathrm{IH}}$ is greater than or equal to two.*

PROOF. By Theorem 1 in [3], two spatial trivalent graphs K and K' are neighborhood equivalent if and only if $d_{\mathrm{IH}}(K, K')$ is finite, which implies the first statement.

Since an IH-move applied to the trivial spatial handcuff graph K_ϕ is unique up to isotopies of S^3, we have

$$K = K_\theta \Leftrightarrow d_{\mathrm{IH}}(K, K_\phi) = 1 .$$

Thus $\deg(K_\phi) = 1$.

For an integer $n > 1$, let K_n be the spatial trivalent graph depicted in Figure 3. We note that K_n and K_m are distinct if $n \neq m$, since their constituent links are distinct. Since the trivial spatial θ-curve K_θ is obtained from K_n by a single IH-move, we have $d_{\mathrm{IH}}(K_n, K_\theta) = 1$, which implies that $\deg(K_\theta) = \infty$. Hence the IH-complex $\mathcal{C}_{\mathrm{IH}}$ is not locally finite.

By Figure 4, we have $d_{\mathrm{IH}}(K_n, K_{n+1}) = 1$. Then a family of three vertices $\{K_\theta, K_n, K_{n+1}\}$ spans a 2-simplex in $\mathcal{C}_{\mathrm{IH}}$, which implies that $\dim(\mathcal{C}_{\mathrm{IH}}) \geq 2$. □

In Proposition 10, we also have $\mathrm{diam}(\mathcal{C}_{\mathrm{IH}}) = \infty$ for the IH-complex, where the diameter $\mathrm{diam}(\mathcal{C}_{\mathrm{IH}})$ is the largest IH-distance between all pairs of spatial trivalent graphs. By Proposition 1, each connected component of $\mathcal{C}_{\mathrm{IH}}$ indicates a handlebody-link [3], which is a disjoint union of handlebodies embedded in the 3-sphere S^3. Baader [1] showed that, for any pair of two knots of Gordian distance two, there exist infinitely many non-equivalent knots whose Gordian distance to each of the pair is one. On the other hand, by Proposition 1, there exists a unique spatial trivalent graph K such that $d_{\mathrm{IH}}(K_\phi, K) = d_{\mathrm{IH}}(K, K_n) = 1$, where we note that $d_{\mathrm{IH}}(K_\phi, K_n) = 2$.

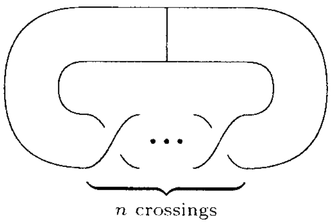


FIGURE 3

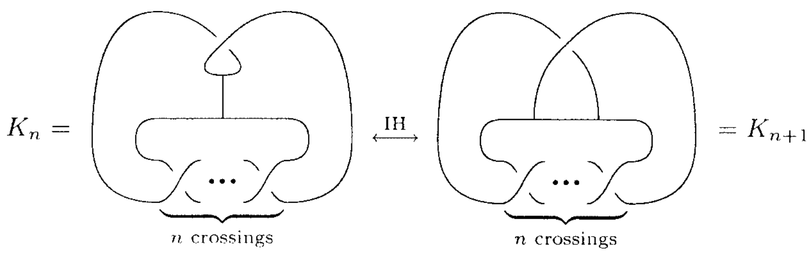


FIGURE 4

3. The flow expansion of the IH-complex

We recall a flow [4] of a spatial trivalent graph. Let $\mathcal{E}(K)$ be the set of edges of a spatial trivalent graph K. Let $\mathcal{O}_e$ be the set of two orientations of an edge $e \in \mathcal{E}(K)$. A map $\varphi_e : \mathcal{O}_e \to \mathbf{Z}$ is a *flow* of an edge e if $\varphi_e(-o) = -\varphi_e(o)$, where $-o$ is the inverse of the orientation o. A flow φ_e is represented by a pair $(o, s) \in \mathcal{O}_e \times \mathbf{Z}$ up to the equivalence relation $(o, s) \sim (-o, -s)$. See Figure 5, where an element of $\mathbf{Z}$ is represented with an underline.

We fix an orientation $o_e \in \mathcal{O}_e$ for each edge e of a spatial trivalent graph K. A collection $\varphi = \{\varphi_e\}_{e \in \mathcal{E}(K)}$ is a *flow* of K if we have

$$\sum_{e \in \mathcal{E}_{\mathrm{in}}(v)} \varphi_e(o_e) = \sum_{e \in \mathcal{E}_{\mathrm{out}}(v)} \varphi_e(o_e)$$

at any vertex v, where

$$\mathcal{E}_{\mathrm{in}}(v) := \{e \mid e \text{ is an edge incident to } v \text{ such that } o_e \text{ points to } v\},$$

$$\mathcal{E}_{\mathrm{out}}(v) := \{e \mid e \text{ is an edge incident to } v \text{ such that } -o_e \text{ points to } v\}.$$

We remark that the definition of a flow of K does not depend on the choice of the orientations o_e. We denote by $\Phi(K)$ the set of flows of K. We remark that a flow of K represents a homology class in $H_1(K; \mathbf{Z})$. Then we have $\Phi(K) \cong H_1(K; \mathbf{Z})$.

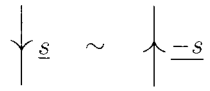


FIGURE 5

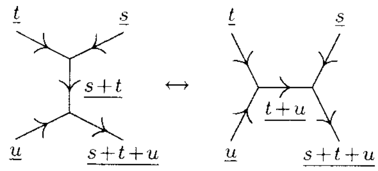


FIGURE 6

A *flowed spatial trivalent graph* (K, φ) is a pair of a spatial trivalent graph K and a flow $\varphi \in \Phi(K)$. Two flowed spatial trivalent graphs are assumed to be the same if one can be transformed into the other by an ambient isotopy preserving a flow. We denote by $\mathcal{K}^\Phi$ the set of flowed spatial trivalent graphs. For $\varphi \in \Phi(K)$, we define $-\varphi := \{-\varphi_e\}_{e \in \mathcal{E}(K)}$. Two flowed spatial trivalent graphs (K, φ) and $(K, -\varphi)$ are distinct in general.

A *flowed IH-move* is a local spatial move on flowed spatial trivalent graphs as described in Figure 6. For $(K, \varphi), (K', \varphi') \in \mathcal{K}^\Phi$, we define the *flowed IH-distance* $d^\Phi_{\mathrm{IH}}((K, \varphi), (K', \varphi'))$ by the minimal number of flowed IH-moves needed to transform (K, φ) into (K', φ'). We set $d^\Phi_{\mathrm{IH}}((K, \varphi), (K', \varphi')) := \infty$ if we do not have such a number. We remark that d^Φ_{IH} is a distance function on $\mathcal{K}^\Phi$.

The *flow expansion* $\mathcal{C}^\Phi_{\mathrm{IH}}$ of the IH-complex $\mathcal{C}_{\mathrm{IH}}$ is the simplicial complex defined by the following condition:

- The vertex set of $\mathcal{C}^\Phi_{\mathrm{IH}}$ consists of all flowed spatial trivalent graphs.
- A family of $n + 1$ vertices $\{(K_0, \varphi^{(0)}), (K_1, \varphi^{(1)}), \ldots, (K_n, \varphi^{(n)})\}$ spans an n-simplex if and only if $d^\Phi_{\mathrm{IH}}((K_i, \varphi^{(i)}), (K_j, \varphi^{(j)})) = 1$ for any $i, j \in \{0, 1, \ldots, n\}$ such that $i \neq j$.

A *path* between two vertices v and v' in a simplicial complex $\mathcal{C}$ is a sequence $\overline{v_0v_1}, \overline{v_1v_2}, \ldots, \overline{v_{n-1}v_n}$ of 1-simplices such that $v_0 = v$ and $v_n = v'$, where $\overline{v_iv_{i+1}}$ is a 1-simplex between the vertices v_i and v_{i+1}. We define the *length* $l(\omega)$ of a path $\omega = \overline{v_0v_1}, \overline{v_1v_2}, \ldots, \overline{v_{n-1}v_n}$ by the number n. When vertices v and v' are connected by a path ω, we write $v \sim_\omega v'$. Then we have

$$d_{\mathrm{IH}}(K, K') = \min\{l(\omega) \mid K \sim_\omega K' \text{ in } \mathcal{C}_{\mathrm{IH}}\},$$
$$d^\Phi_{\mathrm{IH}}((K, \varphi), (K', \varphi')) = \min\{l(\omega) \mid (K, \varphi) \sim_\omega (K', \varphi') \text{ in } \mathcal{C}^\Phi_{\mathrm{IH}}.$$

Let $\mathcal{C}$ and $\mathcal{C}'$ be simplicial complexes. Let $p : \mathcal{C} \to \mathcal{C}'$ be a simplicial map. For a path $\omega = \overline{v_0v_1}, \overline{v_1v_2}, \dots, \overline{v_{n-1}v_n}$ in $\mathcal{C}$, we define the path $p(\omega)$ in $\mathcal{C}'$ by the sequence $\overline{p(v_0)p(v_1)}, \overline{p(v_1)p(v_2)}, \dots, \overline{p(v_{n-1})p(v_n)}$ of 1-simplices in $\mathcal{C}'$, where we delete $\overline{p(v_i)p(v_{i+1})}$ if $p(v_i) = p(v_{i+1})$.

Let $p : \mathcal{C}_{\mathrm{IH}}^{\Phi} \to \mathcal{C}_{\mathrm{IH}}$ be the simplicial map induced by the map sending $(K, \varphi) \in \mathcal{K}^{\Phi}$ to $K \in \mathcal{K}$. We note that the map p is surjective.

LEMMA 2. *Let ω be a path between spatial trivalent graphs K and K' in $\mathcal{C}_{\mathrm{IH}}$. For a flow $\varphi \in \Phi(K)$, there exists a flow $\varphi' \in \Phi(K')$ and a path $\widetilde{\omega}$ between (K, φ) and (K', φ') in $\mathcal{C}_{\mathrm{IH}}^{\Phi}$ such that $p(\widetilde{\omega}) = \omega$ and $l(\widetilde{\omega}) = l(\omega)$.*

PROOF. Without loss of generality, we may assume that $l(\omega) = 1$. For the path ω, there exists an IH-move between K and K'. (Such an IH-move is not unique in general.) By applying the corresponding flowed IH-move, we obtain a flowed spatial trivalent graph (K', φ') from (K, φ). Let $\widetilde{\omega}$ be the 1-simplex $\overline{(K, \varphi)(K', \varphi')}$ in $\mathcal{C}_{\mathrm{IH}}^{\Phi}$. Then $p(\widetilde{\omega}) = \omega$ and $l(\widetilde{\omega}) = l(\omega)$. □

As a corollary of Lemma 2, we have the following lemma.

LEMMA 3. *Let K and K' be spatial trivalent graphs. Fix $\varphi' \in \Phi(K')$. Then we have*

$$d_{\mathrm{IH}}(K, K') = \min_{\varphi \in \Phi(K)} d_{\mathrm{IH}}^{\Phi}((K, \varphi), (K', \varphi'))\,.$$

PROOF. By Lemma 2, we have

$$d_{\mathrm{IH}}(K, K') \geq \min_{\varphi \in \Phi(K)} d_{\mathrm{IH}}^{\Phi}((K, \varphi), (K', \varphi'))\,.$$

Since $p : \mathcal{C}_{\mathrm{IH}}^{\Phi} \to \mathcal{C}_{\mathrm{IH}}$ is a simplicial map, we have $l(p(\omega)) \leq l(\omega)$ for any path ω in $\mathcal{C}_{\mathrm{IH}}^{\Phi}$. Therefore

$$d_{\mathrm{IH}}(K, K') = \min_{\varphi \in \Phi(K)} d_{\mathrm{IH}}^{\Phi}((K, \varphi), (K', \varphi'))\,.$$

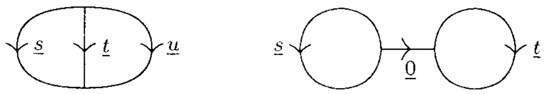
□

4. Connected components of the flow expansion

The *volume* $|\varphi_e|$ of the flow φ_e of an edge e is defined by the absolute value $|\varphi_e(o)|$ ($=|\varphi_e(-o)|$). For $\varphi \in \Phi(K)$, we denote by $[\varphi]$ the multiset consisting of the volumes of the flows of all edges of K. Let $(K_\theta, \theta_{s,t,u})$ and $(K_\phi, \phi_{s,t})$ be the flowed spatial trivalent graphs depicted in Figure 7. Then we have

$$[\theta_{1,-2,1}] = \{1, 2, 1\}\,, \quad [\phi_{1,1}] = \{1, 1, 0\}\,.$$

For a set $A = \{i_1, \dots, i_n\}$ of integers, we denote by $\gcd(A)$ or $\gcd(i_1, \dots, i_n)$ the greatest common divisor of the integers in A. Here we put $\gcd(0, \dots, 0) := 0$.



$(K_\theta, \theta_{s,t,u})$ $(K_\phi, \phi_{s,t})$

FIGURE 7

LEMMA 4. *If flowed spatial trivalent graphs (K, φ) and (K', φ') belong to the same connected component of $\mathcal{C}_{\mathrm{IH}}^{\Phi}$, then we have* $\gcd([\varphi]) = \gcd([\varphi'])$.

PROOF. Without loss of generality, we may assume that (K, φ) and (K', φ') are related by a single flowed IH-move. The equalities

$$\gcd(s, t, u, s+t) = \gcd(s, t, u) = \gcd(s, t, u, t+u)$$

imply the equality $\gcd([\varphi]) = \gcd([\varphi'])$ (see Figure 6). $\square$

We set $\mathcal{K}^{\Phi,n} := \{(K, \varphi) \in \mathcal{K}^{\Phi} \mid \gcd([\varphi]) = n\}$. Let $\mathcal{C}_{\mathrm{IH}}^{\Phi,n}$ be the subcomplex of $\mathcal{C}_{\mathrm{IH}}^{\Phi}$ spanned by the elements of $\mathcal{K}^{\Phi,n}$. The *zero flow* $\mathbf{0} \in \Phi(K)$ is the collection $\{0_e\}_{e \in \mathcal{E}(K)}$ of the zero maps $0_e : \mathcal{O}_e \to \mathbf{Z}$. For $n \in \mathbf{Z}$ and $\varphi \in \Phi(K)$, we define $n\varphi := \{n\varphi_e\}_{e \in \mathcal{E}(K)}$. Then we have $-\varphi = (-1)\varphi$ and $\mathbf{0} = 0\varphi$.

PROPOSITION 5. *For any positive integer n, the simplicial complex $\mathcal{C}_{\mathrm{IH}}^{\Phi,n}$ is isomorphic to $\mathcal{C}_{\mathrm{IH}}^{\Phi,1}$. The simplicial complex $\mathcal{C}_{\mathrm{IH}}^{\Phi,0}$ is isomorphic to $\mathcal{C}_{\mathrm{IH}}$.*

PROOF. We define the map $f : \mathcal{K}^{\Phi,1} \to \mathcal{K}^{\Phi,n}$ by $f(K, \varphi) = (K, n\varphi)$. The map f is a bijection and induces an isomorphism from $\mathcal{C}_{\mathrm{IH}}^{\Phi,1}$ to $\mathcal{C}_{\mathrm{IH}}^{\Phi,n}$.

We define the map $f : \mathcal{K}^{\Phi,0} \to \mathcal{K}$ by $f(K, \varphi) = K$. Since φ is the zero flow if and only if $\gcd([\varphi]) = 0$, the map f is a bijection and induces an isomorphism from $\mathcal{C}_{\mathrm{IH}}^{\Phi,0}$ to $\mathcal{C}_{\mathrm{IH}}$. $\square$

Let $\mathcal{C}_{\mathrm{IH}}^{\Phi}(K, \varphi) \subset \mathcal{C}_{\mathrm{IH}}^{\Phi}$ be the connected component containing $(K, \varphi) \in \mathcal{K}^{\Phi}$. Set $\mathcal{C}_{\mathrm{IH}}^{\Phi,n}(K) := \{\mathcal{C}_{\mathrm{IH}}^{\Phi}(K, \varphi) \mid \gcd([\varphi]) = n\}$. By Lemmas 2 and 4, we have $\mathcal{C}_{\mathrm{IH}}^{\Phi,n}(K) = \mathcal{C}_{\mathrm{IH}}^{\Phi,n}(K')$ if $K \sim_\omega K'$ in $\mathcal{C}_{\mathrm{IH}}$. Then we note that $\mathcal{C}_{\mathrm{IH}}^{\Phi,n}(K)$ is an invariant for handlebody-links.

By Proposition 5, we have $\#\mathcal{C}_{\mathrm{IH}}^{\Phi,0}(K) = 1$ and $\#\mathcal{C}_{\mathrm{IH}}^{\Phi,n}(K) = \#\mathcal{C}_{\mathrm{IH}}^{\Phi,1}(K)$ for a positive integer n, where $\#S$ is the number of the elements of a set S. We remark that Lemma 4 implies that

$$(K, \varphi) \sim_\omega (K, \varphi') \Rightarrow \gcd([\varphi]) = \gcd([\varphi']) .$$

If $\#\mathcal{C}_{\mathrm{IH}}^{\Phi,1}(K) = 1$, then

$$(K, \varphi) \sim_\omega (K, \varphi') \Leftrightarrow \gcd([\varphi]) = \gcd([\varphi']) .$$

PROPOSITION 6. *For the trivial spatial handcuff graph K_ϕ, we have*

$$\#\mathcal{C}_{\mathrm{IH}}^{\Phi,1}(K_\phi) = 1\,.$$

PROOF. Since $\Phi(K_\phi) = \{\phi_{s,t} \mid s, t \in \mathbf{Z}\}$ and $\gcd([\phi_{s,t}]) = \gcd(s, t)$, it is sufficient to show that there exists a sequence of IH-moves between $(K_\phi, \phi_{s,t})$ and $(K_\phi, \phi_{s',t'})$ if $\gcd(s, t) = \gcd(s', t')$. Let (s, t) and (s', t') be pairs of integers such that $\gcd(s, t) = \gcd(s', t')$. Then they are related by the following operations:

$$(i, j) \leftrightarrow (j, i)\,, \quad (i, j) \leftrightarrow (i, -j)\,, \quad (i, j) \leftrightarrow (i, i + j)\,.$$

We note that

$$(K_\phi, \phi_{i,j}) = (K_\phi, \phi_{j,i})\,, \quad (K_\phi, \phi_{i,j}) = (K_\phi, \phi_{i,-j})\,.$$

We can transform $(K_\phi, \phi_{i,j})$ into $(K_\phi, \phi_{i,i+j})$ by applying IH-moves as follows:

$$(K_\phi, \phi_{i,j}) \overset{\mathrm{IH}}{\leftrightarrow} (K_\theta, \theta_{i,-i-j,j}) = (K_\theta, \theta_{i,j,-i-j}) \overset{\mathrm{IH}}{\leftrightarrow} (K_\phi, \phi_{i,-i-j}) = (K_\phi, \phi_{i,i+j})\,.$$

Therefore there exists a path connecting $(K_\phi, \phi_{s,t})$ and $(K_\phi, \phi_{s',t'})$ if $\gcd(s, t) = \gcd(s', t')$. □

5. A lower bound for the IH-distance

Let (X, d_X) and (Y, d_Y) be metric spaces. For a map $f : X \to Y$, the Lipschitz constant $Lip(f)$ is the minimal number λ such that

$$d_Y(f(x_1), f(x_2)) \le \lambda\, d_X(x_1, x_2)$$

for any $x_1, x_2 \in X$. Let $d_{\mathbf{Z}}$ be the standard distance function on $\mathbf{Z}$ defined by $d_{\mathbf{Z}}(x, y) = |x - y|$.

For the multiset $[\varphi]$, we define the sequence $\{[\varphi]_i\}$ by ordering the elements of $[\varphi]$ such that $[\varphi]_i \ge [\varphi]_{i+1}$ for any i. For the flowed spatial trivalent graphs $(K_\theta, \theta_{s,t,u})$ and $(K_\phi, \phi_{s,t})$ depicted in Figure 7, we have

$$[\theta_{1,-2,1}] = \{1, 2, 1\}\,, \quad [\theta_{1,-2,1}]_1 = 2\,, \quad [\theta_{1,-2,1}]_2 = 1\,, \quad [\theta_{1,-2,1}]_3 = 1\,,$$
$$[\phi_{1,1}] = \{1, 1, 0\}\,, \quad [\phi_{1,1}]_1 = 1\,, \quad [\phi_{1,1}]_2 = 1\,, \quad [\phi_{1,1}]_3 = 0\,.$$

Let $N_F : \mathcal{K}^\Phi \to \mathbf{Z}$ be the map sending (K, φ) to the minimal non-negative integer n such that $[\varphi]_1 \le F_{n+1}$, $[\varphi]_2 \le F_n$, where F_n is the n-th Fibonacci number: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.

LEMMA 7. *For any $(K, \varphi), (K', \varphi') \in \mathcal{K}^\Phi$, we have*

$$d_{\mathrm{IH}}^{\Phi}((K, \varphi), (K', \varphi')) \ge |N_F(K, \varphi) - N_F(K', \varphi')|.$$

Furthermore, we have $Lip(N_F) = 1$.

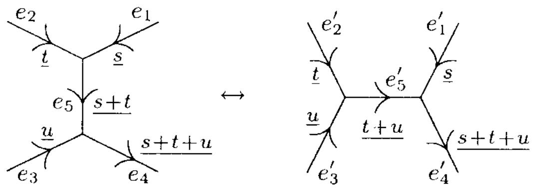


FIGURE 8

PROOF. Let (K, φ) and (K', φ') be flowed spatial trivalent graphs related by a single flowed IH-move around the edges $e_1, \dots, e_5$ and $e_1', \dots, e_5'$ depicted in Figure 8.

Put $n := N_F(K, \varphi)$. We show that, for $e' \in \mathcal{E}(K')$,

$$|\varphi'_{e'}| \le \begin{cases} F_{n+2} & \text{if } e' = e_5' \,, \\ F_{n+1} & \text{otherwise} \,. \end{cases} \tag{1}$$

If $e' \neq e_5'$, then

$$|\varphi'_{e'}| = |\varphi_e| \le [\varphi]_1 \le F_{n+1}$$

for some $e \in \mathcal{E}(K)$. We assume that $e' = e_5'$. If $e_2 = e_3$, then $t = -u$ and

$$|\varphi'_{e'}| = |t + u| = 0 \le F_{n+2} \,.$$

If $e_2 \neq e_3$, then

$$|\varphi'_{e'}| = |t + u| \le |t| + |u| \le [\varphi]_1 + [\varphi]_2 \le F_{n+2} \,.$$

Then we have the equality (1). Therefore $[\varphi']_1 \le F_{n+2}$ and $[\varphi']_2 \le F_{n+1}$, which imply that $N_F(K', \varphi') \le n + 1 = N_F(K, \varphi) + 1$. By the same argument, we have $N_F(K, \varphi) \le N_F(K', \varphi') + 1$. Hence we have

$$|N_F(K, \varphi) - N_F(K', \varphi')| \le 1$$

for any $(K, \varphi), (K', \varphi') \in \mathcal{K}^\Phi$ such that $d^\Phi_{\mathrm{IH}}((K, \varphi), (K', \varphi')) = 1$. This implies that

$$d^\Phi_{\mathrm{IH}}((K, \varphi), (K', \varphi')) \ge |N_F(K, \varphi) - N_F(K', \varphi')|$$

for any $(K, \varphi), (K', \varphi') \in \mathcal{K}^\Phi$.

By this inequality, we have $Lip(N_F) \le 1$. We have the equality

$$d^\Phi_{\mathrm{IH}}((K_\theta, \theta_{1,-2,1}), (K_\phi, \phi_{1,1})) = |N_F(K_\theta, \theta_{1,-2,1}) - N_F(K_\phi, \phi_{1,1})|,$$

which follows from the equalities

$$d^\Phi_{\mathrm{IH}}((K_\theta, \theta_{1,-2,1}), (K_\phi, \phi_{1,1})) = 1 \,, \quad N_F(K_\theta, \theta_{1,-2,1}) = 2 \,, \quad N_F(K_\phi, \phi_{1,1}) = 1 \,.$$

Hence we have $Lip(N_F) \ge 1$, which completes the proof. $\square$

THEOREM 8. *Let v be a flowed spatial trivalent graph invariant which is invariant under flowed IH-moves. Let K and K' be spatial trivalent graphs. Fix $\varphi' \in \Phi(K')$. Then we have*

$$d_{\mathrm{IH}}(K, K') \geq \min_{\substack{\varphi \in \Phi(K) \\ v(K,\varphi)=v(K',\varphi')}} |N_F(K, \varphi) - N_F(K', \varphi')| .$$

PROOF. By Lemma 3, we have

$$d_{\mathrm{IH}}(K, K') = \min_{\varphi \in \Phi(K)} d_{\mathrm{IH}}^{\Phi}((K, \varphi), (K', \varphi')) .$$

Since $d_{\mathrm{IH}}^{\Phi}((K, \varphi), (K', \varphi')) = \infty$ if $v(K, \varphi) \neq v(K', \varphi')$, we have

$$\begin{aligned} d_{\mathrm{IH}}(K, K') &= \min_{\substack{\varphi \in \Phi(K) \\ v(K,\varphi)=v(K',\varphi')}} d_{\mathrm{IH}}^{\Phi}((K, \varphi), (K', \varphi')) \\ &\geq \min_{\substack{\varphi \in \Phi(K) \\ v(K,\varphi)=v(K',\varphi')}} |N_F(K, \varphi) - N_F(K', \varphi')| , \end{aligned}$$

where the last inequality follows from Lemma 7. □

REMARK 9. Let f be a map from $(\mathcal{K}^{\Phi}, d_{\mathrm{IH}}^{\Phi})$ to a metric space (X, d_X) such that $Lip(f) = 1$. By the same argument as in the proof of Theorem 8, we have

$$d_{\mathrm{IH}}(K, K') \geq \min_{\substack{\varphi \in \Phi(K) \\ v(K,\varphi)=v(K',\varphi')}} d_X(f(K, \varphi), f(K', \varphi')) .$$

6. A family which gives all IH-distances

We construct a sequence $(K_i, \varphi^{(i)})$ ($i \in \mathbf{Z}_{\geq 0}$) of flowed spatial θ-curves such that $[\varphi^{(i)}] = \{F_{i+2}, F_{i+1}, F_i\}$ as follows: Let $(K_\psi, \psi_{s,t,u})$ be the flowed spatial θ-curve represented by the diagram $D_{s,t,u}$ depicted in Figure 9, where we ignore the characters a and b. Put $K_0 := K_\psi$, $\varphi^{(0)} := \psi_{F_2,F_1,F_0}$. For $i > 0$, let $(K_i, \varphi^{(i)})$ be a flowed spatial θ-curve obtained from $(K_{i-1}, \varphi^{(i-1)})$ by a single flowed IH-move as shown in Figure 10. Then we have the following proposition.

PROPOSITION 10. *For $i, j \geq 0$, we have $d_{\mathrm{IH}}(K_i, K_j) = |i - j|$, which implies that the diameter of the IH-complex $\mathcal{C}_{\mathrm{IH}}$ is infinite.*

For a proof of this proposition, we recall the definition of the quandle coloring, and give a lemma. A *quandle* [5, 7] is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms:

Q_1. For any $a \in X$, $a * a = a$,

Q_2. For any $a \in X$, the map $S_a : X \to X$ defined by $S_a(x) = x * a$ is a bijection,

Q_3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

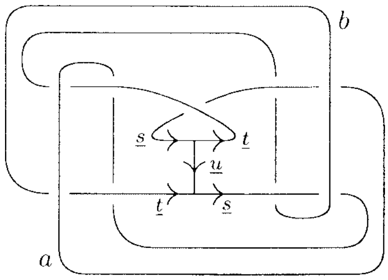


FIGURE 9

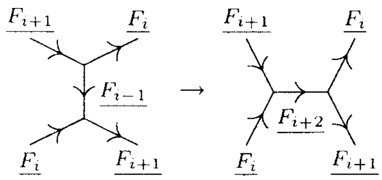


FIGURE 10

For $i \in \mathbf{Z}$, we define $a *^i b := S_b^i(a)$.

Let D be a diagram of a flowed spatial graph (K, φ). We denote by $\mathcal{A}(D)$ the set of arcs of D, where an arc is a piece of a curve whose endpoints are undercrossings or vertices. We choose an orientation $o_e \in \mathcal{O}_e$ for each edge $e \in \mathcal{E}(K)$. Then $(K, \{o_e\}_{e \in \mathcal{E}(K)}, \varphi)$ is a flowed oriented spatial graph. For an arc α which originates from an edge e, we put $o_\alpha := o_e$, $\varphi_\alpha := \varphi_e$. To represent an orientation o_e in D, we may use the co-orientation obtained by rotating the orientation o_e $\pi/2$ counterclockwise. We denote it by the same symbol o_α. We denote by χ_0 the over-arc at a crossing χ of D. We denote by χ_1, χ_2 the under-arcs at χ such that the co-orientation o_{χ_0} points to χ_2.

An X-*coloring* of D is a map $C : \mathcal{A}(D) \to X$ satisfying the following conditions (Figure 11):

C_1. For a crossing χ, we have

$$C(\chi_1) *^s C(\chi_0) = C(\chi_2),$$

where $s = \varphi_{\chi_0}(o_{\chi_0})$.

C_2. For a vertex ω, we have

$$C(\omega_1) = \cdots = C(\omega_d),$$

where $\omega_1, \ldots, \omega_d$ are the arcs incident to ω.

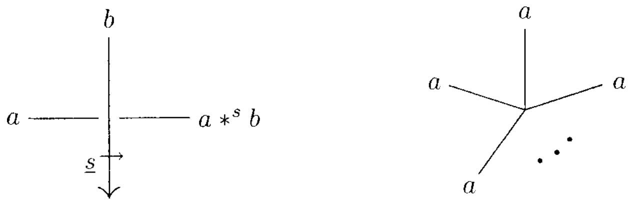


FIGURE 11

We note that an X-coloring C does not depend on the choice of the orientations o_e. An X-coloring is *trivial* if the map C is a constant map. For a diagram D of a flowed spatial graph (K, φ), we set

$$v_X(D) := \begin{cases} 1 & \text{if } D \text{ possesses a nontrivial } X\text{-coloring}, \\ 0 & \text{otherwise}. \end{cases}$$

Then v_X is a flowed spatial trivalent graph invariant which is invariant under flowed IH-moves. For the details, we refer the reader to [4].

Set $X := \mathbf{Z}[x, x^{-1}]/(2x-1)$. Let $* : X \times X \to X$ be the binary operation defined by $a * b = xa + (1-x)b$ for $a, b \in X$. Then $(X, *)$ is a quandle, which is an Alexander quandle. Put $w := v_X$. We set

$$\Phi(K)_w := \{\varphi \in \Phi(K) \mid w(K, \varphi) = 1\}.$$

LEMMA 11. *For $i \geq 0$, we have $\Phi(K_i)_w = \{\varphi^{(i)}\}$.*

PROOF. We first show that $\Phi(K_0)_w = \{\varphi^{(0)}\}$. The coloring relation obtained from Figure 9 is $a *^t b = b *^s a$, that is, Figure 9 gives an X-coloring of the diagram $D_{s,t,u}$ if and only if $a *^t b = b *^s a$ for $a, b \in X$. Since the equality $a *^t b = b *^s a$ is equivalent to $(x^s + x^t - 1)(a - b) = 0$ in X, the diagram $D_{s,t,u}$ has a nontrivial X-coloring if and only if $s = t = 1$, which implies $\Phi(K_0)_w = \{\varphi^{(0)}\}$.

Since w is invariant under flowed IH-moves,

$$\#\Phi(K_i)_w = \#\Phi(K_0)_w = 1.$$

By the construction of the sequence, $\varphi^{(i)} \in \Phi(K_i)_w$. Therefore we have

$$\Phi(K_i)_w = \{\varphi^{(i)}\}.$$

□

PROOF OF PROPOSITION 10. By the construction of the sequence, we have

$$d_{\mathrm{IH}}(K_i, K_j) \leq |i - j|$$

for $i, j \geq 0$. On the other hand, by Theorem 8 and Lemma 11, we have

$$
\begin{aligned}
d_{\mathrm{IH}}(K_i, K_j) &\geq \min_{\substack{\varphi \in \Phi(K) \\ w(K,\varphi)=w(K',\varphi')}} |N_F(K_i, \varphi) - N_F(K_j, \varphi^{(j)})| \\
&= \min_{\varphi \in \Phi(K_i)_w} |N_F(K_i, \varphi) - N_F(K_j, \varphi^{(j)})| \\
&= \min_{\varphi \in \{\varphi^{(i)}\}} |N_F(K_i, \varphi) - N_F(K_j, \varphi^{(j)})| \\
&= |(i+1) - (j+1)| = |i - j| \,.
\end{aligned}
$$

This completes the proof. □

ACKNOWLEDGMENT. The authors would like to thank Seiichi Kamada, Yuya Koda, and Ryo Nikkuni for their helpful comments.

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Present Addresses:
ATSUSHI ISHII
INSTITUTE OF MATHEMATICS,
UNIVERSITY OF TSUKUBA,
TSUKUBA, IBARAKI, 305–8571 JAPAN.
e-mail: aishii@math.tsukuba.ac.jp

KENGO KISHIMOTO
GRADUATE SCHOOL OF SCIENCE,
OSAKA, CITY UNIVERSITY
OSAKA, 558–8585 JAPAN.
e-mail: k-kishi@sci.osaka-cu.ac.jp