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Convergence Rate for a Continued Fraction Expansion Related to Fibonacci Type Sequences

Gabriela Ileana SEBE

"Politehnica" University of Bucharest

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Abstract. Chan ([2], [3]) considered some continued fraction expansions related to random Fibonacci-type sequences. A Wirsing-type approach to the Perron-Frobenius operator of the associated transformation under its invariant measure allows us to study the optimality of the convergence rate. Actually, we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem.

1. Introduction

Let $x \in [0, 1)$ and let k be a fixed integer greater than or equal to 2. Chan [3] proved that x can be written as

$$x = \cfrac{k^{-a_1}}{1 + \cfrac{(k-1)k^{-a_2}}{1 + \cfrac{(k-1)k^{-a_3}}{1 + \cdots}}} = [a_1, a_2, \dots]_k \,, \tag{1}$$

where the "digits" $a_m = a_m(x)$ are natural integers. This expansion is a generalization of the infinite expansion

$$\cfrac{2^{-a_1}}{1 + \cfrac{2^{-a_2}}{1 + \cdots}} = [a_1, a_2, \dots]_2 \,. \tag{2}$$

The case $k = 2$ was first studied in [1] and [2] and it was motivated by the work of Viswanath [14] on random Fibonacci sequences. Chan [3] considered the random Fibonacci-type sequences, $\{Q_m\}$, defined by $Q_{-1} = 0$, $Q_0 = 1$, $a_0(x) = 0$, and

$$Q_m(x) = k^{a_m(x)} Q_{m-1}(x) + (k-1)k^{a_{m-1}(x)} Q_{m-2}(x) \,, \quad m \geq 1 \,,$$

for all $x = [a_1, a_2, \dots]_k \in [0, 1)$.

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Chan [3] has also studied the transformation underlying the continued fraction expansion (1). Precisely, he defined the interval map $T_k : [0, 1) \to [0, 1)$ by $T_k 0 = 0$ and $T_k x = T_k[a_1, a_2, a_3, \ldots]_k = [a_2, a_3, a_4, \ldots]_k$ for $x \neq 0$. One can think of T_k as a shift map, as it shifts the digits of x. There is another way to define T_k for $x \neq 0$. With $\lfloor \cdot \rfloor$ denoting the floor function, set

$$a(x) = \left\lfloor \frac{\log(x^{-1})}{\log k} \right\rfloor, \quad x \neq 0 .$$

Then we have

$$T_k(x) = \frac{1}{k-1}\left(\frac{k^{-a(x)}}{x} - 1\right), \quad x \neq 0 . \tag{3}$$

To get [3], observe that $x = \frac{k^{-a(x)}}{1+(k-1)T_k x}$.

The ergodic properties of these transformations have been studied in [3]. Actually, Chan has obtained the explicit form of the invariant probability density for T_k, $k \geq 3$. It should be said that, given an interval map, in general, it is difficult to obtain the explicit form of its invariant probability density; for some non-trivial examples, see, e.g., [4], [10] and [12]. In [3], it was proved that T_k is ergodic with respect to the measure ν_k defined by

$$\nu_k(A) = c_k \int_A \frac{dx}{((k-1)x+1)((k-1)x+k)} , \quad A \in \mathcal{B}_I , \tag{4}$$

where $\mathcal{B}_I$ is the σ-algebra of Borel subsets of the unit interval $I = [0, 1]$. Here, the normalization constant

$$c_k = (k-1)^2 / \log(k^2/(2k-1))$$

is chosen so that $\nu_k(I) = 1$.

Let us note that ν_k is T_k-invariant, that is, $\nu_k(T_k^{-1}(A)) = \nu_k(A)$ for any $A \in \mathcal{B}_I$.

It should be stressed that the ergodic theorem (see [5], [9] and [11]) does not yield rates of convergence for mixing properties, so that a Gauss-Kuzmin theorem is needed.

Following the treatment in the case of the regular continued fraction (see [7]), the Gauss-Kuzmin-Lévy problem for the transformation T_k, $k \geq 3$, can be approached in terms of the associated Perron-Frobenius operator.

The outline of this paper is as follows. In Section 2 we derive this ope-rator under different probability measures on $\mathcal{B}_I$. We focus our study on the Perron-Frobenius operator of T_k under the invariant measure ν_k induced by the limit distribution function. Let us recall that using well-known general results (see Iosifescu and Grigorescu ([6], pp. 202 and 262–266)), we can derive the asymptotic behaviour of this operator. In Section 3, we use a Wirsing-type approach (see [15]) to get close to the optimal convergence rate. The strategy is to restrict the domain of the Perron-Frobenius operator of T_k under its invariant measure ν_k to the Banach space of functions which have a continuous derivative on I. Actually, in Theorem 1 of Section

3, we obtain upper and lower bounds of convergence rate, respectively $O(w_k^n)$ and $O(v_k^n)$ as $n \to \infty$, with $k \geq 3$, which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem. The last section collects some concluding remarks.

2. The associated Perron-Frobenius operator

Let μ be a probability measure on $\mathcal{B}_I$ such that $\mu(T_k^{-1}(A)) = 0$ whenever $\mu(A) = 0$, $A \in \mathcal{B}_I$, where T_k is defined in (3). In particular, this condition is satisfied if T_k is μ-preserving, that is, $\mu T_k^{-1} = \mu$. It is known, see [7, Section 2.1], that the Perron-Frobenius operator P_μ of T_k under μ is defined as the bounded linear operator on $L_\mu^1 = \{f : I \to \mathbf{C} | \int_I |f| d\mu < \infty\}$ which takes $f \in L_\mu^1$ into $P_\mu f \in L_\mu^1$ with

$$\int_A P_\mu f d\mu = \int_{T_k^{-1}(A)} f d\mu, \quad A \in \mathcal{B}_I .$$

In particular, the Perron-Frobenius operator P_λ of T_k under the Lebesgue measure λ is given by

$$P_\lambda(x) = \frac{d}{dx} \int_{T_k^{-1}([0,x])} f d\lambda \quad \text{a.e. in } I .$$

PROPOSITION 1. *The Perron-Frobenius operator $P_{\nu_k} = U_k$ of T_k under ν_k is given a.e. in I by the equation*

$$U_k f(x) = \sum_{i \in \mathbf{N}} p_k^i(x) f(u_k^i(x)), \quad f \in L_{\nu_k}^1, \quad k \in \mathbf{N}, \quad k \geq 2, \tag{5}$$

where

$$p_k^i(x) = \frac{\gamma^{i+1}(k-1)((k-1)x+1)((k-1)x+k)}{((k-1)x+(k-1)\gamma^i+1)((k-1)x+(k-1)\gamma^{i+1}+1)},$$
$$u_k^i(x) = \frac{\gamma^i}{(k-1)x+1}, \quad i \in \mathbf{N}, \quad x \in I, \tag{6}$$

with $\gamma = 1/k$.

PROOF. Let $T_k^i : I_i \to I$ denote the restriction of T_k to the interval $I_i = (k^{-i-1}, k^{-i}]$, $i \in \mathbf{N}$, that is,

$$T_k^i(u) = \frac{1}{k-1}\left(\frac{k^{-i}}{u} - 1\right), \quad u \in I_i .$$

For any $f \in L_{\nu_k}^1$ and any $A \in \mathcal{B}_I$ we have

$$\int_{T_k^{-1}(A)} f d\nu_k = \sum_{i \in \mathbf{N}} \int_{T_k^{-1}(A \cap I_i)} f d\nu_k = \sum_{i \in \mathbf{N}} \int_{(T_k^i)^{-1}(A)} f d\nu_k . \tag{7}$$

For any $i \in \mathbf{N}$, by the change of variable

$$x = (T_k^i)^{-1}(y) = \frac{k^{-i}}{(k-1)y+1}$$

we successively get

$$\begin{aligned}\int_{(T_k^i)^{-1}(A)} f d\nu_k &= c_k \int_{(T_k^i)^{-1}(A)} f(x) \frac{dx}{((k-1)x+1)((k-1)x+k)} \\ &= \int_A f\left(\frac{k^{-i}}{(k-1)y+1}\right) \\ &\quad \times \frac{k^{-i}(k-1)((k-1)y+1)((k-1)y+k)}{((k-1)y+(k-1)k^{-i}+1)(k((k-1)y+1)+(k-1)k^{-i})} \nu_k(dy) \\ &= \int_A f(u_k^i(y)) p_k^i(y) \nu_k(dy) .\end{aligned} \tag{8}$$

Now, (5) follows from (7) and (8). □

PROPOSITION 2. *Let μ be a probability measure on $\mathcal{B}_I$. Assume that $\mu \ll \lambda$ and let $h = d\mu/d\lambda$. Then*

$$\mu(T_k^{-n}(A)) = \int_A \frac{U_k^n f_k(x)}{((k-1)x+1)((k-1)x+k)} dx \tag{9}$$

for any $n \in \mathbf{N}$, $k \in \mathbf{N}$, $k \geq 2$ and $A \in \mathcal{B}_I$, where

$$f_k(x) = ((k-1)x+1)((k-1)x+k)h(x) , \quad x \in I .$$

PROOF. For $n = 0$, equation (9) reduces to

$$\mu(A) = \int_A h(x)dx , \quad A \in \mathcal{B}_I ,$$

which is obviously true. Assume that (9) holds for some $n \in \mathbf{N}$. Then

$$\begin{aligned}\mu(T_k^{-(n+1)}(A)) &= \mu(T_k^{-n}(T_k^{-1}(A)) \\ &= \int_{T_k^{-1}(A)} \frac{U_k^n f_k(x)}{((k-1)x+1)((k-1)x+k)} dx = \frac{1}{c_k} \int_{T_k^{-1}(A)} U_k^n f_k d(\nu_k) .\end{aligned}$$

By the very definition of the Perron-Frobenius operator U_k we have

$$\int_{T_k^{-1}(A)} U_k^n f_k d\nu_k = \int_A U_k^{n+1} f_k d\nu_k .$$

Therefore,

$$\mu(T_k^{-(n+1)}(A)) = \frac{1}{c_k}\int_A U_k^{n+1} f_k d\nu_k = \int_A \frac{U_k^{n+1} f_k(x)dx}{((k-1)x+1)((k-1)x+k)}$$

and the proof is complete. □

3. A Wirsing-type approach

Let μ be a probability measure on $\mathcal{B}_I$ such that $\mu \ll \lambda$. For any $n \in \mathbf{N}$ put

$$F_k^n(x) = \mu(T_k^n < x)\,, \quad x \in I\,,$$

where T_k^0 is the identity map. As $(T_k^n < x) = T_k^{-n}((0,x))$, by Proposition 2 we have

$$F_k^n(x) = \int_0^x \frac{U_k^n f_k^0(u)}{((k-1)u+1)((k-1)u+k)} du\,, \tag{10}$$

with $f_k^0(x) = ((k-1)x+1)((k-1)x+k)(F^0)'(x)$, $x \in I$, where $(F^0)' = d\mu/d\lambda$.

In this section we will assume that $(F^0)' \in C^1(I)$. So, we study the behaviour of U_k^n as $n \to \infty$, assuming that the domain of U_k is $C^1(I)$, the collection of all functions $f : I \to \mathbf{C}$ which have a continuous derivative.

Let $f \in C^1(I)$. Then the series (5) can be differentiated term-by-term, since the series of derivatives is uniformly convergent. Putting $\Delta_i = \gamma^i - \gamma^{2i}$, $i \in \mathbf{N}$, we get

$$p_k^i(x) = (k-1)\left[\gamma^{i+1} + \frac{\Delta_i}{(k-1)x+(k-1)\gamma^i+1} - \frac{\Delta_{i+1}}{(k-1)x+(k-1)\gamma^{i+1}+1}\right],$$

$$\begin{aligned}(U_k f)'(x) &= \sum_{i\in\mathbf{N}}\left[(p_k^i)'(x) f(u_k^i(x)) - p_k^i(x)\frac{\gamma^i(k-1)}{((k-1)x+1)^2} f'(u_k^i(x))\right]\\
&= (k-1)\sum_{i\in\mathbf{N}}\left[(k-1)\left(\frac{\Delta_i+1}{((k-1)x+(k-1)\gamma^{i+1}+1)^2}\right.\right.\\
&\qquad \left. - \frac{\Delta_i}{((k-1)x+(k-1)\gamma^i+1)^2}\right) f(u_k^i(x))\\
&\qquad \left. - p_k^i(x)\frac{\gamma^i}{((k-1)x+1)^2} f'(u_k^i(x))\right] \qquad (11)\\
&= -(k-1)\sum_{i\in\mathbf{N}}\left[\frac{(k-1)\Delta_{i+1}}{((k-1)x+(k-1)\gamma^{i+1}+1)^2}\right.\end{aligned}$$

$$\times\left(f\left(\frac{\gamma^{i+1}}{(k-1)x+1}\right)-f\left(\frac{\gamma^{i}}{(k-1)x+1}\right)\right)$$
$$+\,p_k^i(x)\frac{\gamma^i}{((k-1)x+1)^2}f'(u_k^i(x))\Bigg],\quad x\in I\,.$$

Thus, we can write

$$(U_k f)'(x) = -V_k f'\,,\quad f\in C^1(I)\,,$$

where $V_k : C(I)\to C(I)$ is defined by

$$V_k g(x)=\sum_{i\in\mathbf{N}}\left(\frac{(k-1)^2\Delta_{i+1}}{((k-1)x+(k-1)\gamma^{i+1}+1)^2}\int_{\frac{\gamma^i}{(k-1)x+1}}^{\frac{\gamma^{i+1}}{(k-1)x+1}} g(u)du\right.$$
$$\left.+\,p_k^i(x)\frac{\gamma^i(k-1)}{((k-1)x+1)^2}g\left(\frac{\gamma^i}{(k-1)x+1}\right)\right),\quad g\in C(I)\,,\quad x\in I\,.$$

Clearly,

$$(U_k^n f)' = (-1)^n V_k^n f'\,,\quad n\in\mathbf{N}_+\,,\quad f\in C^1(I)\,. \tag{12}$$

We are going to show that V_k^n takes certain functions into functions with very small values when $n\in\mathbf{N}_+$ is large.

PROPOSITION 3. *There are positive constants* $v_k < w_k < 1$ *and a real-valued function* $\varphi_k\in C(I)$ *such that*

$$v_k\varphi_k \le V_k\varphi_k \le w_k\varphi_k\,,\quad k\in\mathbf{N}\,,\quad k\ge 2\,.$$

PROOF. Let $h_k:\mathbf{R}_+\to\mathbf{R}$, with $k\in\mathbf{N}$, $k\ge 2$, be a continuous bounded function such that $\lim_{x\to\infty} h_k(x)<\infty$. We look for a function $g_k:(0,1]\to\mathbf{R}$ such that $U_k g_k = h_k$, assuming that the equation

$$U_k g_k(x)=\sum_{i\in\mathbf{N}} p_k^i(x) g_k\left(\frac{\gamma^i}{(k-1)x+1}\right)=h_k(x) \tag{13}$$

holds for $x\in\mathbf{R}_+$. Then (13) yields

$$\frac{h_k(x)}{(k-1)x+k}-\frac{h_k(kx+1)}{k(k-1)x+2k-1}$$
$$=\frac{(k-1)((k-1)x+1)}{((k-1)x+k)(k(k-1)x+2k-1)}g_k\left(\frac{1}{(k-1)x+1}\right),\quad x\in\mathbf{R}_+\,.$$

Hence

$$g_k(u)=\frac{1}{k-1}\left[\left(k\left(\frac{1}{u}-1\right)+2k-1\right)h_k\left(\frac{1}{k-1}\left(\frac{1}{u}-1\right)\right)\right.$$

$$-\left(\frac{1}{u}+k-1\right)h_k\left(\frac{k}{k-1}\left(\frac{1}{u}-1\right)+1\right)\Bigg], \quad u\in(0,1],$$

and we indeed have $U_k g_k = h_k$ since

$$\begin{aligned}
U_k g_k(x) &= \sum_{i\in\mathbf{N}} \frac{p_k^i(x)\gamma^i}{(k-1)((k-1)x+1)} \\
&\quad\times\left[\left(k\left(\frac{(k-1)x+1}{\gamma^i}-1\right)+2k-1\right)h_k\left(\frac{1}{k-1}\left(\frac{(k-1)x+1}{\gamma^i}-1\right)\right)\right.\\
&\quad\left.-\left(\frac{(k-1)x+1}{\gamma^i}+k-1\right)h_k\left(\frac{k}{k-1}\left(\frac{(k-1)x+1}{\gamma^i}-1\right)+1\right)\right]\\
&= \frac{(k-1)x+k}{k}\sum_{i\in\mathbf{N}}\frac{\gamma^{2i}}{((k-1)x+(k-1)\gamma^i+1)((k-1)x+(k-1)\gamma^{i+1}+1)}\\
&\quad\times\left[\left(\frac{(k-1)x+1}{\gamma^{i+1}}+k-1\right)h_k\left(\frac{1}{k-1}\left(\frac{(k-1)x+1}{\gamma^i}-1\right)\right)\right.\\
&\quad\left.-\left(\frac{(k-1)x+1}{\gamma^i}+k-1\right)h_k\left(\frac{1}{k-1}\left(\frac{(k-1)x+1}{\gamma^{i+1}}-1\right)\right)\right]\\
&= h_k(x), \quad x\in\mathbf{R}_+ .
\end{aligned}$$

In particular, for any fixed $a_k \in I$ we consider the function $h_{a_k} : \mathbf{R}_+ \to \mathbf{R}$ defined by

$$h_{a_k}(x) = \frac{1}{e_k x + a_k + 1}, \quad x \in \mathbf{R}_+,$$

where the coefficient e_k will be specified later. By the above, the function $g_{a_k} : (0,1] \to \mathbf{R}$ defined as

$$\begin{aligned}
g_{a_k}(x) &= \frac{x}{k-1}\left[\left(k\left(\frac{1}{x}-1\right)+2k-1\right)h_{a_k}\left(\frac{1}{k-1}\left(\frac{1}{x}-1\right)\right)\right.\\
&\quad\left.-\left(\frac{1}{x}+k-1\right)h_{a_k}\left(\frac{k}{k-1}\left(\frac{1}{x}-1\right)+1\right)\right]\\
&= (k-1)\frac{[(k-1)(a_k+1)+(k-2)e_k]x^2+(k+1)e_k x}{[((k-1)(a_k+1)-e_k)x+e_k][((k-1)(e_k+a_k+1)-ke_k)x+ke_k]},\\
&\quad x\in(0,1],
\end{aligned}$$

satisfies

$$U_k g_{a_k}(x) = h_{a_k}(x), \quad x \in I.$$

Setting

$$\varphi_{a_k}(x) = g'_{a_k}(x) = (k-1)e_k^2$$

$$\times \frac{(k^2-1)[(k-1)(a_k+1)-e_k]x^2+2k[(k-1)(a_k+1)+(k-2)e_k]x+k(k+1)e_k}{[((k-1)(a_k+1)-e_k)x+e_k]^2[((k-1)(e_k+a_k+1)-ke_k)x+ke_k]^2}$$

we have

$$V_k\varphi_{a_k}(x) = -(U_k g_{a_k})'(x) = \frac{e_k}{(e_k x + a_k + 1)^2}, \quad x \in I.$$

We choose a_k by asking that

$$(\varphi_{a_k}/V_k\varphi_{a_k})(0) = (\varphi_{a_k}/V_k\varphi_{a_k})(1).$$

Since

$$(\varphi_{a_k}/V_k\varphi_{a_k})(0) = \frac{(k^2-1)(a_k+1)^2}{ke_k^2}$$

and

$$(\varphi_{a_k}/V_k\varphi_{a_k})(1) = \frac{e_k}{(k-1)^2(a_k+1)^2}[(k^2+2k-1)(a_k+1)+(2k-1)e_k],$$

this amounts to the equation

$$E_k(a_k) = (k+1)(k-1)^3(a_k+1)^4 - k(k^2+2k-1)e_k^3(a_k+1) - k(2k-1)e_k^4 = 0.$$

We choose the coefficient e_k such that the equation $E_k(x) = 0$, $x \in I$, yields a unique solution $a_k \in I$. Asking that

$$E_k(0) < 0, \; E_k(1) > 0, \quad \text{and} \quad \frac{dE_k}{da_k} > 0, \quad k \geq 3,$$

we may take $e_k = \sqrt[3]{k}$. For this unique acceptable solution $a_k \in I$, the function $\varphi_{a_k}/V\varphi_{a_k}$ attains its maximum equal to $\frac{(k^2-1)(a_k+1)^2}{ke_k^2}$ at $x = 0$ and $x = 1$, and has a minimum $m(a_k) = (\varphi_{a_k}/V\varphi_{a_k})(x^k_{\min}) > 1$. It follows that for $\varphi_k = \varphi_{a_k}$ we have

$$\frac{ke_k^2\varphi_k}{(k^2-1)(a_k+1)^2} \leq V_k\varphi_k \leq \frac{\varphi_k}{m(a_k)},$$

that is, $v_k\varphi_k \leq V_k\varphi_k \leq w_k\varphi_k$, where

$$v_k = \frac{ke_k^2}{(k^2-1)(a_k+1)^2} \quad \text{and} \quad w_k = \frac{1}{m(a_k)}.$$

□

COROLLARY 1. *Let $f_k^0 \in C^1(I)$ such that $(f_k^0)' > 0$. Put $\alpha_k = \min_{x\in I} \varphi_k(x)/(f_k^0)'(x)$ and $\beta_k = \max_{x\in I} \varphi_k(x)/(f_k^0)'(x)$. Then*

$$\frac{\alpha_k}{\beta_k} v_k^n (f_k^0)' \leq V_k^n (f_k^0)' \leq \frac{\beta_k}{\alpha_k} w_k^n (f_k^0)', \quad n \in \mathbf{N}_+. \tag{14}$$

PROOF. Since V_k is a positive operator, we have

$$v_k^n \varphi_k \le V_k^n \varphi_k \le w_k^n \varphi_k\,, \quad n \in \mathbf{N}_+\,.$$

Noting that $\alpha_k (f_k^0)' \le \varphi_k \le \beta_k (f_k^0)'$, we can write

$$\frac{\alpha_k}{\beta_k} v_k^n (f_k^0)' \le \frac{1}{\beta_k} v_k^n \varphi_k \le \frac{1}{\beta_k} V_k^n \varphi_k \le V_k^n (f_k^0)' \le \frac{1}{\alpha_k} V_k^n \varphi_k \le$$

$$\le \frac{1}{\alpha_k} w_k^n \varphi_k \le \frac{\beta_k}{\alpha_k} w_k^n (f_k^0)', \quad n \in \mathbf{N}_+\,,$$

which shows that (14) holds. □

THEOREM 1 (Near-optimal solution to Gauss-Kuzmin-Lévy problem). *Let $f_k^0 \in C^1(I)$ such that $(f_k^0)' > 0$ and let μ be a probability measure on $\mathcal{B}_I$ such that $\mu \ll \lambda$. For any $n \in \mathbf{N}_+$ and $x \in I$ we have*

$$\frac{k\alpha_k \min_{x\in I}(f_k^0)'(x)}{2\beta_k c_k^2} v_k^n G_k(x)(1 - G_k(x))$$

$$\le |\mu(T_k^n < x) - G_k(x)| \le \frac{k(2k-1)\beta_k \max_{x\in I}(f_k^0)'(x)}{2\alpha_k c_k^2} w_k^n G_k(x)(1 - G_k(x))$$

where α_k, β_k, v_k and w_k are defined in Proposition 3 *and Corollary* 1, *and*

$$G_k(x) = \frac{c_k}{(k-1)^2} \log\left(\frac{k((k-1)x+1)}{(k-1)x+k}\right).$$

PROOF. For any $n \in \mathbf{N}$ and $x \in I$ set $d_n(G_k(x)) = \mu(T_k^n < x) - G_k(x)$. Then by (10) we have

$$d_n(G_k(x)) = \int_0^x \frac{U_k^n f_k^0(u)}{((k-1)u+1)((k-1)u+k)} du - G_k(x)\,.$$

Differentiating twice with respect to x yields

$$d_n'(G(x)) \frac{c_k}{((k-1)x+1)((k-1)x+k)} = \frac{U_k^n f_k^0(x)}{((k-1)x+1)((k-1)x+k)}$$
$$- \frac{c_k}{((k-1)x+1)((k-1)x+k)}\,,$$

$$(U_k^n f_k^0(x))' = c_k^2 \frac{d_n''(G_k(x))}{((k-1)x+1)((k-1)x+k)}\,, \quad n \in \mathbf{N}\,, \quad x \in I\,.$$

Hence by (12) we have

$$d_n''(G_k(x)) = \frac{(-1)^n((k-1)x+1)((k-1)x+k)}{c_k^2} V_k^n (f_k^0)'(x)\,, \quad n \in \mathbf{N}\,, \quad x \in I\,.$$

Since $d_n(0) = d_n(1) = 0$, a well-known interpolation formula yields

$$d_n(x) = -\frac{x(1-x)}{2} d_n''(\theta)\,, \quad n \in \mathbf{N}\,, \quad x \in I\,,$$

for a suitable $\theta = \theta(n, x) \in I$. Therefore

$$\begin{aligned}&\mu(T_k^n < x) - G_k(x)\\ &\quad = \frac{(-1)^{n+1}}{c_k^2}((k-1)\theta_k + 1)((k-1)\theta_k + k) V_k^n (f_k^0)'(\theta_k) \frac{G_k(x)(1-G_k(x))}{2}\end{aligned}$$

for any $n \in \mathbf{N}$ and $x \in I$, and another suitable $\theta_k = \theta_k(n, x) \in I$. The result stated follows now from Corollary 1. □

4. Final remarks

Let us consider the case $k = 3$. The equation $E_3(x) = 0$, with $e_3 = \sqrt[3]{3} = 1.44224957$, has as unique acceptable solution $a = a_3 = 0.722946965$. For this value of a the function $\varphi_a / V\varphi_a$ attains its maximum equal to 3.805675163 at $x = 0$ and $x = 1$, and has a minimum $m(a) = (\varphi_a/V\varphi_a)(0.023133079) = 3.77804431$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_3^n)$ and $O(v_3^n)$ as $n \to \infty$, with $v_3 > 0.262765464$ and $w_3 < 0.264687208$.

Finally, let us consider the case $k = 5$. The equation $E_5(x) = 0$, with $e_5 = \sqrt[3]{5} = 1.709975947$, has as unique acceptable solution $a = a_5 = 0.428487617$. For this value of a the function $\varphi_a / V\varphi_a$ attains its maximum equal to 3.349763881 at $x = 0$ and $x = 1$ and has a minimum $m(a) = (\varphi_a/V\varphi_a)(0.008438422) = 3.31939294$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_5^n)$ and $O(v_5^n)$ as $n \to \infty$, with $v_5 > 0.298528504$ and $w_5 < 0.301259904$.

To conclude, the determination of the exact convergence rate remains an open question. We may derive it using the same strategy as in [8] and [13] for the case $k = 2$.

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Present Address:
DEPARTMENT OF MATHEMATICS I,
"POLITEHNICA" UNIVERSITY OF BUCHAREST,
SPLAIUL INDEPENDENŢEI 313,
060042 BUCHAREST, ROMANIA.
e-mail: igsebe@yahoo.com