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Classes of Infinitely Divisible Distributions on $\mathbf{R}^d$ Related to the Class of Selfdecomposable Distributions

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Abstract. This paper studies new classes of infinitely divisible distributions on $\mathbf{R}^d$. Firstly, the connecting classes with a continuous parameter between the Jurek class and the class of selfdecomposable distributions are revisited. Secondly, the range of the parameter is extended to construct new classes and characterizationreds in terms of stochastic integrals with respect to Lévy processes are given. Finally, the nested subclasses of those classes are discussed and characterized in two ways: One is by stochastic integral representations and another is in terms of Lévy measures.

1. Introduction

Let $I(\mathbf{R}^d)$ be the class of all infinitely divisible distributions on $\mathbf{R}^d$ and $I_{\log}(\mathbf{R}^d) = \{\mu \in I(\mathbf{R}^d) : \int_{|x|>1} \log|x|\mu(dx) < \infty\}$. Let $\widehat{\mu}(z)$, $z \in \mathbf{R}^d$, be the characteristic function of $\mu \in I(\mathbf{R}^d)$.

In this paper, we first revisit the classes in $I(\mathbf{R}^d)$ connecting the class of selfdecomposable distributions ($L(\mathbf{R}^d)$, say) and the Jurek class (the class of s-selfdecomposable distributions, ($U(\mathbf{R}^d)$, say), see Jurek (1985)). Those connecting classes were already studied by O'Connor (1979) in $I(\mathbf{R}^1)$ and by Jurek (1988) in $I(E)$, where E is a Banach space. Throughout this paper, we treat the case $I(\mathbf{R}^d)$. Although there are several equivalent definitions of $L(\mathbf{R}^d)$ and $U(\mathbf{R}^d)$, we use here their definitions in terms of Lévy measures. Then we study more general classes including the classes above and nested subclasses of those classes.

The Lévy-Khintchine representation of $\widehat{\mu}$ we use in this paper is

$$\widehat{\mu}(z) = \exp\left\{-2^{-1}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbf{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\},$$

where A is a nonnegative-definite symmetric $d\times d$ matrix, $\gamma \in \mathbf{R}^d$ and ν is the Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbf{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty$. We call (A, ν, γ) the Lévy-Khintchine triplet of μ and we write $\mu = \mu_{(A,\nu,\gamma)}$ when we want to emphasize the triplet.

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The polar decomposition of the Lévy measure ν of $\mu \in I(\mathbf{R}^d)$, with $0 < \nu(\mathbf{R}^d) \leq \infty$, is the following: There exist a measure λ on $S = \{\xi \in \mathbf{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \leq \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0, \infty))$, $0 < \nu_\xi((0, \infty)) \leq \infty$ for each $\xi \in S$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr)\,, \quad B \in \mathcal{B}(\mathbf{R}^d \setminus \{0\})\,. \tag{1.1}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of a measurable function $c(\xi)$ and $\frac{1}{c(\xi)}$, respectively, with $0 < c(\xi) < \infty$. We say that μ or ν has a polar decomposition (λ, ν_ξ) and ν_ξ is called a radial component of ν. (See, e.g., Barndorff-Nielsen et al. (2006), Lemma 2.1.)

The connecting classes between $U(\mathbf{R}^d)$ and $L(\mathbf{R}^d)$ mentioned above are also characterized by mappings with a parameter from $I(\mathbf{R}^d)$ into $I(\mathbf{R}^d)$. We extend the range of the parameter and first study the classes defined by these mappings. These mappings are the special cases studied in Sato (2006b) as will be mentioned later.

We start with following classes, where the classes $U(\mathbf{R}^d)$ and $L(\mathbf{R}^d)$ are two known special classes.

DEFINITION 1.1 (The class $K_\alpha(\mathbf{R}^d)$). Let $\alpha < 2$. We say that $\mu \in I(\mathbf{R}^d)$ belongs to the class $K_\alpha(\mathbf{R}^d)$ if $\nu = 0$ or $\nu \neq 0$ and, in case $\nu \neq 0$, ν_ξ in (1.1) satisfies

$$\nu_\xi(dr) = r^{-\alpha-1}\ell_\xi(r)dr\,, \quad r > 0\,, \tag{1.2}$$

where $\ell_\xi(r)$ is nonincreasing in $r \in (0, \infty)$ for λ-a.e. ξ and is measurable in ξ for each $r > 0$, and $\lim_{r\to\infty} \ell_\xi(r) = 0$.

REMARK 1.2. (i) Because of the condition that $\lim_{r\to\infty} \ell_\xi(r) = 0$, $K_\alpha(\mathbf{R}^d)$, $0 < \alpha < 2$, does not include the class of α-stable distributions, but does include the class of $(\alpha+\varepsilon)$-stable distributions for any $\varepsilon \in (0, 2 - \alpha)$. It also includes tempering α-stable distributions, which are defined by (1.2) with a completely monotone function $\ell_\xi(r)$ on $(0, \infty)$ such that $\lim_{r\to 0} \ell_\xi(r) = 1$ and $\lim_{r\to\infty} \ell_\xi(r) = 0$. (See Rosiński (2007).)

(ii) Let ν be the Lévy measure of $\mu \in I(\mathbf{R}^d)$ and $\alpha > 0$. Since $\int_{\mathbf{R}^d} |x|^\delta \mu(ds) < \infty$ if and only if $\int_{|x|>1} |x|^\delta \nu(dx) < \infty$, (see, e.g. Sato (1999) Theorem 25.3,) $\mu \in K_\alpha(\mathbf{R}^d)$ has the finite δ-moment for any $0 < \delta < \alpha$. This fact is the same as for α-stable distributions.

REMARK 1.3. (i) The Jurek class $U(\mathbf{R}^d)$ is $K_{-1}(\mathbf{R}^d)$ and the class of selfdecomposable distributions $L(\mathbf{R}^d)$ is $K_0(\mathbf{R}^d)$.

(ii) Let $\alpha < \beta < 2$. Then $K_\beta(\mathbf{R}^d) \subset K_\alpha(\mathbf{R}^d)$. This is trivial from the definition.

Therefore, $K_\alpha(\mathbf{R}^d)$, $-1 \leq \alpha \leq 0$, are connecting classes with a continuous parameter α between the classes $U(\mathbf{R}^d)$ and $L(\mathbf{R}^d)$, as mentioned in the beginning of this section.

This paper is organized as follows. In Section 2, some known results related to the classes $K_\alpha(\mathbf{R}^d)$ are mentioned. In Section 3, we give a complete proof for the decomposability of the distributions in $K_\alpha(\mathbf{R}^d)$, $\alpha < 0$. In Section 4, we define mappings $\Phi_\alpha, \alpha \in \mathbf{R}$, in terms of stochastic integrals with respect to Lévy processes related to the classes $K_\alpha(\mathbf{R}^d)$ and determine those domains and ranges. The proofs for the ranges are given in Section 5. In Section 6, we construct nested subclasses of the ranges of Φ_α by iterating the mapping Φ_α. Then we firstly determine the domains $\mathfrak{D}(\Phi_\alpha^{m+1})$, $m = 1, 2, 3, \ldots$, and secondly characterize the ranges of the mappings Φ_α^{m+1} in two ways: One is by stochastic integral representations and another is in terms of Lévy measures.

2. Known results

In this section, we explain several results from O'Connor (1979) and Jurek (1988).

1. (Characterization by the decomposability)

O'Connor (1979) defined the classes $K_\alpha(\mathbf{R}^1)$, $-1 < \alpha < 0$, as in Definition 1.1, and proved that $\mu \in K_\alpha(\mathbf{R}^1)$ if and only if for any $c \in (0, 1)$ there exists $\mu_c \in I(\mathbf{R}^1)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(cz)^{c^{-\alpha}} \widehat{\mu}_c(z)\,. \tag{2.1}$$

His proof used Lévy measures. However, his proof for getting the convexity of Lévy density on $(-\infty, 0)$ and the concavity on $(0, \infty)$ (in the proof of his Theorem 3 in O'Connor (1979)) is not clear to the authors of this paper. So, we will give our proof in Section 3, extending the range of α to $(-\infty, 0)$.

Jurek (1988) defined the classes $U_\alpha(E)$, $-1 \le \alpha \le 0$, where E is a Banach space, as the classes of limiting distributions as follows. $\mu \in U_\alpha(E)$ if and only if there exists a sequence $\{\mu_j\} \subset I(E)$ such that

$$\lim_{n\to\infty} n^{-1}(\mu_1 * \mu_2 * \cdots * \mu_n)^{*n^\alpha} = \mu\,. \tag{2.2}$$

He then showed the decomposability (2.1) as a consequence of (2.2). So, as a result, we see that $K_\alpha(\mathbf{R}^d) = U_\alpha(\mathbf{R}^d)$, but there is no proof by using Lévy measures in Jurek (1988). This is another reason why we will give our proof in Section 3. Our proof will use Lévy measures in the same way as in the proof of Theorem 15.10 of Sato (1999) for selfdecomposability.

2. (Characterization by the stochastic integrals with respect to Lévy processes)

Let $-1 \le \alpha < 0$. Jurek (1988) showed that $\mu \in U_\alpha(E)$ if and only if there exists a Lévy process $\{X_t\}$ on E such that

$$\mu = \mathcal{L}\left(\int_0^1 t^{-1/\alpha} dX_t\right), \tag{2.3}$$

where $\mathcal{L}(X)$ is the law of a random variable X. For the case $\alpha = 0$, the following is known (Wolfe (1972) and others). $\mu \in K_0(\mathbf{R}^d)$ if and only if there exists a Lévy process $\{X_t\}$ on $\mathbf{R}^d$ satisfying $E[\log^+ |X_1|] < \infty$ such that

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right),$$

where $\log^+ |x| = (\log |x| \vee 0)$.

REMARK 2.1. (2.3) can have a different form. Change the variables from t to s by $t = 1 + \alpha s$. Then

$$\mu = \mathcal{L}\left(- \int_0^{-1/\alpha} (1 + \alpha s)^{-1/\alpha} dX_{1+\alpha s} \right).$$

If we define another Lévy process $\{\widetilde{X}_t\}$ by $\widetilde{X}_s = -X_{1+\alpha s}$, then we have

$$\mu = \mathcal{L}\left(\int_0^{-1/\alpha} (1 + \alpha s)^{-1/\alpha} d\widetilde{X}_s \right). \tag{2.4}$$

(2.4) will be seen in Definition 4.1 with $\alpha < 0$ below and this expression is more natural when we consider the case $\alpha = 0$ as we will see in Remark 4.7 later.

3. Decomposability of distributions in $K_\alpha(\mathbf{R}^d)$, $\alpha < 0$

As mentioned before, the classes $U(\mathbf{R}^d)$ and $L(\mathbf{R}^d)$ have characterizations in terms of characteristic functions. Namely, $\mu \in U(\mathbf{R}^d)$ if and only if for any $c \in (0, 1)$, there exists $\mu_c(z) \in I(\mathbf{R}^d)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(cz)^c \widehat{\mu}_c(z)\,,$$

and $\mu \in L(\mathbf{R}^d)$ if and only if for any $c \in (0, 1)$, there exists $\mu_c(z) \in I(\mathbf{R}^d)$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(cz)\widehat{\mu}_c(z)\,.$$

As we announced in Section 2, we give our proof of characterization of $K_\alpha(\mathbf{R}^d)$, $\alpha < 0$, in a similar way as follows.

THEOREM 3.1. *Let $\alpha < 0$. $\mu \in K_\alpha(\mathbf{R}^d)$ if and only if for any $c \in (0, 1)$, there exists $\mu_c \in I(\mathbf{R}^d)$ such that*

$$\widehat{\mu}(z) = \widehat{\mu}(cz)^{c^{-\alpha}} \widehat{\mu}_c(z)\,.$$

PROOF. (The "only if" part) It is enough to consider the case with $A = O$ and $\gamma = 0$. Suppose $\mu \in K_\alpha(\mathbf{R}^d)$ and the polar decomposition of the Lévy measure of μ is (λ, ν_ξ), with $\nu_\xi(dr) = r^{-\alpha-1}\ell_\xi(r)dr$. Then we have

$$\widehat{\mu}(z) = \exp\left\{ \int_S \lambda(d\xi) \int_0^\infty \left(e^{i\langle z, r\xi\rangle} - 1 - \frac{i\langle z, r\xi\rangle}{1 + r^2} \right) \frac{1}{r^{\alpha+1}} \ell_\xi(r)dr \right\}.$$

Thus,

$$\widehat{\mu}(cz)^{c^{-\alpha}} = \exp\left\{ c^{-\alpha} \int_S \lambda(d\xi) \int_0^\infty \left(e^{i\langle z, cr\xi\rangle} - 1 - \frac{i\langle z, cr\xi\rangle}{1 + r^2} \right) \frac{1}{r^{\alpha+1}} \ell_\xi(r)dr \right\}$$

$$= \exp\left\{\int_S \lambda(d\xi)\int_0^\infty \left(e^{i\langle z,u\xi\rangle} - 1 - \frac{i\langle z,u\xi\rangle}{1+(u/c)^2}\right)\frac{1}{u^{\alpha+1}}\ell_\xi\left(\frac{u}{c}\right)du\right\}$$

$$= \widehat{\mu}(z)\exp\left\{\int_S \lambda(d\xi)\int_0^\infty i\langle z,u\xi\rangle\left(\frac{1}{1+u^2} - \frac{1}{1+(u/c)^2}\right)\frac{1}{u^{\alpha+1}}\ell_\xi\left(\frac{u}{c}\right)du\right\}$$

$$\times \exp\left\{-\int_S \lambda(d\xi)\int_0^\infty \left(e^{i\langle z,u\xi\rangle} - 1 - \frac{i\langle z,u\xi\rangle}{1+u^2}\right)\frac{1}{u^{\alpha+1}}\left(\ell_\xi(u) - \ell_\xi\left(\frac{u}{c}\right)\right)du\right\}$$

$$=: \widehat{\mu}(z)e^{i\langle z,a_c\rangle}(\widehat{\rho_c}(z))^{-1},$$

where

$$a_c = \int_S \xi\lambda(d\xi)\int_0^\infty u^{-\alpha}\left(\frac{1}{1+u^2} - \frac{1}{1+(u/c)^2}\right)\ell_\xi\left(\frac{u}{c}\right)du$$

and

$$\widehat{\rho_c}(z) = \exp\left\{\int_S \lambda(d\xi)\int_0^\infty \left(e^{i\langle z,u\xi\rangle} - 1 - \frac{i\langle z,u\xi\rangle}{1+u^2}\right)\frac{1}{u^{\alpha+1}}\left(\ell_\xi(u) - \ell_\xi\left(\frac{u}{c}\right)\right)du\right\}.$$

We have to check the finiteness of a_c and that $\rho_c \in I(\mathbf{R}^d)$.

Since ν is a Lévy measure, we have $\int_S \lambda(d\xi)\int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) < \infty$, which implies

$$\int_S \lambda(d\xi)\int_0^1 r^{-\alpha+1}\ell_\xi(r)dr < \infty \text{ and } \int_S \lambda(d\xi)\int_1^\infty r^{-\alpha-1}\ell_\xi(r)dr < \infty.$$

Furthermore, this concludes

$$|a_c| \le \int_S |\xi|\lambda(d\xi)\int_0^\infty u^{-\alpha}\left|\frac{1}{1+u^2} - \frac{1}{1+(u/c)^2}\right|\ell_\xi(u/c)du$$

$$= \int_S \lambda(d\xi)\int_0^\infty c^{1-\alpha}v^{-\alpha}\left|\frac{1}{1+(cv)^2} - \frac{1}{1+v^2}\right|\ell_\xi(v)dv$$

$$= \int_S \lambda(d\xi)\int_0^\infty \frac{c^{1-\alpha}v^{1-\alpha}}{(1+(cv)^2)}\ell_\xi(v)dv$$

$$\le c^{1-\alpha}\int_S \lambda(d\xi)\int_0^1 v^{1-\alpha}\ell_\xi(v)dv + c^{1-\alpha}\int_S \lambda(d\xi)\int_1^\infty \frac{v^{1-\alpha}}{1+(cv)^2}\ell_\xi(v)dv < \infty.$$

This shows the finiteness of a_c.

With respect to ρ_c, since $0 < c < 1$ and ℓ_ξ is nonincreasing, we have $h_\xi(u) := u^{-\alpha-1}(\ell_\xi(u) - \ell_\xi(u/c)) \ge 0$. Thus, $\nu_\rho(B) = \int_S \lambda(d\xi)\int_0^\infty 1_B(r\xi)h_\xi(r)dr$ is a nonnegative measure. Furthermore, we have

$$\int_{\mathbf{R}^d}(r^2 \wedge 1)\nu_\rho(dr) = \int_S \lambda(d\xi)\int_0^\infty (r^2 \wedge 1)r^{-\alpha-1}(\ell_\xi(r) - \ell_\xi(r/c))dr < \infty,$$

because $\int_S \lambda(d\xi)\int_0^\infty (r^2 \wedge 1)r^{-\alpha-1}\ell_\xi(r)dr < \infty$. Therefore, ν_ρ is a Lévy measure, and

$\rho_c \in I(\mathbf{R}^d)$ by the uniqueness of Lévy-Khintchine representation. Thus, if we put $\widehat{\mu}_c(z) = \widehat{\rho}_c(z)e^{-i\langle z,a_c\rangle}$, we have

$$\widehat{\mu}(z) = \widehat{\mu}(cz)^{c^{-\alpha}}\widehat{\mu}_c(z)\,.$$

The "only if" part is now proved.

(The "if" part) Conversely, suppose that $\mu \in I(\mathbf{R}^d)$ satisfies that for any $c \in (0,1)$, there exists $\mu_c(z) \in I(\mathbf{R}^d)$ such that $\widehat{\mu}(z) = \widehat{\mu}(cz)^{c^{-\alpha}}\widehat{\mu}_c(z)$. Since

$$\widehat{\mu}(z) = \exp\left\{-2^{-1}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbf{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\},$$

we have

$$\begin{aligned}\widehat{\mu}(cz)^{c^{-\alpha}} &= \exp\left\{-2^{-1}c^{-\alpha}\langle cz, Acz\rangle + ic^{-\alpha}\langle \gamma, cz\rangle \right.\\ &\qquad \left. + \int_{\mathbf{R}^d}\left(e^{i\langle cz,x\rangle} - 1 - \frac{i\langle cz, x\rangle}{1+|x|^2}\right)c^{-\alpha}\nu(dx)\right\}\\ &= \exp\left\{-2^{-1}\langle z, c^{2-\alpha}Az\rangle + i\langle c^{1-\alpha}\gamma, z\rangle \right.\\ &\qquad + \int_{\mathbf{R}^d}\left(e^{i\langle z,y\rangle} - 1 - \frac{i\langle z, y\rangle}{1+|y|^2}\right)c^{-\alpha}\nu\left(\frac{dy}{c}\right)\\ &\qquad \left. + i\left\langle z, \int_{\mathbf{R}^d} y\left(\frac{1}{1+|y|^2} - \frac{1}{1+|y/c|^2}\right)c^{-\alpha}\nu\left(\frac{dy}{c}\right)\right\rangle\right\}.\end{aligned}$$

Since $\mu \in I(\mathbf{R}^d)$, $\widehat{\mu}(z) \neq 0$ for any $z \in \mathbf{R}^d$. Then we have

$$\begin{aligned}\widehat{\mu}_c(z) = \exp&\left\{ - 2^{-1}\langle z, A_c z\rangle + i\langle \gamma_c, z\rangle \right.\\ &\left. + \int_{\mathbf{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\left(\nu(dx) - c^{-\alpha}\nu\left(\frac{dx}{c}\right)\right)\right\},\end{aligned}$$

where $A_c = (1 - c^{2-\alpha})A$ and

$$\gamma_c = (1 - c^{1-\alpha})\gamma - \int_{\mathbf{R}^d} y\left(\frac{1}{1+|y|^2} - \frac{1}{1+|y/c|^2}\right)c^{-\alpha}\nu\left(\frac{dy}{c}\right).$$

Since $\mu_c \in I(\mathbf{R}^d)$, $\nu^c(B) := \nu(B) - c^{-\alpha}\nu(c^{-1}B)$ is a Lévy measure for any $c \in (0,1)$. Recall that the polar decomposition of ν is $\nu(B) = \int_S \lambda(d\xi)\int_0^\infty 1_B(r\xi)\nu_\xi(dr)$. Then,

$$\begin{aligned}\nu^c(B) &= \int_S \lambda(d\xi)\int_0^\infty (1_B(r\xi)\nu_\xi(dr) - 1_{c^{-1}B}(r\xi)c^{-\alpha}\nu_\xi(dr))\\ &= \int_S \lambda(d\xi)\int_0^\infty 1_B(r\xi)\left(\nu_\xi(dr) - c^{-\alpha}\nu_\xi\left(\frac{dr}{c}\right)\right).\end{aligned}$$

It remains to show that

$$\nu_\xi(dr) = r^{-\alpha-1}\ell_\xi(r)dr$$

for some nonincreasing function ℓ_ξ measurable in ξ. For that, we consider a measure $r^\alpha \nu_\xi(dr)$ on $(0, \infty)$ and let

$$H_\xi(x) := \int_{e^{-x}}^{\infty} r^\alpha \nu_\xi(dr)\,.$$

Here $H_\xi(x)$ is measurable in ξ. We also put

$$\begin{aligned} H_\xi^c(x) :&= H_\xi(x) - H_\xi(x + \log c) \\ &= \int_{e^{-x}}^{\infty} r^\alpha \nu_\xi(dr) - \int_{e^{-x}/c}^{\infty} r^\alpha \nu_\xi(dr) \\ &= \int_{e^{-x}}^{\infty} r^\alpha \left(\nu_\xi(dr) - c^{-\alpha}\nu_\xi\left(\frac{dr}{c}\right)\right)\,. \end{aligned}$$

Since $\nu^c(dr)$ is a Lévy measure, $H_\xi^c(x)$ is nonnegative and is nondecreasing for $\lambda-$almost every ξ. Moreover, $H_\xi(x)$ is convex on $(-\infty, \infty)$ as shown below.

Let $s \in \mathbf{R}$, $u > 0$ and $c \in (0, 1)$. Then $H_\xi^c(s + u) \geq H_\xi^c(s)$, and thus

$$H_\xi(s + u) - H_\xi(s + u + \log c) \geq H_\xi(s) - H_\xi(s + \log c) \geq 0,$$

which is

$$H_\xi(s + u) - H_\xi(s) \geq H_\xi(s + u + \log c) - H_\xi(s + \log c) \geq 0\,. \tag{3.1}$$

Then H_ξ is convex for $\lambda-$almost every ξ, as in Sato (1999) pp. 95–96. Furthermore, repeating the argument in p.96 of Sato (1999) we can write

$$H_\xi(x) = \int_{-\infty}^{x} h_\xi(t)dt\,,$$

where $h_\xi(t)$ is some left-continuous nondecreasing function in u. Hence $h_x(t)$ is measurable in ξ. Now put

$$H_\xi(-\log x) = \int_{-\infty}^{-\log x} h_\xi(t)dt = \int_{x}^{\infty} h_\xi(-\log r)r^{-1}dr\,,$$

then, the definition of H, we have

$$\int_{x}^{\infty} r^\alpha \nu_\xi(dr) = \int_{x}^{\infty} h_\xi(-\log r)r^{-1}dr\,,$$

which implies

$$\nu_\xi(dr) = r^{-\alpha-1}h_\xi(-\log r)dr\,.$$

Since h_ξ is nondecreasing, we have $h_\xi(-\log r)$ is a nonincreasing function, and putting $\ell_\xi(r) = h_\xi(-\log r)$, we complete the proof. □

4. Mappings defined by stochastic integrals related to $K_\alpha(\mathbf{R}^d)$

We are now going to study mappings defined by the stochastic integrals with respect to Lévy processes related to $K_\alpha(\mathbf{R}^d)$.

Let $\alpha \in \mathbf{R}$ and

$$\varepsilon_\alpha(u) = \int_u^1 x^{-\alpha-1} dx\,, \quad 0 < u \le 1\,, \tag{4.1}$$

Then, when $\alpha \neq 0$,

$$\varepsilon_\alpha(u) = \alpha^{-1}(u^{-\alpha} - 1)\,, \quad 0 < u < 1\,,$$

and when $\alpha = 0$,

$$\varepsilon_0(u) = \log u^{-1}\,, \quad 0 < u < 1\,,$$

Let $\varepsilon_\alpha^*(t)$ be the inverse function of $\varepsilon_\alpha(u)$, that is, $t = \varepsilon_\alpha(u)$ if and only if $u = \varepsilon_\alpha^*(t)$. Note that

$$\varepsilon_\alpha(0) = \begin{cases} (-\alpha)^{-1}\,, & \alpha < 0\,, \\ \infty\,, & \alpha \ge 0\,. \end{cases}$$

Then, when $\alpha \neq 0$,

$$\varepsilon_\alpha^*(t) = \begin{cases} (1+\alpha t)^{-1/\alpha}\,, & 0 < t < \varepsilon_\alpha(0)\,, \\ 0\,, & t \ge \varepsilon_\alpha(0)\,, \end{cases}$$

and when $\alpha = 0$,

$$\varepsilon_0^*(t) = e^{-t}\,, \quad t > 0\,.$$

Let $\{X_t^{(\mu)}\}$ be the Lévy process on $\mathbf{R}^d$ with the distribution $\mu \in I(\mathbf{R}^d)$ at $t = 1$.

DEFINITION 4.1. Let $\alpha \in \mathbf{R}$. We define mappings $\Phi_\alpha : \mathfrak{D}(\Phi_\alpha) \to I(\mathbf{R}^d)$ by

$$\Phi_\alpha(\mu) = \mathcal{L}\left(\int_0^{\varepsilon_\alpha(0)} \varepsilon_\alpha^*(t) dX_t^{(\mu)} \right),$$

where $\mathfrak{D}(\Phi_\alpha)$ is the domain of the mapping Φ_α.

REMARK 4.2. Let $-\infty < \beta < \alpha < \infty$. As in Sato (2006b) write the mapping as

$$\Phi_{\beta,\alpha}(\mu) = \mathcal{L}\left(\int_0^\infty f_{\beta,\alpha}(s) X_s^{(\mu)} \right),$$

where $f_{\beta,\alpha}(s)$ is the inverse function of

$$s = (\Gamma(\alpha - \beta))^{-1} \int_t^1 (1-u)^{\alpha-\beta-1} u^{-\alpha-1} du\,.$$

Our mappings in this paper Φ_α are special cases of $\Phi_{\beta,\alpha}$ with $\beta = \alpha - 1$. Sato (2006b) discussed the domains of $\Phi_{\beta,\alpha}$, but not the ranges of them, and commented that description of the range of $\Phi_{\beta,\alpha}$ is to be made. Our concern here is their ranges, although not for general $\beta < \alpha$, because our motivation of this study started with the classes $K_\alpha(\mathbf{R}^d)$.

Regarding the domains of Φ_α, we have the following result from Theorems 2.4 and 2.8 of Sato (2006b).

PROPOSITION 4.3 (Domains of Φ_α).

(i) *When* $\alpha < 0$, $\mathfrak{D}(\Phi_\alpha) = I(\mathbf{R}^d)$.
(ii) *When* $\alpha = 0$, $\mathfrak{D}(\Phi_\alpha) = I_{\log}(\mathbf{R}^d)$.
(iii) *When* $0 < \alpha < 1$, $\mathfrak{D}(\Phi_\alpha) = \{\mu \in I(\mathbf{R}^d) : \int_{\mathbf{R}^d} |x|^\alpha \mu(dx) < \infty\} =: I_\alpha(\mathbf{R}^d)$.
(iv) *When* $\alpha = 1$, $\mathfrak{D}(\Phi_1) = \{\mu \in I(\mathbf{R}^d) : \int_{\mathbf{R}^d} |x|\mu(dx) < \infty$, $\lim_{T\to\infty} \int_1^T t^{-1} dt \int_{|x|>t} x\nu(dx)$ *exists in* $\mathbf{R}^d$, $\int_{\mathbf{R}^d} x\mu(dx) = 0\} =: I_1^*(\mathbf{R}^d)$.
(v) *When* $1 < \alpha < 2$, $\mathfrak{D}(\Phi_\alpha) = \{\mu \in I(\mathbf{R}^d) : \int_{\mathbf{R}^d} |x|^\alpha \mu(dx) < \infty$, $\int_{\mathbf{R}^d} x\mu(dx) = 0\} =: I_\alpha^0(\mathbf{R}^d)$.
(vi) *When* $\alpha \geq 2$, $\mathfrak{D}(\Phi_\alpha) = \{\delta_0\}$, *where* δ_0 *is the distribution with the total mass at* 0.

Note that when $\alpha < 0$, the interval of the integral is finite, so the stochastic integral exists for any $\mu \in I(\mathbf{R}^d)$ by a result in Sato (2006a). Because of (vi) above, we are only interested in the case $\alpha < 2$. So, from now on, we assume that $\alpha < 2$.

REMARK 4.4. O'Connor (1979) mentioned the definition of Φ_α, $-1 < \alpha < 2$, and stated without proofs that $\mathfrak{D}(\Phi_\alpha) = I_\alpha(\mathbf{R}^1)$, $0 < \alpha < 1$, and $\mathfrak{D}(\Phi_\alpha) = I_\alpha^0(\mathbf{R}^1)$, $1 < \alpha < 2$, but he did not mention the case $\alpha = 1$. Actually, as we will see, the case $\alpha = 1$ is the most difficult case to handle.

REMARK 4.5 (Ranges). We know

$$\Phi_0(I_{\log}(\mathbf{R}^d)) = L(\mathbf{R}^d) \quad \text{(Wolfe (1982) and others)}\,.$$

In Jurek (1985), it is shown that

$$U(\mathbf{R}^d) = \left\{\mathcal{L}\left(\int_0^1 t\,dX_t^{(\mu)}\right),\ \mu \in I(\mathbf{R}^d)\right\}.$$

But this is trivially the same as $\Phi_{-1}(I(\mathbf{R}^d))$.

In the following denote the mapped distribution by $\widetilde{\mu} = \Phi_\alpha(\mu) = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ with a polar decomposition $(\widetilde{\lambda}, \widetilde{\nu}_\xi)$. We want to prove

THEOREM 4.6. *The ranges of the mappings Φ_α are,*

(i) *when* $\alpha < 0$, $\Phi_\alpha(I(\mathbf{R}^d)) = K_\alpha(\mathbf{R}^d)$,

(ii) *when* $\alpha = 0$, $\Phi_0(I_{\log}(\mathbf{R}^d)) = K_0(\mathbf{R}^d)$,

(iii) *when* $0 < \alpha < 1$, $\Phi_\alpha(I_\alpha(\mathbf{R}^d)) = K_\alpha(\mathbf{R}^d)$,

(iv) *when* $\alpha = 1$, $\Phi_\alpha(I_1^*(\mathbf{R}^d)) =$
$\{\widetilde{\mu} \in K_1(\mathbf{R}^d)$: *for* $\widetilde{\ell}_\xi(r)$ *with* $\widetilde{\nu}_\xi(dr) = r^{-2}\widetilde{\ell}_\xi(r)dr$,
$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 tdt\int_S \xi\widetilde{\lambda}(d\xi)\int_0^\infty \frac{r^2}{1+t^2r^2}d\widetilde{\ell}_\xi(r+)$ *exists in* $\mathbf{R}^d$ *and equals* $\widetilde{\gamma}\}$,

(v) *when* $1 < \alpha < 2$, $\Phi_\alpha(I_\alpha^0(\mathbf{R}^d)) = \{\widetilde{\mu} \in K_\alpha(\mathbf{R}^d)\ :\ \int_{\mathbf{R}^d} x\widetilde{\mu}(dx) = 0\}$.

Although (ii) is known, we have written it just for the completeness of the theorem. We give the proof of Theorem 4.6 in the next section.

We end this section with mentioning the continuity of $\Phi_\alpha(\mu)$ in α near 0 from below for each fixed $\mu \in I_{\log}(\mathbf{R}^d)$. (The continuity in $\alpha \in [-1, 0)$ for fixed $\mu \in I(\mathbf{R}^d)$ is trivial.)

REMARK 4.7. Now, let α tend to 0 from below. As to the interval of the integral, we have

$$\int_0^{-1/\alpha} \to \int_0^\infty \quad \text{as } \alpha \uparrow 0$$

and as to the integrand, we have

$$(1+\alpha t)^{-1/\alpha} \to e^{-t} \quad \text{as } \alpha \uparrow 0\,.$$

So, the question is whether $\lim_{\alpha\uparrow 0}\Phi_\alpha(\mu) = \Phi_0(\mu)$, $\mu \in I_{\log}(\mathbf{R}^d)$, holds or not. But, this is true, if we apply the dominated convergence theorem to the cumulants of $\mathcal{L}(\Phi_\alpha(\mu))$.

This remark explains why our expression (2.4) is more natural, when we consider the case $\alpha = 0$ as mentioned in Remark 2.1.

5. Proof of Theorem 4.6

Suppose $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\alpha)$, $-\infty < \alpha < 2$. Then we have that the mapped distribution $\widetilde{\mu} = \Phi_\alpha(\mu) = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ satisfies

$$\widetilde{A} = (2-\alpha)^{-1}A\,, \tag{5.1}$$

$$\widetilde{\nu}(B) = \int_0^1 \nu(s^{-1}B)s^{-\alpha-1}ds\,, \tag{5.2}$$

$$\begin{aligned}\widetilde{\gamma} &= \lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 t^{-\alpha}dt\left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+t^2|x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) \\ &= \lim_{T\uparrow\varepsilon_\alpha(0)}\int_0^T \varepsilon_\alpha^*(s)ds\left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+|\varepsilon_\alpha^*(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)\,.\end{aligned} \tag{5.3}$$

The reasons are follows. The derivation of $\widetilde{A}$ is that

$$\widetilde{A} = \int_0^{\varepsilon_\alpha(0)} \varepsilon_\alpha^*(t)^2 A dt = \int_1^0 s^2 A d\varepsilon_\alpha(s) = (2-\alpha)^{-1} A\,.$$

(5.2) is shown as follows. By using Proposition 2.6 of Sato (2006b), we have

$$\begin{aligned}\widetilde{\nu}(B) &= \int_0^{\varepsilon_\alpha(0)} dt \int_{\mathbf{R}^d} 1_B(x\varepsilon_\alpha^*(t))\nu(dx)\\ &= \int_0^1 (-d\varepsilon_\alpha(s)) \int_{\mathbf{R}^d} 1_B(xs)\nu(dx)\\ &= \int_0^1 s^{-\alpha-1} ds \int_{\mathbf{R}^d} 1_{s^{-1}B}(x)\nu(dx)\\ &= \int_0^1 \nu(s^{-1}B) s^{-\alpha-1} ds\,.\end{aligned}$$

$\widetilde{\nu}$ is similarly obtained, but by the change of variables $t \to \varepsilon_\alpha^*(s)$ we get two representations for $\widetilde{\gamma}$. We sometimes use the zero mean condition,

$$\gamma = -\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx)\,. \tag{5.4}$$

We need the following lemma. Denote

$$\log^* x := \begin{cases} 1 & \text{if } 0 < x \le 1\,,\\ \log x & \text{if } x > 1\,.\end{cases}$$

LEMMA 5.1. *Let $-\infty < \alpha < 2$ and let $\widetilde{\nu}$ be a Lévy measure. Then there exists a Lévy measure ν satisfying* (5.2) *such that*

$$\begin{cases} \int_{\mathbf{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty\,, & \textit{when } \alpha < 0\,,\\ \int_{\mathbf{R}^d}(|x|^2 \wedge 1)\log^*|x|\nu(dx) < \infty\,, & \textit{when } \alpha = 0\,,\\ \int_{\mathbf{R}^d}(|x|^2 \wedge |x|^\alpha)\nu(dx) < \infty\,, & \textit{when } 0 < \alpha < 2\end{cases} \tag{5.5}$$

if and only if $\widetilde{\nu}$ is represented as

$$\widetilde{\nu}(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1_B(u\xi) u^{-\alpha-1} \widetilde{\ell}_\xi(u) du\,, \quad B \in \mathcal{B}(\mathbf{R}^d)\,, \tag{5.6}$$

where $\widetilde{\lambda}$ is a measure on S and $\widetilde{\ell}_\xi(u)$ is a function measurable in ξ and for $\widetilde{\lambda}$-a.e. *ξ. nonincreasing in $u \in (0,\infty)$, not identically zero and* $\lim_{u\to\infty} \widetilde{\ell}_\xi(u) = 0$.

This lemma follows from similar arguments as those used in Lemma 4.4 in Sato (2006b).

PROOF OF LEMMA 5.1. (The "only if" part) Assume that the Lévy measure ν satisfies (5.2) and (5.5). The polar decomposition gives us

$$\int_S \lambda(d\xi)\int_0^\infty (r^2\wedge 1)\nu_\xi(dr) < \infty\,, \quad \text{when } \alpha \le 0 \tag{5.7}$$

$$\int_S \lambda(d\xi)\int_0^\infty (r^2\wedge r^\alpha)\nu_\xi(dr) < \infty\,, \quad \text{when } \alpha > 0\,. \tag{5.8}$$

Then we have for $B \in \mathcal{B}(\mathbf{R}^d)$

$$\begin{aligned}\widetilde{\nu}(B) &= \int_0^1 \nu(s^{-1}B)s^{-\alpha-1}ds\\ &= \int_0^1\int_S \lambda(d\xi)\int_0^\infty \nu_\xi(dr)1_{s^{-1}B}(r\xi)s^{-\alpha-1}ds\\ &= \int_S\lambda(d\xi)\int_0^\infty r^\alpha\nu_\xi(dr)\int_0^r 1_B(u\xi)u^{-\alpha-1}du\\ &=: \int_S \lambda(d\xi)\int_0^\infty 1_B(u\xi)u^{-\alpha-1}\widetilde{\ell}_\xi(u)du\,,\end{aligned}$$

where

$$\widetilde{\ell}_\xi(u) = \int_u^\infty r^\alpha \nu_\xi(dr)\,. \tag{5.9}$$

Therefore $\widetilde{\ell}_\xi(u)$ is measurable in ξ, and for λ-a.e. ξ. nonincreasing in u, and $\lim_{u\to\infty}\widetilde{\ell}_\xi(u) = 0$ from (5.7) and (5.8).

(The "if" part) Suppose that $\widetilde{\nu}$ satisfies (5.6). Let $\widetilde{\ell}_\xi(u+)$ be the right-continuous function defined by $\lim_{t\uparrow u}\widetilde{\ell}_\xi(t) = \widetilde{\ell}_\xi(u+)$. Then since $-\widetilde{\ell}_\xi(u+)$ is a right-continuous increasing function, there exists a measure $\widetilde{Q}_\xi$ on $(0,\infty)$ satisfying

$$\widetilde{Q}_\xi((r,s]) = -\widetilde{\ell}_\xi(s+) + \widetilde{\ell}_\xi(r+)\,,$$

and we put

$$\nu_\xi(dr) = r^{-\alpha}\widetilde{Q}_\xi(dr)\,.$$

Furthermore, define

$$\nu(B) = \int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(r\xi)\nu_\xi(dr)\,.$$

Let $\lambda = \widetilde{\lambda}$. Then for the case $\alpha < 0$ we have

$$\begin{aligned}\int_0^\infty (|x|^2\wedge 1)\nu(dx) &= \int_S \lambda(d\xi)\int_0^\infty (r^2\wedge 1)\nu_\xi(dr)\\ &= \int_S \lambda(d\xi)\Big(\int_0^1 r^{2-\alpha}\widetilde{Q}_\xi(dr) + \int_1^\infty r^{-\alpha}\widetilde{Q}_\xi(dr)\Big)\,.\end{aligned}$$

Since $\widetilde{\nu}$ is a Lévy measure, we have $\int_S \lambda(d\xi)\int_0^\infty (r^2\wedge 1)r^{-\alpha-1}\widetilde{\ell}_\xi(r+)dr<\infty$. Note that

$$\begin{aligned}0<\int_S \lambda(d\xi)\int_0^1 r^{1-\alpha}\widetilde{\ell}_\xi(r+)dr &= \int_S \lambda(d\xi)\int_0^1 r^{1-\alpha}\int_r^\infty \widetilde{Q}_\xi(dx)dr\\ &= \int_S \lambda(d\xi)\int_0^\infty \widetilde{Q}_\xi(dx)\int_0^{x\wedge 1} r^{1-\alpha}dr\\ &= (2-\alpha)^{-1}\int_S \lambda(d\xi)\int_0^1 x^{2-\alpha}\widetilde{Q}_\xi(dx)+(2-\alpha)^{-1}\int_S \lambda(d\xi)\widetilde{\ell}_\xi(1+)<\infty\end{aligned}$$

and

$$\begin{aligned}0<\int_S \lambda(d\xi)\int_1^\infty r^{-\alpha-1}\widetilde{\ell}_\xi(r+)dr &= \int_S \lambda(d\xi)\int_1^\infty r^{-\alpha-1}\int_r^\infty \widetilde{Q}_\xi(dx)dr\\ &= \int_S \lambda(d\xi)\int_1^\infty \widetilde{Q}_\xi(dx)\int_1^x r^{-\alpha-1}dr\\ &= \alpha^{-1}\int_S \lambda(d\xi)\int_1^\infty (1-x^{-\alpha})\widetilde{Q}_\xi(dx)\\ &= \alpha^{-1}\int_S \lambda(d\xi)\widetilde{\ell}_\xi(1+)-\alpha^{-1}\int_S \lambda(d\xi)\int_1^\infty x^{-\alpha}\widetilde{Q}_\xi(dx)<\infty\,.\end{aligned}$$

By the first inequality, $\int_S \lambda(d\xi)\widetilde{\ell}_\xi(1+)>0$ is finite and we see that

$$0<\int_S \lambda(d\xi)\int_0^1 x^{2-\alpha}\widetilde{Q}_\xi(dx)<\infty \quad\text{and}\quad 0<\int_S \lambda(d\xi)\int_1^\infty x^{-\alpha}\widetilde{Q}_\xi(dx)<\infty\,,$$

which imply (5.5). For the remaining cases $\alpha=0$ and $0<\alpha<2$, similar logic as in the case $\alpha<0$ works and we concludes (5.5). □

PROOF OF THEOREM 4.6. As in Sato (2006a), we use the notation $C_\#^+$ for the class of nonnegative bounded continuous functions on $\mathbf{R}^d$ vanishing on a neighborhood of the origin.

(i), (ii) and (iii) ($-\infty<\alpha<1$) (The "only if" part) Suppose that $\widetilde{\mu}\in\Phi_\alpha(\mathfrak{D}(\Phi_\alpha))$ and $\widetilde{\mu}=\Phi_\alpha(\mu)$, $\mu=\mu_{(A,\nu,\gamma)}$. When $\nu\neq 0$, since $\mu\in\mathfrak{D}(\Phi_\alpha)$, (5.6) holds by Lemma 5.1 so that $\widetilde{\mu}\in K_\alpha(\mathbf{R}^d)$.

(The "if" part) Suppose $\widetilde{\mu}=\widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}\in K_\alpha(\mathbf{R}^d)$. If $\widetilde{\mu}$ is Gaussian then putting $A=(2-\alpha)\widetilde{A}$, $\nu=0$, and $\gamma=(1-\alpha)\widetilde{\gamma}$, we have $\mu=\mu_{(A,\nu,\gamma)}\in\mathfrak{D}(\Phi_\alpha)$ and $\widetilde{\mu}=\Phi_\alpha(\mu)$. If $\widetilde{\mu}$ is non-Gaussian, then we have (5.5) by Lemma 5.1. We put $A=(2-\alpha)\widetilde{A}$ and

$$\gamma=(1-\alpha)\Bigl(\widetilde{\gamma}+\int_0^{\varepsilon_\alpha(0)}\varepsilon_\alpha^*(t)dt\int_{\mathbf{R}^d}x\Bigl(\frac{1}{1+|x|^2}-\frac{1}{1+|\varepsilon_\alpha^*(t)x|^2}\Bigr)\nu(dx)\Bigr).$$

Although the parametrization of α is different, the argument similar to the proof of (2.35) in Sato (2006b) works and it follows from (5.5) that

$$\int_0^{\varepsilon_\alpha(0)} \varepsilon_\alpha^*(t)dt \int_{\mathbf{R}^d} |x| \left| \frac{1}{1+|x|^2} - \frac{1}{1+|\varepsilon_\alpha^*(t)x|^2} \right| \nu(dx) < \infty .$$

Thus $\mu = \mu_{(A,\mu,\gamma)} \in \mathfrak{D}(\Phi_\alpha)$ and $\Phi_\alpha(\mu) = \widetilde{\mu}$.

(iv) ($\alpha = 1$) (The "only if" part) Suppose that $\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} = \Phi(\mu) \in \Phi_1(\mathfrak{D}(\Phi_1))$ and $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_1)$. First assume that $\widetilde{\mu}$ is Gaussian. Then for given $\varphi \in C_\#^+$, $0 = \int_0^1 \int_{\mathbf{R}^d} \varphi(sx)s^{-2}\nu(dx)ds$, which implies $0 = s^{-2}\int_{\mathbf{R}^d} \varphi(sx)\nu(dx)$ a.e. Since by the dominated convergence theorem $s^{-2}\int_{\mathbf{R}^d} \varphi(sx)\nu(dx)$ is continuous in s, letting $s = 1$, we have $\nu = 0$. Furthermore, from Proposition 4.3 (iv) with (5.4) $\gamma = 0$ and hence $\widetilde{\gamma} = 0$. When $\widetilde{\mu}$ is non-Gaussian, ν satisfies (5.5) with $\alpha = 1$, and (5.3) and (5.4) imply that

$$-\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 tdt \int_{\mathbf{R}^d} \frac{x|x|^2}{1+t^2|x|^2} \nu(dx) \tag{5.10}$$

exists in $\mathbf{R}^d$ and equals $\widetilde{\gamma}$. Thus, $\widetilde{\mu} \in \{\widetilde{\mu} \in K_1(\mathbf{R}^d)$: for $\widetilde{\ell}_\xi(r)$ with $\widetilde{\nu}_\xi(dr) = r^{-2}\widetilde{\ell}_\xi(r)dr$, $\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 tdt \int_S \xi\widetilde{\lambda}(d\xi)\int_0^\infty \frac{r^2}{1+t^2r^2}d\widetilde{\ell}_\xi(r+)$ exists in $\mathbf{R}^d$ and equals $\widetilde{\gamma}\}$.

(The "if" part) Suppose $\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in \{\widetilde{\mu} \in K_1(\mathbf{R}^d)$: for $\widetilde{\ell}_\xi(r)$ with $\widetilde{\nu}_\xi(dr) = r^{-2}\widetilde{\ell}_\xi(r)dr$, $\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 tdt \int_S \xi\widetilde{\lambda}(d\xi)\int_0^\infty \frac{r^2}{1+t^2r^2}d\widetilde{\ell}_\xi(r+)$ exists in $\mathbf{R}^d$ and equals $\widetilde{\gamma}\}$. If $\widetilde{\mu}$ is centered Gaussian, then $\widetilde{\mu} \in \Phi_1(\mathfrak{D}(\Phi_1))$ from Proposition 4.3. If $\widetilde{\mu}$ is non-Gaussian and satisfies (5.6) and (5.10), then by Lemma 5.1 a measure ν exists and satisfies (5.2) and (5.5) with $\alpha = 1$. Let $\gamma = -\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx)$ and $A = \widetilde{A}$. It follows from the existence of (5.10) and $\int_{|x|>1} |x|\nu(dx) < \infty$ that

$$\lim_{T\to\infty} \int_1^T t^{-1}dt \int_{|x|>t} x\nu(dx) < \infty$$

as in the proof of Theorem 2.8 of Sato (2006b). Thus $\mu \in \mathfrak{D}(\Phi_1)$. Furthermore (5.10) implies

$$\widetilde{\gamma} = \lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 t^{-1}dt \left(-\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx) + \int_{\mathbf{R}^d} x\left(\frac{1}{1+|tx|^2} - \frac{1}{1+|x|^2} \right)\nu(dx) \right),$$

which equals the right-hand side of (5.3). Therefore $\Phi_1(\mu) = \widetilde{\mu}$ and $\widetilde{\mu} \in \Phi_1(\mathfrak{D}(\Phi_1))$.

(v) ($1 < \alpha < 2$) (The "only if" part) Assume that $\widetilde{\mu} = \Phi_\alpha(\mu)$ with some $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\alpha)$. The Gaussian case is the same as that of the proof for (ii). If $\widetilde{\mu}$ is non-Gaussian, then it follows from Lemma 5.1 that there exists $\widetilde{\nu}$ satisfying (5.6). Since $\mu \in \mathfrak{D}(\Phi_\alpha)$, ν and γ satisfy $\int_{|x|>1}|x|^\alpha\nu(dx) < \infty$ and (5.4), respectively. Then as in the proof of Theorem 2.4 (iii) of Sato (2006b), $\widetilde{\gamma}$ exists and equals to

$$\widetilde{\gamma} = -\int_0^\infty \varepsilon_\alpha^*(t)dt \int_{\mathbf{R}^d} \frac{x|\varepsilon_\alpha^*(t)x|^2}{1+|\varepsilon_\alpha^*(t)x|^2}\nu(dx) = -\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\widetilde{\nu}(dx), \tag{5.11}$$

which is

$$\int_{\mathbf{R}^d} x\widetilde{\mu}(dx) = 0\,. \tag{5.12}$$

Hence $\widetilde{\mu} \in \{\widetilde{\mu} \in K_\alpha(\mathbf{R}^d) : \int_{\mathbf{R}^d} x\widetilde{\mu}(dx) = 0\}$.

(The "if" part) Suppose $\widetilde{\mu} = \widetilde{\mu}_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in \{\widetilde{\mu} \in K_\alpha(\mathbf{R}^d) : \int_{\mathbf{R}^d} x\widetilde{\mu}(dx) = 0\}$. The Gaussian case is obvious. Suppose $\widetilde{\mu}$ be non-Gaussian. Due to Lemma 5.1 a measure ν with $\nu(\{0\}) = 0$ exists and satisfies (5.2) and (5.5). It follows from (5.2) that

$$\begin{aligned}
\int_{\mathbf{R}^d} \frac{|x|^3}{1+|x|^2}\widetilde{\nu}(dx) &= \int_0^1 t^{2-\alpha}dt \int_{\mathbf{R}^d} \frac{|x|^3}{1+t^2|x|^2}\nu(dx) \\
&\le \int_{|x|\le 1} |x|^3\nu(dx) \int_0^1 t^{2-\alpha}dt + \int_{|x|>1} |x|^3\nu(dx) \int_0^{1/|x|} t^{2-\alpha}dt \\
&\quad + \int_{|x|>1} |x|\nu(dx) \int_{1/|x|}^1 t^{-\alpha}dt \\
&= (3-\alpha)^{-1} \int_{|x|\le 1} |x|^3\nu(dx) + (3-\alpha)^{-1} \int_{|x|>1} |x|^\alpha\nu(dx) \\
&\quad + (1-\alpha)^{-1} \int_{|x|>1} \left(|x| - |x|^\alpha\right)\nu(dx) < \infty\,.
\end{aligned}$$

Hence we have $\int_{|x|>1} |x|\widetilde{\nu}(dx) < \infty$ which is equivalent to $\int_{\mathbf{R}^d} |x|\widetilde{\mu}(dx) < \infty$ and (5.11) holds. Let $\gamma = -\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\nu(dx)$, $A = (2-\alpha)\widetilde{A}$ and $\mu = \mu_{(A,\nu,\gamma)}$. Then $\mu \in \mathfrak{D}(\Phi_\alpha)$ by Proposition 4.4 (v). Further

$$\int_0^\infty \varepsilon_\alpha^*(t)dt\left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+|\varepsilon_\alpha^*(t)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) = -\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2}\widetilde{\nu}(dx)\,,$$

which equals $\widetilde{\gamma}$. Hence (5.3) is true and $\Phi_\alpha(\mu) = \widetilde{\mu}$, namely $\widetilde{\mu} \in \Phi_\alpha(\mathfrak{D}(\Phi_\alpha))$. □

6. Nested subclasses of $\Phi_\alpha(\mathfrak{D}(\Phi_\alpha))$

Φ_α-mapping allows us to construct nested subclasses of $\Phi_\alpha(\mathfrak{D}(\Phi_\alpha))$ defined by the iterated mappings Φ_α^{m+1}, $m = 1, 2, \ldots$. This is the topic in this section. We will see the domains $\mathfrak{D}(\Phi_\alpha^{m+1})$ in Subsection 6.1 and characterize the ranges $\Phi_\alpha^{m+1}(\mathfrak{D}(\Phi_\alpha^{m+1}))$ by both stochastic integral representations and the Lévy-Khintchine triplet, which are respectively given in Subsections 6.2 and 6.3.

6.1. Domains of Φ_α^{m+1}

THEOREM 6.1. *Let* $m = 1, 2, \ldots$

(i) *When* $\alpha < 0$, $\mathfrak{D}(\Phi_\alpha^{m+1}) = I(\mathbf{R}^d)$.

(ii) *When* $\alpha = 0$,

$$\mathfrak{D}(\Phi_0^{m+1}) = \left\{\mu \in I(\mathbf{R}^d) : \int_{|x|>1} (\log|x|)^{m+1}\mu(dx) < \infty\right\} =: I_{\log^{m+1}}(\mathbf{R}^d)\,.$$

(iii) *When* $0 < \alpha < 1$,

$$\mathfrak{D}(\Phi_\alpha^{m+1}) = \left\{\mu \in I(\mathbf{R}^d) : \int_{|x|>1} |x|^\alpha\,(\log|x|)^m\,\mu(dx) < \infty\right\} =: I_{\alpha,\log^m}(\mathbf{R}^d)\,.$$

(iv) *When* $\alpha = 1$,

$$\mathfrak{D}(\Phi_1^{m+1}) = \left\{\mu \in I(\mathbf{R}^d) : \int_{|x|>1} |x|\,(\log|x|)^m\,\mu(dx) < \infty,\ \int_{\mathbf{R}^d} x\mu(dx) = 0\,,\right.$$
$$\left.\lim_{T\to\infty}\int_1^T t^{-1}dt \int_{|x|>t} x(\log(|x|/t))^m\nu(dx) \textit{ exists in } \mathbf{R}^d\right\} =: I^*_{1,\log^m}(\mathbf{R}^d)\,.$$

(v) *When* $1 < \alpha < 2$,

$$\mathfrak{D}(\Phi_\alpha^{m+1}) = \left\{\mu \in I(\mathbf{R}^d) : \int_{|x|>1} |x|^\alpha\,(\log|x|)^m\,\mu(dx) < \infty,\ \int_{\mathbf{R}^d} x\mu(dx) = 0\right\}$$
$$=: I^0_{\alpha,\log^m}(\mathbf{R}^d)\,.$$

PROOF OF THEOREM 6.1. Since when $\alpha < 0$, the integral for $\Phi_\alpha^{m+1}(\mu)$ is not improper integral, it is easy to see that $\mathfrak{D}(\Phi_\alpha^{m+1}) = I(\mathbf{R}^d)$, (see Sato (2006a)). When $\alpha = 0$, Jurek (1985) determined $\mathfrak{D}(\Phi_0^{m+1})$ as above.

We are now going to prove (iii), (iv) and (v). First, note that

$$\int_{1/x}^1 u^{-1}(\log ux)^m du = (m+1)^{-1}(\log x)^{m+1} \quad \text{for } m = 0,1,2,\ldots. \tag{6.1}$$

Now, Theorem 6.1 (iii), (iv) and (v) are true for $m = 0$ as seen in Proposition 4.3 (iii), (iv) and (v). Suppose that it is true for some integer $m \geq 0$, as the induction hypothesis. Suppose $0 < \alpha < 1$. Then

$$\mathfrak{D}(\Phi_\alpha^{m+2}) = \left\{\mu \in \mathfrak{D}(\Phi_\alpha) : \int_{|x|>1} |x|^\alpha(\log|x|)^m\widetilde{\nu}(dx) < \infty,\right.$$
$$\left.\text{where } \widetilde{\nu} \text{ is the Lévy measure of } \widetilde{\mu} = \Phi_\alpha(\mu)\right\}.$$

Recall from (5.2) that

$$\widetilde{\nu}(B) = \int_0^1 \nu(s^{-1}B)s^{-\alpha-1}ds\,.$$

Thus,

$$\int_{|x|>1} |x|^\alpha (\log|x|)^m \widetilde{\nu}(dx) = \int_{|x|>1} |x|^\alpha (\log|x|)^m \int_0^1 \nu(s^{-1}dx)s^{-\alpha-1}ds$$
$$= \int_0^1 s^{-\alpha-1}ds \int_{|x|>1} |x|^\alpha (\log|x|)^m \nu(s^{-1}dx)$$
$$= \int_{|y|>1} |y|^\alpha \nu(dy) \int_{1/|y|}^1 s^{-1}(\log|sy|)^m ds\,.$$

Then by (6.1),

$$\int_{|x|>1} |x|^\alpha (\log|x|)^m \widetilde{\nu}(dx) < \infty$$

if and only if

$$\int_{|x|>1} |x|^\alpha (\log|x|)^{m+1} \nu(dx) < \infty\,,$$

and we conclude that $\mathfrak{D}(\Phi_\alpha^{m+2}) = I_{\alpha,\log^{m+1}}(\mathbf{R}^d)$.

When $1 < \alpha < 2$, there is no problem for the moment condition, and the condition, $\int_{\mathbf{R}^d} x\mu(dx) = 0$, always holds. Thus we get $\mathfrak{D}(\Phi_\alpha^{m+2}) = I^0_{\alpha,\log^{m+1}}(\mathbf{R}^d)$.

Finally we prove (iv). So, suppose $\alpha = 1$. Also suppose it is true for some integer $m \geq 0$. We have

$$\mathfrak{D}(\Phi_1^{m+2}) = \{\mu \in \mathfrak{D}(\Phi_1) : \int_{|x|>1} |x|\,(\log|x|)^m\,\widetilde{\nu}(dx) < \infty,\ \int_{\mathbf{R}^d} x\mu(dx) = 0\,,$$
$$\lim_{T\to\infty} \int_1^T t^{-1}dt \int_{|x|>t} x(\log(|x|t^{-1}))^m \widetilde{\nu}(dx) \text{ exists in } \mathbf{R}^d\,, \tag{6.2}$$
$$\text{where } \widetilde{\nu} \text{ is the Lévy measure of } \Phi_1(\mu)\}\,.$$

Since the moment condition can be given by the same way as for the case $1 < \alpha < 2$, in order to reach the conclusion, it remains to show that

$$\lim_{T\to\infty} \int_1^T t^{-1}dt \int_{|x|>t} x(\log(|x|t^{-1}))^{m+1} \nu(dx) \text{ exists in } \mathbf{R}^d\,. \tag{6.3}$$

We have

$$\int_{|y|>t} y(\log(|y|t^{-1}))^m \widetilde{\nu}(dy) = \int_{|y|>t} y(\log(|y|t^{-1}))^m \int_0^1 \nu(s^{-1}dy)s^{-2}ds$$
$$= \int_0^1 s^{-1}ds \int_{|sx|>t} x(\log(|sx|t^{-1}))^m \nu(dx)$$

$$= \int_{|x|>t} x\nu(dx) \int_{t/|x|}^{1} s^{-1}(\log(s|x|t^{-1}))^m ds$$

$$= (m+1)^{-1} \int_{|x|>t} x(\log(|x|t^{-1}))^{m+1}\nu(dx).$$

Hence (6.2) is equivalent to (6.3). This completes the proof. □

6.2. Characterizations of $\Phi_\alpha^{m+1}(\mathfrak{D}(\Phi_\alpha^{m+1}))$ by stochastic integral representations. Let $\alpha < 2$, $m = 1, 2, \dots$,

$$g_{\alpha,m}(s) = (m!)^{-1}s^{-\alpha-1}(\log s^{-1})^m, \quad 0 < s \le 1,$$

$$\varepsilon_{\alpha,m}(u) = \int_u^1 g_{\alpha,m}(s)ds, \quad 0 < u \le 1,$$

and let $\varepsilon^*_{\alpha,m}(t)$ be the inverse function of $\varepsilon_{\alpha,m}(x)$ such that $t = \varepsilon_{\alpha,m}(u)$ if and only if $u = \varepsilon^*_{\alpha,m}(t)$. Note that when $\alpha < 0$, $\varepsilon_{\alpha,m}(0) = (-\alpha)^{-(m+1)}$ and when $0 \le \alpha < 2$, $\varepsilon_{\alpha,m}(0) = \infty$. $\varepsilon_{\alpha,0}(u)$ is (4.1). We consider a mapping for $\mu \in \mathfrak{D}(\Phi_\alpha^{m+1})$ defined by the stochastic integral

$$\overline{\mu}_m = \mathcal{L}\left(\int_0^{\varepsilon_{\alpha,m}(0)} \varepsilon^*_{\alpha,m}(t)dX_t^{(\mu)}\right), \tag{6.4}$$

when the integral exists. By Proposition 2.6 of Sato (2006b), if it exists, then the Lévy-Khintchine triplet $(\overline{A}_m, \overline{\nu}_m, \overline{\gamma}_m)$ of $\overline{\mu}_m$ is given as

$$\overline{A}_m = \int_0^{\varepsilon_{\alpha,m}(0)} \varepsilon^*_{\alpha,m}(t)^2 A dt = (m!)^{-1}\int_0^1 s^{1-\alpha}(\log s^{-1})^m A ds = (2-\alpha)^{-(m+1)}A, \tag{6.5}$$

$$\overline{\nu}_m(B) = \int_0^{\varepsilon_{\alpha,m}(0)} dt \int_{\mathbf{R}^d} 1_B(\varepsilon^*_{\alpha,m}(t)x)\nu(dx) \tag{6.6}$$

$$\begin{aligned}\overline{\gamma}_m &= \lim_{\varepsilon\downarrow 0}(m!)^{-1}\int_\varepsilon^1 t^{-\alpha}(\log t^{-1})^m dt \\ &\qquad \left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+t^2|x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) \\ &= \lim_{T\uparrow\varepsilon_{\alpha,m}(0)}\int_0^T \varepsilon^*_{\alpha,m}(s)ds\left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+|\varepsilon^*_{\alpha,m}(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right).\end{aligned} \tag{6.7}$$

In the following, we show that $\Phi_\alpha^{m+1}(\mu)$ is equal to $\overline{\mu}_m$ in (6.4) when $\alpha \ne 1$ and $\mu \in \mathfrak{D}(\Phi_\alpha^{m+1})$.

THEOREM 6.2. *Let $\alpha \in (-\infty, 1) \cup (1, 2)$ and $m \in \mathbf{N}$. Suppose $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\alpha^{m+1})$. Then the distribution $\overline{\mu}_m$ in* (6.4) *is definable and $\Phi_\alpha^{m+1}(\mu) = \overline{\mu}_m$.*

REMARK 6.3. The following are known.

(i) (Jurek (2004)) Let $\alpha = -1$ and $\mu \in I(\mathbf{R}^d)$. Then

$$\Phi_{-1}^{m+1}(\mu) = \mathcal{L}\left(\int_0^1 \tau_m^*(t)dX_t^{(\mu)}\right),$$

where $\tau_m(u) = \int_0^u g_{-1,m}(s)ds$, $0 < u \le 1$ and $\tau_m^*(t)$ is its inverse. However, by changing variable t to $1-t$, we see that

$$\mathcal{L}\left(\int_0^1 \tau_m^*(t)dX_t^{(\mu)}\right) = \mathcal{L}\left(\int_0^1 \varepsilon_{-1,m}^*(t)dX_t^{(\mu)}\right).$$

(ii) (Jurek (1983)) When $\alpha = 0$,

$$\varepsilon_{0,m}^*(t) = e^{-((m+1)!t)^{(m+1)^{-1}}}. \tag{6.8}$$

In our setting, we can get (6.4) as follows. By a standard calculation, we see that

$$\varepsilon_{0,m}(u) = ((m+1)!)^{-1}(\log u^{-1})^{m+1}$$

and thus (6.4) is given by taking the inverse function of $t = \varepsilon_{0,m}(u)$.

To prove Theorem 6.2 for $\alpha \in (0,1) \cup (1,2)$, we need a lemma.

LEMMA 6.4. (i) *Let $\alpha < 1$ and $\varepsilon \in (0,1)$. When $0 \le \alpha < 1$, assume that for some $n \in \mathbf{N}$, $\int_{|x|>1} |x|^\alpha (\log|x|)^n \mu(dx) < \infty$. Then there exists an $M_\varepsilon > 0$ independent of $s \in [\varepsilon, 1]$ such that*

$$\int_0^1 t^{-\alpha}(\log t^{-1})^n dt \int_{\mathbf{R}^d} |x| \left| \frac{1}{1+s^2t^2|x|^2} - \frac{1}{1+|x|^2} \right| \nu(dx) < M_\varepsilon. \tag{6.9}$$

(ii) *Let $1 < \alpha < 2$ and $\varepsilon \in (0,1)$. If for some $n \in \mathbf{N}$, $\int_{|x|>1} |x|^\alpha (\log|x|)^n \mu(dx) < \infty$, then there exists an $M_\varepsilon > 0$ independent of $s \in [\varepsilon, 1]$ such that*

$$\int_0^1 t^{-\alpha}(\log t^{-1})^n dt \int_{\mathbf{R}^d} \frac{t^2|x|^3}{1+s^2t^2|x|^2} \nu(dx) < M_\varepsilon. \tag{6.10}$$

PROOF. (i) Let c_1 and c_2 be some positive constants. Then

$$\begin{aligned}
&\int_0^1 t^{-\alpha}(\log t^{-1})^n dt \int_{\mathbf{R}^d} |x| \left| \frac{1}{1+s^2t^2|x|^2} - \frac{1}{1+|x|^2} \right| \nu(dx) \\
&\quad \le \int_0^1 t^{-\alpha}(\log t^{-1})^n dt \left(\int_{|x|\le 1} \frac{|x|^3}{(1+s^2t^2|x|^2)(1+|x|^2)} \nu(dx) \right. \\
&\qquad \left. + c_1 \int_{|x|>1, |tx|\le 1} |x|\nu(dx) + c_2 \int_{|x|>1, |tx|>1} t^{-2}|x|^{-1}\nu(dx) \right) \\
&\quad = \int_0^1 t^{-\alpha}(\log t^{-1})^n dt \int_{|x|\le 1} |x|^3 \nu(dx)
\end{aligned}$$

$$+ c_1 \int_{|x|>1} |x|\nu(dx) \int_0^{1/|x|} t^{-\alpha}(\log t^{-1})^n dt$$
$$+ c_2 \int_{|x|>1} |x|^{-1}\nu(dx) \int_{1/|x|}^1 t^{-\alpha-2}(\log t^{-1})^n dt\,.$$

By the integral by parts formula, $\int_0^{1/|x|} t^{-\alpha}(\log t^{-1})^n dt$ is shown to be constructed by a linear combination of $|x|^{\alpha-1}(\log|x|)^k$ with $k = 0, 1, \dots, n$ and $\int_{1/|x|}^1 t^{-\alpha-2}(\log t^{-1})^n dt$ is shown to be constructed by a linear combination of $|x|^{\alpha+1}(\log|x|)^k$ with $k = 0, 1, \dots, n$ and a constant term. Thus the assumed moment condition gives the conclusion.

(ii) Observe that

$$\int_0^1 t^{2-\alpha}(\log t^{-1})^n dt \int_{\mathbf{R}^d} \frac{|x|^3}{1+s^2t^2|x|^2}\nu(dx)$$
$$\le \int_0^1 t^{2-\alpha}(\log t^{-1})^n dt \int_{|x|\le 1} |x|^3\nu(dx)$$
$$+ \int_{|x|>1} |x|^3\nu(dx) \int_0^{1/|x|} t^{2-\alpha}(\log t^{-1})^n dt$$
$$+ \frac{1}{s^2}\int_{|x|>1} |x|\nu(dx) \int_{1/|x|}^1 t^{-\alpha}(\log t^{-1})^n dt\,.$$

Here by the integral by parts formula, $\int_0^{1/|x|} t^{2-\alpha}(\log t^{-1})^n dt$ is shown to be a linear combination of $|x|^{\alpha-3}(\log|x|)^k$ with $k = 0, 1, \dots, n$ and $\int_{1/|x|}^1 t^{-\alpha}(\log t^{-1})^n dt$ is shown to be constructed by a linear combination of $|x|^{\alpha-1}(\log|x|)^k$ with $k = 0, 1, \dots, n$ and a constant term. Then the assumed moment conditions give the result. □

We are now ready to prove Theorem 6.2.

PROOF OF THEOREM 6.2. Denote the Lévy-Khintchine triplet of $\Phi_\alpha^{m+1}(\mu)$, $\mu \in \mathfrak{D}(\Phi_\alpha^{m+1})$, by $(\widetilde{A}^{(m+1)}, \widetilde{\nu}^{(m+1)}, \widetilde{\gamma}^{(m+1)})$. We first show that, for $B \in \mathcal{B}(\mathbf{R}^d)$,

$$\widetilde{\nu}^{(m+1)}(B) = \int_0^1 \nu(s^{-1}B)g_{\alpha,m}(s)ds\,, \quad m \in \mathbf{N}\,. \tag{6.11}$$

Let $m = 1$. Then we have by (5.2),

$$\widetilde{\nu}^{(2)}(B) = \int_0^1 \widetilde{\nu}^{(1)}(s^{-1}B)s^{-\alpha-1}ds$$
$$= \int_0^1 s^{-\alpha-1}ds \int_0^1 \nu((ts)^{-1}B)t^{-\alpha-1}dt$$
$$= \int_0^1 \nu(u^{-1}B)u^{-\alpha-1}du \int_u^1 s^{-1}ds$$

$$= \int_0^1 \nu(u^{-1}B) g_{\alpha,1}(u) du\,.$$

Thus (6.11) is true for $m = 1$. Next suppose

$$\widetilde{\nu}^{(m)}(B) = \int_0^1 \nu(s^{-1}B) g_{\alpha,m-1}(s) ds$$

for some integer $m \geq 1$. Then by (5.2) again,

$$\begin{aligned}
\widetilde{\nu}^{(m+1)}(B) &= \int_0^1 \widetilde{\nu}^{(m)}(s^{-1}B) s^{-\alpha-1} ds \\
&= \int_0^1 s^{-\alpha-1} ds \int_0^1 \nu((us)^{-1}B) g_{\alpha,m-1}(u) du \\
&= ((m-1)!)^{-1} \int_0^1 s^{-\alpha-1} ds \int_0^1 \nu((us)^{-1}B) u^{-\alpha-1} (\log u^{-1})^{m-1} du \\
&= ((m-1)!)^{-1} \int_0^1 u^{-\alpha-1} (\log u^{-1})^{m-1} du \int_0^u \nu(t^{-1}B)(tu^{-1})^{-\alpha-1}(u^{-1}) dt \\
&= ((m-1)!)^{-1} \int_0^1 \nu(t^{-1}B) t^{-\alpha-1} dt \int_t^1 u^{-1} (\log u^{-1})^{m-1} du \\
&= (m!)^{-1} \int_0^1 \nu(t^{-1}B) t^{-\alpha-1} (\log t^{-1})^m dt \\
&= \int_0^1 \nu(t^{-1}B) g_{\alpha,m}(t) dt\,,
\end{aligned}$$

and thus we get (6.11). Note that if $\overline{\mu}_m$ in (6.4) is definable, its Lévy measure $\overline{\nu}$ should be (6.6). On the other hand, we have

$$\begin{aligned}
\int_0^1 \nu(u^{-1}B) g_{\alpha,m}(u) du &= \int_0^1 g_{\alpha,m}(u) du \int_{\mathbf{R}^d} 1_{u^{-1}B}(x) \nu(dx) \\
&= \int_0^1 (-d\varepsilon_{\alpha,m}(u)) \int_{\mathbf{R}^d} 1_B(ux) \nu(dx) \\
&= \int_0^{\varepsilon_{\alpha,m}(0)} dt \int_{\mathbf{R}^d} 1_B(\varepsilon^*_{\alpha,m}(t)x) \nu(dx) \\
&= \overline{\nu}_m(B)\,,
\end{aligned}$$

which is equal to $\widetilde{\nu}^{(m+1)}(B)$ and finite.

$\widetilde{A}^{(m+1)}$ is directly calculated due to (5.1) and is given by $(2-\alpha)^{-(m+1)}A = \overline{A}_m$, say.

As for $\widetilde{\gamma}^{(m+1)}$, we show that $\widetilde{\gamma}^{(m+1)}$ equals $\overline{\gamma}_m$ in (6.7) by the induction argument. When $m = 0$, this holds due to (5.3). Next suppose $\widetilde{\gamma}^{(m)} = \overline{\gamma}_{m-1}$ for some integer $m \geq 1$. It follows from (5.3) that

$$\widetilde{\gamma}^{(m+1)} = \lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 t^{-\alpha} dt \left(\widetilde{\gamma}^{(m)} + \int_{\mathbf{R}^d} x \left(\frac{1}{1+t^2|x|^2} - \frac{1}{1+|x|^2} \right) \widetilde{\nu}^{(m)}(dx) \right) .$$

First consider the case $\alpha < 1$. When $0 \le \alpha < 1$, since $\mu \in \mathfrak{D}(\Phi_\alpha^{m+1})$, we have $\int_{|x|>1} |x|^\alpha (\log |x|)^m \mu(dx) < \infty$. Thus by Lemma 6.4 (i) with $n = m-1$, we have

$$\begin{aligned}\widetilde{\gamma}^{(m+1)} = \lim_{\varepsilon\downarrow 0} ((m-1)!)^{-1} \int_\varepsilon^1 s^{-\alpha} ds \int_0^1 t^{-\alpha} (\log t^{-1})^{m-1} dt \\ \times \left(\gamma + \int_{\mathbf{R}^d} x \left(\frac{1}{1+s^2t^2|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) .\end{aligned}$$

Since we can use Fubini's theorem due to Lemma 6.4 (i), we can directly show the finiteness of the following integral:

$$\begin{aligned}&\int_0^1 s^{-\alpha} ds \int_0^1 t^{-\alpha} (\log t^{-1})^{m-1} dt \left(\gamma + \int_{\mathbf{R}^d} x \left(\frac{1}{1+s^2t^2|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) \\ &= \int_0^1 \int_0^1 (st)^{-\alpha} (\log t^{-1})^{m-1} dt ds \left(\gamma + \int_{\mathbf{R}^d} x \left(\frac{1}{1+s^2t^2|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) \\ &= \int_0^1 \int_0^s u^{-\alpha} (\log s/u)^{m-1} s^{-1} du ds \left(\gamma + \int_{\mathbf{R}^d} x \left(\frac{1}{1+u^2|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) \\ &= \int_0^1 u^{-\alpha} du \int_u^1 (\log s/u)^{m-1} s^{-1} ds \left(\gamma + \int_{\mathbf{R}^d} x \left(\frac{1}{1+u^2|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) \\ &= m^{-1} \int_0^1 u^{-\alpha} (\log u^{-1})^m du \left(\gamma + \int_{\mathbf{R}^d} x \left(\frac{1}{1+u^2|x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) ,\end{aligned}$$

which is finite by Lemma 6.4 (i) with $n = m$. Thus we see that $\widetilde{\gamma}^{(m+1)}$ is the same as $\overline{\gamma}_m$ in (6.7) when $\alpha < 1$.

When $1 < \alpha < 2$, due to the zero mean condition in Proposition 4.3, it suffices to see the convergence of

$$\widetilde{\gamma}^{(m+1)} = \lim_{\varepsilon\downarrow 0} (m!)^{-1} \int_\varepsilon^1 t^{-\alpha} (\log t^{-1})^m dt \int_{\mathbf{R}^d} \frac{xt^2|x|^2}{1+t^2|x|^2} \nu(dx) . \tag{6.12}$$

Now by Lemma 6.4 (ii) with $n = m$, the expression of

$$\widetilde{\gamma}^{(m)} = ((m-1)!)^{-1} \int_0^1 t^{-\alpha} (\log t^{-1})^{m-1} dt \int_{\mathbf{R}^d} \frac{xt^2|x|^2}{1+t^2|x|^2} \nu(dx)$$

yields

$$\widetilde{\gamma}^{(m+1)} = -\lim_{\varepsilon\downarrow 0} ((m-1)!)^{-1} \int_\varepsilon^1 s^{-\alpha} ds \left(\int_0^1 t^{-\alpha} (\log t^{-1})^{m-1} dt \int_{\mathbf{R}^d} \frac{xs^2t^2|x|^2}{1+s^2t^2|x|^2} \nu(dx) \right) .$$

By a similar argument as in the case $\alpha < 1$, in order to see the convergence of the integral in $\widetilde{\gamma}^{(m+1)}$ it suffices to study the quantity

$$\int_0^1 u^{-\alpha}(\log u^{-1})^m du \left(\gamma + \int_{\mathbf{R}^d} x\left(\frac{1}{1+u^2|x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right).$$

However since we have Lemma 6.4 (2) with $n = m+1$, this is finite. Thus we conclude (6.12).

Altogether we have shown that $(\widetilde{A}^{(m+1)}, \widetilde{\nu}^{(m+1)}, \widetilde{\gamma}^{(m+1)}) = (\overline{A}_m, \overline{\nu}_m, \overline{\gamma}_m)$, each of which appears in (6.5), (6,6) and (6.7), respectively, and thus $(\overline{A}_m, \overline{\nu}_m, \overline{\gamma}_m)$ is the Lévy-Khintchine triplet of some distribution in $I(\mathbf{R}^d)$, which should be $\overline{\mu}_m$ by Proposition 2.6 of Sato (2006b). We thus conclude that $\Phi_\alpha^{m+1}(\mu) = \overline{\mu}_m$ for $\mu \in \mathfrak{D}(\Phi_\alpha^{m+1})$. □

6.3. Characterizations of $\Phi_\alpha^{m+1}(\mathfrak{D}(\Phi_\alpha^{m+1}))$by the Lévy-Khintchine triplet. We consider the range $\Phi_\alpha^{m+1}(\mathfrak{D}(\Phi_\alpha^{m+1}))$ for $m = 1, 2, \ldots$ Let $-\infty < \alpha < 2$ and suppose $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\alpha^{m+1})$. Then, when $\alpha \neq 1$, due to Theorem 6.2, the mapped distribution $\Phi_\alpha^{m+1}(\mu) = \widetilde{\mu}_{(\widetilde{A}^{(m+1)}, \widetilde{\nu}^{(m+1)}, \widetilde{\gamma}^{(m+1)})}$ satisfies

$$\widetilde{A}^{(m+1)} = (2-\alpha)^{-(m+1)}A\,, \tag{6.13}$$

$$\widetilde{\nu}^{(m+1)}(B) = (m!)^{-1}\int_0^1 \nu(s^{-1}B)s^{-\alpha-1}(\log s^{-1})^m ds\,, \tag{6.14}$$

$$\begin{aligned}\widetilde{\gamma}^{(m+1)} &= \lim_{\varepsilon\downarrow 0}(m!)^{-1}\int_\varepsilon^1 t^{-\alpha}(\log t^{-1})^m dt\left(\gamma - \int_{\mathbf{R}^d} x\left(\frac{1}{1+|x|^2} - \frac{1}{1+t^2|x|^2}\right)\nu(dx)\right)\\ &= \lim_{T\uparrow\varepsilon_{\alpha,m}(0)}\int_0^T \varepsilon^*_{\alpha,m}(s)ds\left(\gamma - \int_{\mathbf{R}^d} x\left(\frac{1}{1+|x|^2} - \frac{1}{1+|\varepsilon^*_{\alpha,m}(s)x|^2}\right)\nu(dx)\right).\end{aligned} \tag{6.15}$$

When $\alpha = 1$, we cannot use Theorem 6.2. However, as in the proof of Theorem 6.2, we can use the induction method for the Lévy measures even when $\alpha = 1$, and thus (6.14) also holds for $\alpha = 1$ as well as (6.13). For $\widetilde{\gamma}^{(m+1)}$, we have an alternative expression (6.18) below. Now with the aid of the representation $(\widetilde{A}^{(m+1)}, \widetilde{\nu}^{(m+1)}, \widetilde{\gamma}^{(m+1)})$ above, we can specify the range.

THEOREM 6.5. *Let $-\infty < \alpha < 2$ and $m = 1, 2, \ldots$ Then $\widetilde{\mu}_{m+1} = \widetilde{\mu}_{(\widetilde{A}^{(m+1)}, \widetilde{\nu}^{(m+1)}, \widetilde{\gamma}^{(m+1)})} \in \Phi_\alpha^{m+1}(\mathfrak{D}(\Phi_\alpha^{m+1}))$ if and only if one of the following conditions depending on α is satisfied.*

(i) $(-\infty < \alpha < 1)$ *$\widetilde{\mu}_{m+1}$ is Gaussian, or $\widetilde{\mu}_{m+1}$ is non-Gaussian and*

$$\widetilde{\nu}^{(m+1)}(B) = \int_S \widetilde{\lambda}(d\xi)\int_0^\infty 1_B(u\xi)u^{-\alpha-1}\widetilde{h}_\xi^{(m)}(u)du\,, \quad B \in \mathcal{B}(\mathbf{R}^d)\,. \tag{6.16}$$

Here $\widetilde{\lambda}$ *is a measure on S and* $\widetilde{h}_\xi^{(m)}(u)$ *is a measurable function in* ξ *such that satisfies*

$$\widetilde{h}_\xi^{(m)}(u) = ((m-1)!)^{-1} \int_u^\infty x^{-1} (\log(x/u))^{m-1} \widetilde{\ell}_\xi(x)dx \tag{6.17}$$

where $\widetilde{\ell}_\xi(u)$ *is a function measurable in* ξ *and for* $\lambda - a.e.\ \xi$. *nonincreasing in* $u \in (0,\infty)$, *not identically zero and* $\lim_{u\to\infty} \widetilde{\ell}_\xi(u) = 0$.

(ii) $(\alpha = 1)$ $\widetilde{\mu}_{m+1}$ *is centered Gaussian, or* $\widetilde{\mu}_{m+1}$ *is non-Gaussian and* $\widetilde{\nu}^{(m+1)}$ *satisfies* (6.16), (6.17) *with* $\alpha = 1$ *and*

$$-\lim_{\varepsilon\downarrow 0}((m-1)!)^{-1} \int_\varepsilon^1 s^{-1}ds \int_0^1 t^{-1}(\log t^{-1})^{m-1}dt \int_{\mathbf{R}^d} \frac{(st)^2 x|x|^2}{1+(st)^2|x|^2}\nu(dx) \tag{6.18}$$

exists in $\mathbf{R}^d$ *and equals* $\widetilde{\gamma}^{(m+1)}$. *Here the measure* ν *is the one in* (6.14).

(iii) $(1 < \alpha < 2)$ $\widetilde{\mu}_{m+1}$ *is centered Gaussian, or* $\widetilde{\mu}_{m+1}$ *is non-Gaussian and* $\widetilde{\nu}^{(m+1)}$ *has expression* (6.16), (6.17) *and*

$$\int_{\mathbf{R}^d} x\widetilde{\mu}_{m+1}(dx) = 0\,. \tag{6.19}$$

As seen in the proof of Lemma 6.6, the function $\widetilde{\ell}_\xi(x)$ is given by

$$\widetilde{\ell}_\xi(x) = \int_x^\infty r^\alpha \nu_\xi(dr)\,,$$

where ν_ξ is the radial component of the Lévy measure ν of $\mu \in \mathfrak{D}(\Phi_\alpha^{m+1})$.

A function $f(t)$ defined for $t > 0$ is called m-times monotone where m is an integer, $m \geq 2$, if $(-1)^k f^{(k)}(t)$ is nonnegative, nonincreasing and convex for $t > 0$, and for $k = 0, 1, 2, \ldots, m-2$. When $m = 1$, $f(t)$ will simply be nonnegative and nonincreasing.

Note that $\widetilde{h}_\xi^{(m)}(u)$ is m-times monotone. In order to see this, we have only to differentiate it in the following way.

$$\begin{aligned}
\frac{d}{ds}\widetilde{h}_\xi^{(m)}(s) &= -\frac{1}{s}\int_s^\infty \frac{1}{(m-2)!x}(\log(x/s))^{m-2}\,dx \int_x^\infty r^\alpha \nu_\xi(dr) < 0\,,\\
\frac{d^2}{ds^2}\widetilde{h}_\xi^{(m)}(s) &= \frac{1}{s^2}\int_s^\infty \frac{1}{(m-2)!x}(\log(x/s))^{m-2}\,dx \int_x^\infty r^\alpha \nu_\xi(dr)\\
&\quad + \frac{1}{s^2}\int_s^\infty \frac{1}{(m-3)!x}(\log(x/s))^{m-3}\,dx \int_x^\infty r^\alpha \nu_\xi(dr) > 0\,.
\end{aligned}$$

The differentiation continues to $m-1$ times, but $(d/ds)^{m-1}\widetilde{h}_\xi^{(m)}(s)$ includes the term

$$(-s)^{1-m}\int_s^\infty x^{-1}dx \int_x^\infty r^\alpha \nu_\xi(dr)$$

and hence $(d/ds)^m \widetilde{h}_\xi^{(m)}(s)$ includes the term

$$(-s)^{-m} \int_s^\infty r^\alpha \nu_\xi(dr)\,.$$

Then since we have no information about absolute continuity of the measure $\nu_\xi(dr)$ and differentiability of $\int_s^\infty r^\alpha \nu_\xi(dr)$ can not be guaranteed, we can not assert any stronger results for $\widetilde{h}_\xi^{(m)}(s)$ other than m-times differentiability.

We need the following lemma and here we use the same notation as before. This lemma follows from similar arguments as those used in Lemma 4.4 in Sato (2006b).

LEMMA 6.6. *Let $-\infty < \alpha < 2$ and $m = 1, 2, \ldots$, and let $\widetilde{\nu}$ be a Lévy measure. Then there exists a Lévy measure ν satisfying* (6.14) *such that*

$$\begin{cases} \int_{\mathbf{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty\,, & \text{when } \alpha < 0\,,\\ \int_{\mathbf{R}^d} (|x|^2 \wedge 1)(\log^* |x|)^{m+1}\nu(dx) < \infty\,, & \text{when } \alpha = 0\,,\\ \int_{\mathbf{R}^d} (|x|^2 \wedge |x|^\alpha)(\log^* |x|)^{m}\nu(dx) < \infty\,, & \text{when } 0 < \alpha < 2 \end{cases} \tag{6.20}$$

if and only if $\widetilde{\nu}$ is represented as (6.16).

PROOF OF LEMMA 6.6 (The "only if" part). Assume that the Lévy measure ν satisfy (6.14) and (6.20). The polar decomposition gives

$$\begin{cases} \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) < \infty\,, & \text{when } \alpha < 0\,,\\ \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1)(\log^* r)^{m+1}\nu_\xi(dr) < \infty\,, & \text{when } \alpha = 0\,,\\ \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge r^\alpha)(\log^* r)^{m}\nu_\xi(dr) < \infty\,, & \text{when } \alpha > 0\,. \end{cases} \tag{6.21}$$

Then we have for $B \in \mathcal{B}(\mathbf{R}^d)$

$$\begin{aligned} \widetilde{\nu}(B) &= (m!)^{-1} \int_0^1 t^{-\alpha-1}\nu(t^{-1}B)(\log t^{-1})^m dt \\ &= (m!)^{-1} \int_0^1 t^{-\alpha-1} dt \int_S \lambda(d\xi) \int_0^\infty 1_B(t\xi r)(\log t^{-1})^m \nu_\xi(dr) \\ &= (m!)^{-1} \int_S \lambda(d\xi) \int_0^\infty r^\alpha \nu_\xi(dr) \int_0^r 1_B(s\xi) s^{-\alpha-1}(\log(r/s))^m ds \\ &=: \int_S \lambda(d\xi) \int_0^\infty 1_B(s\xi) s^{-\alpha-1} \widetilde{h}_\xi^{(m)}(s) ds\,, \end{aligned}$$

where

$$\begin{aligned} \widetilde{h}_\xi^{(m)}(s) &= (m!)^{-1} \int_s^\infty (\log(r/s))^m r^\alpha \nu_\xi(dr) \\ &= ((m-1)!)^{-1} \int_s^\infty r^\alpha \nu_\xi(dr) \int_s^r x^{-1}(\log(x/s))^{m-1} dx \end{aligned}$$

$$= ((m-1)!)^{-1} \int_s^\infty x^{-1}(\log(x/s))^{m-1} dx \int_x^\infty r^\alpha \nu_\xi(dr)$$
$$=: ((m-1)!)^{-1} \int_s^\infty x^{-1}(\log(x/s))^{m-1} \widetilde{\ell}_\xi(x) dx\,.$$

Here $\widetilde{\ell}_\xi(u)$ is measurable in ξ and for $\lambda - a.e.\xi$. nonincreasing in $u \in (0,\infty)$, and $\lim_{u\to\infty} \widetilde{\ell}_\xi(u) = 0$ from (6.21).

(The "if" part) Suppose that $\widetilde{\nu}$ satisfies (6.10). We consider the case $-\infty < \alpha < 0$. Then since $h_\xi^{(m)}(r)$ is a continuous decreasing function, we can define a measure $\widetilde{R}_\xi$ on $(0,\infty)$ satisfying $\widetilde{R}_\xi((r,s]) = -\widetilde{h}_\xi^{(m)}(s) + \widetilde{h}_\xi^{(m)}(r)$ and put $\nu_\xi(dr) = r^{-\alpha}\widetilde{R}_\xi(dr)$. Furthermore define

$$\nu(B) = \int_S \widetilde{\lambda}(dr) \int_0^\infty 1_B(r\xi)\nu_\xi(dr)\,.$$

Here the same logic as in the proof of Lemma 5.1 holds and we see that (6.20).

In the following, similar to the proof of Lemma 5.1, we put $\widetilde{R}_\xi([r,\infty)) = \widetilde{\ell}_\xi(r+)$ and $\nu_\xi(dr) = r^{-\alpha}\widetilde{R}_\xi(dr)$. Furthermore define

$$\nu(B) = \int_S \widetilde{\lambda}(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr)\,.$$

Then for the case $\alpha = 0$, let $\lambda = \widetilde{\lambda}$, and we have

$$\begin{aligned}
&\int_{\mathbf{R}^d} (|x|^2 \wedge 1)(\log^* |x|)^{m+1} \nu(dx) \\
&\quad = \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge 1)(\log^* r)^{m+1} \nu_\xi(dr) \\
&\quad = \int_S \lambda(d\xi) \bigg(\int_0^1 r^2 \widetilde{R}_\xi(dr) + \int_1^\infty (\log^* r)^{m+1} \widetilde{R}_\xi(dr) \bigg).
\end{aligned}$$

Since $\widetilde{\nu}$ is a Lévy measure, it follows that

$$\int_S \lambda(d\xi)((m-1)!)^{-1} \int_0^\infty (r^2 \wedge 1) r^{-1} dr \int_r^\infty x^{-1}(\log(x/r))^{m-1} \widetilde{\ell}_\xi(x) dx < \infty\,.$$

Then a simple calculation gives

$$\begin{aligned}
0 &< ((m-1)!)^{-1} \int_0^1 r dr \int_r^\infty x^{-1}(\log(x/r))^{m-1} dx \int_x^\infty \widetilde{R}_\xi(dy) \\
&= ((m-1)!)^{-1} \int_0^1 r dr \int_r^\infty \widetilde{R}_\xi(dy) \int_r^y x^{-1}(\log(x/r))^{m-1} dx \\
&= (m!)^{-1} \int_0^1 r dr \int_r^\infty (\log(y/r))^m \widetilde{R}_\xi(dy)
\end{aligned}$$

$$
\begin{aligned}
&= (m!)^{-1} \int_0^1 \widetilde{R}_\xi(dy) \int_0^y r(\log(y/r))^m dr \\
&\quad + (m!)^{-1} \int_1^\infty \widetilde{R}_\xi(dy) \int_0^1 r(\log(y/r))^m dr < \infty .
\end{aligned}
$$

Since the last two integrals are positive and

$$
\begin{aligned}
&(m!)^{-1} \int_0^1 \widetilde{R}_\xi(dy) \int_0^y r(\log(y/r))^m dr \\
&\qquad = (m!)^{-1} \int_0^1 t(\log t^{-1})^m dt \int_0^1 y^2 \widetilde{R}_\xi(dy) ,
\end{aligned}
$$

the finiteness of $\int_S \lambda(d\xi) \int_0^1 r^2 \widetilde{R}_\xi(dr)$ is shown. Next we see that

$$
\begin{aligned}
0 &< ((m-1)!)^{-1} \int_1^\infty r^{-1} dr \int_r^\infty x^{-1}(\log(x/r))^{m-1} dx \int_x^\infty \widetilde{R}_\xi(dy) \\
&= ((m-1)!)^{-1} \int_1^\infty r^{-1} dr \int_r^\infty \widetilde{R}_\xi(dy) \int_r^y x^{-1}(\log(x/r))^{m-1} dx \\
&= (m!)^{-1} \int_1^\infty r^{-1} dr \int_r^\infty \left[(\log(x/r))^m\right]_r^y \widetilde{R}_\xi(dy) \\
&= (m!)^{-1} \int_1^\infty \widetilde{R}_\xi(dy) \int_1^y r^{-1}(\log(y/r))^m dr \\
&= ((m+1)!)^{-1} \int_1^\infty (\log y)^{m+1} \widetilde{R}_\xi(dy) < \infty .
\end{aligned}
$$

Hence, we have (6.20).

When $0 < \alpha < 2$, let $\lambda = \widetilde{\lambda}$ and we have

$$
\begin{aligned}
&\int_0^\infty (|x|^2 \wedge |x|^\alpha)(\log^* |x|)^m \nu(dx) \\
&\qquad = \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge r^\alpha)(\log^* r)^m \nu_\xi(dr) \\
&\qquad = \int_S \lambda(d\xi) \int_0^\infty (r^2 \wedge r^\alpha)(\log^* r)^m r^{-\alpha} \widetilde{R}_\xi(dr) \\
&\qquad = \int_S \lambda(d\xi) \left(\int_0^1 r^{2-\alpha} \widetilde{R}_\xi(dr) + \int_1^\infty (\log^* r)^m \widetilde{R}_\xi(dr) \right).
\end{aligned}
$$

Since $\widetilde{\nu}$ is a Lévy measure. We have

$$
\int_S \lambda(d\xi)((m-1)!)^{-1} \int_0^\infty (r^2 \wedge 1) r^{-\alpha-1} dr \int_r^\infty x^{-1}(\log(x/r))^{m-1} \widetilde{\ell}_\xi(x) dx < \infty .
$$

Thus

$$\begin{aligned}0 &< ((m-1)!)^{-1}\int_0^1 r^{1-\alpha}dr\int_r^\infty x^{-1}(\log(x/r))^{m-1}dx\int_x^\infty \widetilde{R}_\xi(dy)\\&= ((m-1)!)^{-1}\int_0^1 r^{1-\alpha}dr\int_r^\infty \widetilde{R}_\xi(dy)\int_r^y x^{-1}(\log(x/r))^{m-1}dx\\&= (m!)^{-1}\int_0^1 \widetilde{R}_\xi(dy)\int_0^y r^{1-\alpha}(\log(y/r))^m dr\\&\quad + (m!)^{-1}\int_1^\infty \widetilde{R}_\xi(dy)\int_0^1 r^{1-\alpha}(\log(y/r))^m dr < \infty\,.\end{aligned}$$

The first term in the right-hand side equals

$$(m!)^{-1}\int_0^1 y^{2-\alpha}\widetilde{R}_\xi(dy)\int_0^1 t^{1-\alpha}(\log t^{-1})^m dt\,.$$

Furthermore,

$$\begin{aligned}0 &< ((m-1)!)^{-1}\int_1^\infty r^{-\alpha-1}dr\int_r^\infty x^{-1}(\log(x/r))^{m-1}\widetilde{\ell}_\xi(x)dx\\&= ((m-1)!)^{-1}\int_1^\infty r^{-\alpha-1}dr\int_r^\infty x^{-1}(\log(x/r))^{m-1}dx\int_x^\infty \widetilde{R}_\xi(dy)\\&= ((m-1)!)^{-1}\int_1^\infty r^{-\alpha-1}dr\int_r^\infty \widetilde{R}_\xi(dy)\int_r^y x^{-1}(\log(x/r))^{m-1}dx\\&= (m!)^{-1}\int_1^\infty r^{-\alpha-1}dr\int_r^\infty (\log(y/r))^m\widetilde{R}_\xi(dy)\\&= (m!)^{-1}\int_1^\infty \widetilde{R}_\xi(dy)\int_1^y r^{-\alpha-1}(\log(y/r))^m dr < \infty\,.\end{aligned}$$

Here with the integral by parts formula

$$(m!)^{-1}\int_1^y r^{-\alpha-1}(\log(y/r))^m dr$$

is a linear combination of $(\log y)^k$, $k = 0, \dots, m$, and the coefficient of $(\log y)^m$ is positive. Thus, du to that $\widetilde{R}_\xi([1,\infty)) < \infty$ a.s. ξ, (6.20) holds. □

PROOF OF THEOREM 6.5. (i) ($-\infty < \alpha < 1$) (The "only if" part) Assume that $\widetilde{\mu}_{m+1} \in \Phi_\alpha^{m+1}(\mathfrak{D}(\Phi_\alpha^{m+1}))$. The Gaussian case is obvious. When $\widetilde{\mu}_{m+1}$ is non-Gaussian, then from Lemma 6.6 there exists $\widetilde{\nu}$ satisfying (6.16) and (6.17).

(The "if" part) If $\widetilde{\mu}_{m+1}$ is Gaussian, then putting $A = (2-\alpha)^m\widetilde{A}^{(m+1)}$, $\nu = 0$ and $\gamma = (1-\alpha)^m\widetilde{\gamma}^{(m+1)}$, we have $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\alpha^{m+1})$ and $\widetilde{\mu}_{m+1} = \Phi_\alpha^{m+1}(\mu)$.

If $\widetilde{\mu}_{m+1}$ is non-Gaussian, then (6.16) and (6.17) give the measure ν in Lemma 6.6. We put $A = (2-\alpha)^m \widetilde{A}^{(m+1)}$ and

$$\gamma = (1-\alpha)^m \left(\widetilde{\gamma}^{(m+1)} + (m!)^{-1} \int_0^1 s^{-\alpha} (\log s^{-1})^m ds \int_{\mathbf{R}^d} x \left(\frac{1}{1+|x|^2} - \frac{1}{1+|sx|^2} \right) \nu(dx) \right)$$

The existence of γ is proved as follows. Let c_1 and c_2 be some positive constants. Then

$$\begin{aligned}
&\int_0^1 s^{-\alpha} (\log s^{-1})^m ds \int_{\mathbf{R}^d} |x| \left| \frac{1}{1+|x|^2} - \frac{1}{1+s^2|x|^2} \right| \nu(dx) \\
&\quad \le \int_0^1 s^{-\alpha} (\log s^{-1})^m ds \left(\int_{|x|\le 1} \frac{|x|^3}{(1+|x|^2)(1+s^2|x|^2)} \nu(dx) \right. \\
&\qquad \left. + c_1 \int_{|x|>1, |sx|\le 1} |x| \nu(dx) + c_2 \int_{|x|>1, |sx|>1} s^{-2} |x|^{-1} \nu(dx) \right) \\
&\quad = \int_0^1 s^{-\alpha} (\log s^{-1})^m ds \int_{|x|\le 1} |x|^3 \nu(dx) \\
&\qquad + c_1 \int_{|x|>1} |x| \nu(dx) \int_0^{1/|x|} s^{-\alpha} (\log s^{-1})^m ds \\
&\qquad + c_2 \int_{|x|>1} |x|^{-1} \nu(dx) \int_{1/|x|}^1 s^{-\alpha-2} (\log s^{-1})^m ds\,.
\end{aligned}$$

Here by the integral by parts formula, $\int_0^{1/|x|} s^{-\alpha} (\log s^{-1})^m ds$ is shown to be constructed by a linear combination of $|x|^{\alpha-1} (\log |x|)^k$ with $k = 0, 1, \ldots, m$ and $\int_{1/|x|}^1 s^{-\alpha-2} (\log s^{-1})^m ds$ is shown to be constructed by a linear combination of $|x|^{\alpha+1} (\log |x|)^k$ with $k = 0, 1, \ldots, m$. Then on behalf of (6.20) we can prove the existence of γ. Thus $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\alpha^{m+1})$ and $\Phi_\alpha^{m+1}(\mu) = \widetilde{\mu}_{m+1}$.

(ii) ($\alpha = 1$) (The "only if" part) Suppose that $\widetilde{\mu}_{m+1} = \Phi_1^{m+1}(\mu)$ and $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_1^{m+1})$. First assume that $\widetilde{\mu}_{m+1}$ is Gaussian. Then for given $\varphi \in C_\#^+$ (see the beginning of Proof of Theorem 4.7 for its definition),

$$0 = \int_0^1 s^{-2} (\log s^{-1})^m ds \int_{\mathbf{R}^d} \varphi(s^{-1}x) \nu(dx)\,,$$

which implies $0 = s^{-2} (\log s^{-1})^m \int_{\mathbf{R}^d} \varphi(s^{-1}x) \nu(dx)$. Since by the dominated convergence theorem $s^{-2} (\log s^{-1})^m \int_{\mathbf{R}^d} \varphi(s^{-1}x) \nu(dx)$ is continuous in s, letting $s = 1/2$, we have $\nu = 0$. This together with $\gamma = 0$ (which follows from Proposition 4.3) implies $\widetilde{\gamma}^{(m+1)} = 0$. Hence $\widetilde{\mu}_{m+1}$ is centered Gaussian. If $\widetilde{\mu}_{m+1}$ is non-Gaussian, then Lemma 6.6 assures the existence of a measure $\widetilde{\nu}^{(m+1)}$ such that satisfies (6.16) and (6.17) with $\alpha = 1$. As for $\widetilde{\gamma}^{(m+1)}$, since $\widetilde{\gamma}^{(m)}$ satisfies the condition of $\Phi_1^m(\mu) \in \mathfrak{D}(\Phi_1)$, zero mean condition of the domain $\mathfrak{D}(\Phi_1)$

yields

$$\widetilde{\gamma}^{(m)} = -\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2} \widetilde{\nu}^{(m)}(dx)\,.$$

In fact we can show $\int_{\mathbf{R}^d} \frac{|x|^3}{1+|x|^2} \widetilde{\nu}^{(m)}(dx) < \infty$ by Proposition 4.3 (iv). Then the general result of mapping Φ_1 yields

$$\begin{aligned}\widetilde{\gamma}^{(m+1)} &= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 s^{-1}ds \left(\widetilde{\gamma}^{(m)} - \int_{\mathbf{R}^d} x \left(\frac{1}{1+|x|^2} - \frac{1}{1+s^2|x|^2} \right) \widetilde{\nu}^{(m)}(dx) \right) \\ &= -\lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 s^{-1}ds \int_{\mathbf{R}^d} \frac{x|x|^2}{1+s^2|x|^2} \widetilde{\nu}^{(m)}(dx),\end{aligned}$$

which is equivalent to (6.18).

(The "if" part) If $\widetilde{\mu}_{m+1}$ is centered Gaussian, then $\widetilde{\mu}_{m+1} \in \Phi_\alpha^{m+1}(\mathfrak{D}(\Phi_\alpha^{m+1}))$ by Theorem 4.6. If $\widetilde{\mu}_{m+1}$ is non-Gaussian and satisfies (6.16) and (6.17) with $\alpha = 1$, then Lemma 6.6 assures the existence of a measure ν satisfying (6.14) and (6.20) with $\alpha = 1$. Let $\gamma = -\int_{\mathbf{R}^d} \frac{x|x|^2}{1+|x|^2} \nu(dx)$, $A = \widetilde{A}^{(m+1)}$ and $\mu = \mu_{(A,\nu,\gamma)}$. We see that under the assumption of the moment condition (6.20), the existence of $\widetilde{\gamma}^{(m+1)}$ is equivalent to that of

$$\lim_{T\to\infty} \int_1^T t^{-1}dt \int_{|x|>t} x(\log(|x|t^{-1}))^m \nu(dx) \tag{6.22}$$

which appears in the description of $\mathfrak{D}(\Phi_1^{m+1})$ in Theorem 6.1 (iv). Recall that

(6.23)

$$\widetilde{\gamma}^{(m+1)} = -((m-1)!)^{-1} \lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 s^{-1}ds \int_0^1 t^{-1}(\log t^{-1})^{m-1}dt \int_{\mathbf{R}^d} \frac{xs^2t^2|x|^2}{1+s^2t^2|x|^2} \nu(dx)\,,$$

if exists. We divide the integral range $\mathbf{R}^d$ of ν into $\{|x| \le 1\}$, $\{|x| > 1, st|x| \le 1\}$, and $\{|x| > 1, st|x| > 1\}$, and consider the finiteness of $\widetilde{\gamma}^{(m+1)}$. Observe that

$$\begin{aligned}&\int_0^1 s^{-1}ds \int_0^1 t^{-1}(\log t^{-1})^{m-1}dt \int_{|x|\le 1} \frac{s^2t^2|x|^3}{1+s^2t^2|x|^2} \nu(dx) \\ &\quad \le \int_0^1 sds \int_0^1 t(\log t^{-1})^{m-1}dt \int_{|x|\le 1} |x|^3 \nu(dx) < \infty\,.\end{aligned}$$

Regarding the integral on $\{|x| > 1, st|x| \le 1\}$, we have

$$\begin{aligned}&\int_0^1 ds \int_0^1 (st)^{-1}(\log t^{-1})^{m-1}dt \int_{|x|>1, st|x|\le 1} \frac{(st)^2|x|^3}{1+(st)^2|x|^2} \nu(dx) \\ &\quad = \int_0^1 ds \int_0^s u^{-1}(\log s/u)^{m-1}s^{-1}du \int_{|x|>1, u|x|\le 1} \frac{u^2|x|^3}{1+u^2|x|^2} \nu(dx)\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 u\,du \int_u^1 s^{-1}(\log s/u)^{m-1}du \int_{|x|>1,u|x|\le 1} |x|^3\nu(dx)\\
&= \frac{1}{m}\int_0^1 u(\log u^{-1})^m du \int_{|x|>1,u|x|\le 1} |x|^3\nu(dx)\\
&= \frac{1}{m}\int_{|x|>1} |x|^3\nu(dx)\int_0^{1/|x|} u(\log u^{-1})^m du\,.
\end{aligned}$$

Since $\int_0^{1/|x|} u(\log u^{-1})^m du$ is shown to be a linear combination of $|x|^{-2}(\log|x|)^k$ with $k = 0,\dots,m$, (6.20) assures the finiteness of the integral. Hence for the finiteness of $\widetilde{\gamma}^{(m+1)}$, we only consider the finiteness of

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 s^{-1}ds\int_0^1 t^{-1}(\log t^{-1})^{m-1}dt\int_{|x|>1,st|x|>1}\frac{(st)^2x|x|^2}{1+(st)^2|x|^2}\nu(dx) \tag{6.24}$$

which is equivalent to that of

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 s^{-1}ds\int_0^1 t^{-1}(\log t^{-1})^{m-1}dt\int_{|x|>1,st|x|>1} x\nu(dx)\,. \tag{6.25}$$

To see this, we consider the convergence of the difference of both integrals, namely consider the quantity

$$\begin{aligned}
&\int_0^1 s^{-1}ds\int_0^1 t^{-1}(\log t^{-1})^{m-1}dt\int_{|x|>1,st|x|>1}\frac{|x|}{1+s^2t^2|x|^2}\nu(dx)\\
&\quad= \int_0^1 s^{-1}ds\int_0^s u^{-1}(\log s/u)^{m-1}du\int_{|x|>1,u|x|>1}\frac{|x|}{1+u^2|x|^2}\nu(dx)\\
&\quad= \int_0^1 s^{-1}ds\int_{|x|>1,|x|>1/s}|x|^{-1}\nu(dx)\int_{1/|x|}^s u^{-3}(\log s/u)^{m-1}du\,.
\end{aligned}$$

Here $\int_{1/|x|}^s u^{-3}(\log s/u)^{m-1}du$ is shown to be a linear combination of $|x|^2(\log s|x|)^k$, $k = 0,\dots,m$ and s^{-2}. Noticing $|x|^2(\log s|x|)^k \le |x|^2(\log|x|)^k$, observe that

$$\begin{aligned}
\int_0^1 s^{-1}ds\int_{|x|>1,|x|>1/s}|x|(\log|x|)^k\nu(dx) &= \int_{|x|>1}|x|(\log|x|)^k\nu(dx)\int_{1/|x|}^1 s^{-1}ds\\
&= \int_{|x|>1}|x|(\log|x|)^{k+1}\nu(dx) < \infty
\end{aligned}$$

and that

$$\begin{aligned}
\int_0^1 s^{-3}ds\int_{|x|>1,|x|>1/s}|x|^{-1}\nu(dx) &= \int_{|x|>1}|x|^{-1}\nu(dx)\int_{1/|x|}^1 s^{-3}ds\\
&\le \int_{|x|>1}\frac{1}{2}\left(|x|+|x|^{-1}\right)\nu(dx) < \infty\,,
\end{aligned}$$

and we conclude the convergence. Then by a further calculation, (6.25) is equivalent to

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 s^{-1}ds\int_0^s u^{-1}(\log s/u)^{m-1}du\int_{|x|>1,u|x|>1} x\nu(dx)\\
&\quad=\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 s^{-1}ds\int_{|x|>1} x\nu(dx)\int_{1/|x|}^s u^{-1}(\log s/u)^{m-1}du\\
&\quad=\lim_{\varepsilon\downarrow 0}\frac{1}{m}\int_\varepsilon^1 s^{-1}ds\int_{|x|>1/s} x(\log s|x|)^m\nu(dx)\,,
\end{aligned}$$

which is equivalent to (6.22) by the change of variables formula. In the first equality we use Fubini's theorem with (6.20). As a consequence, we have $\mu\in\mathfrak{D}(\Phi_1^{m+1})$ and the mapping $\Phi_1^{m+1}(\mu)$ recover (6.13), (6.14) and (6.18). Now we conclude that $\Phi_1^{m+1}(\mu)=\widetilde{\mu}_{m+1}$ and $\widetilde{\mu}_{m+1}\in\Phi_1^{m+1}(\mathfrak{D}(\Phi_1^{m+1}))$. In order to recover (6.14), we use the induction method as in the proof of Theorem 6.2.

(iii) $(1<\alpha<2)$ (The "only if" part) Assume that $\widetilde{\mu}_{m+1}=\Phi_\alpha^{m+1}(\mu)$ with some $\mu=\mu_{(A,\nu,\gamma)}\in\mathfrak{D}(\Phi_\alpha^{m+1})$. The Gaussian case is the same as that in the proof for (ii). If $\widetilde{\mu}_{m+1}$ is non-Gaussian, then it follows from Lemma 6.6 that there exists $\widetilde{\nu}^{(m+1)}$ satisfying (6.16) and (6.17). Since $\mu\in\mathfrak{D}(\Phi_\alpha^{m+1})$, ν and γ satisfy $\int_{|x|>1}|x|^\alpha(\log|x|)^m\nu(dx)<\infty$ and $\gamma=-\int_{\mathbf{R}^d}\frac{x|x|^2}{1+|x|^2}\nu(dx)$, respectively. We show the existence of $\widetilde{\gamma}^{(m+1)}$, we have

$$\begin{aligned}
&(m!)^{-1}\int_0^1 t^{2-\alpha}(\log t^{-1})^m dt\int_{\mathbf{R}^d}\frac{|x|^3}{1+t^2|x|^2}\nu(dx)\\
&\qquad\le (m!)^{-1}\int_0^1 t^{2-\alpha}(\log t^{-1})^m dt\int_{|x|\le 1}|x|^3\nu(dx)\\
&\qquad\quad+(m!)^{-1}\int_{|x|>1}|x|^3\nu(dx)\int_0^{1/|x|} t^{2-\alpha}(\log t^{-1})^m dt\\
&\qquad\quad+(m!)^{-1}\int_{|x|>1}|x|\nu(dx)\int_{1/|x|}^1 t^{-\alpha}(\log t^{-1})^m dt\,.
\end{aligned}$$

Here by the integral by parts formula, $\int_0^{1/|x|}t^{2-\alpha}(\log t^{-1})^m dt$ is shown to be a linear combination of $|x|^{\alpha-3}(\log|x|)^k$ with $k=0,1,\dots,m$ and $\int_{1/|x|}^1 t^{-\alpha}(\log t^{-1})^m dt$ is shown to be a linear combination of $|x|^{\alpha-1}(\log|x|)^k$ with $k=0,1,\dots,m$ and a constant. Then form (6.20) $\widetilde{\gamma}$ exists and equals to

$$\begin{aligned}
\widetilde{\gamma}&=-\int_0^\infty \varepsilon_{\alpha,m}^*(t)dt\int_{\mathbf{R}^d}\frac{x|\varepsilon_{\alpha,m}^*(t)x|^2}{1+|\varepsilon_{\alpha,m}^*(t)x|^2}\nu(dx)\\
&=-\int_{\mathbf{R}^d}\frac{x|x|^2}{1+|x|^2}\widetilde{\nu}(dx)\,,
\end{aligned}$$

which is (6.19).

(The "if" part) If $\widetilde{\mu}_{m+1}$ is centered Gaussian with its component $\widetilde{A}^{(m+1)}$. Then by putting $A = (2-\alpha)^{m+1}\widetilde{A}^{(m+1)}$, we have $\mu_{(A,0,0)} \in \mathfrak{D}(\Phi_\alpha^{m+1})$ and $\widetilde{\mu}_{m+1} = \Phi_\alpha^{m+1}(\mu)$. Suppose $\widetilde{\mu}$ be non-Gaussian and satisfy condition of (iii). On behalf of Lemma 6.6, we have a measure ν satisfying (6.14) and (6.20). We investigate the absolute moment of $\widetilde{\nu}$ and see that

$$\begin{aligned}\int_{\mathbf{R}^d} \frac{|x|^3}{1+|x|^2}\widetilde{\nu}^{(m+1)}(dx) &= \int_0^\infty \varepsilon^*_{\alpha,m}(t)dt \int_{\mathbf{R}^d} \frac{|x|^3}{1+|\varepsilon^*_{\alpha,m}(t)x|^2}\nu(dx)\\ &= \frac{1}{m!}\int_0^1 s^{2-\alpha}(\log s^{-1})^m ds \int_{\mathbf{R}^d}\frac{|x|^3}{1+s^2|x|^2}\nu(dx)\,.\end{aligned}$$

Then as we have seen in the preceding paragraph,

$$\int_{\mathbf{R}^d} \frac{|x|^3}{1+|x|^2}\widetilde{\nu}^{(m+1)}(dx) < \infty\,.$$

Thus $\int_{\mathbf{R}^d}|x|\widetilde{\mu}(dx) < \infty$, and hence $\int_{\mathbf{R}^d} x\widetilde{\mu}^{(m+1)}(dx) = 0$. Let $\gamma = -\int_{\mathbf{R}^d}\frac{x|x|^2}{1+|x|^2}\nu(dx)$ and $A = (2-\alpha)^{m+1}\widetilde{A}^{(m+1)}$. Then on behalf of (6.14), Theorem 6.1 (v) is satisfied. Thus $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_\alpha^{m+1})$. □

REMARK 6.7. (A remark on the case $\alpha = 1$) We explain why we did not treat the case $\alpha = 1$ in Theorem 6.2. When $\alpha = 1$, $\overline{\gamma}_m$ in (6.7) with $\gamma = -\int_{\mathbf{R}^d}\frac{x|x|^2}{1+|x|^2}\nu(dx)$ may be different from $\widetilde{\gamma}^{(m+1)}$ in (6.18) given by the mapping Φ_1^{m+1} in the following sense. We have

$$\begin{aligned}\overline{\gamma}_m &= -\lim_{\varepsilon\downarrow 0}(m!)^{-1}\int_\varepsilon^1 t^{-1}(\log t^{-1})^m \int_{\mathbf{R}^d}\frac{t^2x|x|^2}{1+t^2|x|^2}\nu(dx)\\ &= -\lim_{\varepsilon\downarrow 0}((m-1)!)^{-1}\int_\varepsilon^1 t^{-1}dt\int_t^1 s^{-1}(\log s^{-1})^{m-1}ds\int_{\mathbf{R}^d}\frac{t^2x|x|^2}{1+t^2|x|^2}\nu(dx)\\ &= -\lim_{\varepsilon\downarrow 0}((m-1)!)^{-1}\int_\varepsilon^1 s^{-1}(\log s^{-1})^{m-1}ds\int_\varepsilon^s t^{-1}dt\int_{\mathbf{R}^d}\frac{t^2x|x|^2}{1+t^2|x|^2}\nu(dx)\\ &= -\lim_{\varepsilon\downarrow 0}((m-1)!)^{-1}\int_\varepsilon^1 s^{-1}(\log s^{-1})^{m-1}ds\int_{\varepsilon/s}^1 u^{-1}du\int_{\mathbf{R}^d}\frac{(su)^2x|x|^2}{1+(su)^2|x|^2}\nu(dx)\\ &= -\lim_{\varepsilon\downarrow 0}((m-1)!)^{-1}\int_\varepsilon^1 u^{-1}du\int_{\varepsilon/u}^1 s^{-1}(\log s^{-1})^{m-1}ds\int_{\mathbf{R}^d}\frac{(su)^2x|x|^2}{1+(su)^2|x|^2}\nu(dx)\,.\end{aligned}$$

On the other hand,

$$\widetilde{\gamma}^{(m+1)} = -\lim_{\varepsilon\downarrow 0}((m-1)!)^{-1}\int_\varepsilon^1 u^{-1}du\int_0^1 s^{-1}(\log s^{-1})^{m-1}ds\int_{\mathbf{R}^d}\frac{(su)^2x|x|^2}{1+(su)^2|x|^2}\nu(dx)$$

as we have seen in (6.18). In $\overline{\gamma}_m$, the convergence depends on simultaneous convergence of double integrals, whereas in $\widetilde{\gamma}^{(m+1)}$ the convergence is only concerned with the first integral since the second integral exists. Here, we cannot prove $\overline{\gamma}_m = \widetilde{\gamma}^{(m+1)}$ when $\alpha = 1$. If the integral of $\widetilde{\gamma}^{(m+1)}$ or $\overline{\gamma}_m$ would absolutely converge, then due to Fubini's theorem both would be the same. But we cannot see it now. If $\overline{\gamma}_m \neq \widetilde{\gamma}^{(m+1)}$, then the ranges of the two mappings are different.

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References

[1] O. E. BARNDORFF-NIELSEN, M. MAEJIMA and K. SATO, Some classes of multivariate infinitely divisible distributions admitting stochastic integral representations, *Bernoulli* **12** (2006), 1–33.

[2] Z. J. JUREK, The class $L_m(Q)$ of probability measures on Banach spaces, *Bull. Polish Acad. Sci. Math.* **31** (1983), 51–62.

[3] Z.J. JUREK, Relations between the *s*–selfdecomposable and selfdecomposable measures, *Ann. Probab.* **13** (1985), 592–608.

[4] Z. J. JUREK, Random integral representations for classes of limit distributions similar to Levy class L_0. *Probab. Th. Rel. Fields* **78** (1988), 473–490.

[5] Z. J. JUREK, The random integral representation hypothesis revisited : new class of s-selfdecomposable laws, in: *Abstract and Applied Analysis* ; Proc. Intern. Conf. Hanoi, 2002, World Scientific (2004), 495–514.

[6] T. A. O'CONNOR, Infinitely divisible distributions similar to class L distributions, *Z.W.* **50** (1979), 265–271.

[7] J. ROSIŃSKI, Tempering stable processes, *Stoch. Proc. Appl.* **117** (2007), 677–707.

[8] K. SATO, *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press, Cambridge, 1999.

[9] K. SATO, Additive processes and stochastic integrals, *Illinois J. Math.* **50** (2006a) (Doob Volume), 825–851.

[10] K. SATO, Two families of improper stochastic integrals with respect to Lévy processes, *ALEA Lat. Am. J. Probab. Math. Statist.* **1** (2006b), 47–87.

[11] S. J. WOLFE, On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$, *Stoch. Proc. Appl.* **12** (1982), 301–312.

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