TOKYO J. MATH.
VOL. 33, NO. 2, 2010

Compact Minimal CR Submanifolds of a Complex Projective Space with Positive Ricci Curvature

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(Communicated by M. Guest)

Abstract. We give a reduction theorem for the codimension of a compact n-dimensional minimal proper CR submanifold M immersed in a complex projective space CP^m with complex structure J, under the assumption that the Ricci curvature of M is equal to or greater than $n-1$. Moreover, we classify compact n-dimensional minimal CR submanifolds whose Ricci tensor S satisfies $S(X,X) \geq (n-1)g(X,X) + kg(PX,PX)$, $k = 0, 1, 2$, for any vector field X tangent to M, where PX is the tangential part of JX.

1. Introduction

The purpose of the present paper is to study the pinching problem in terms of Ricci curvatures of minimal CR submanifolds immersed in a complex projective space.

Let CP^m denote the complex projective space of real dimension $2m$ (complex dimension m) with constant holomorphic sectional curvature 4 and Kähler structure (J, g). Let M be a real n-dimensional Riemannian manifold isometrically immersed in CP^m with induced metric g. If there exist a differentiable holomorphic distribution $H : x \mapsto H_x \subset T_x(M)$ and complementary orthogonal anti-invariant distribution $H^\perp$, then M is called a CR submanifold. In particular, when M satisfies $JT_x(M)^\perp \subset T_x(M)$ for any point x of M, M is called a generic submanifold. Any real hypersurface is obviously generic.

In [8], Kon proved that if the Ricci tensor S of a compact n-dimensional minimal CR submanifold M of CP^m satisfies $S(X,X) \geq (n-1)g(X,X) + 2g(PX,PX)$, then M is a real projective space RP^n, or a complex projective space $CP^{n/2}$, or a pseudo-Einstein real hypersurface $\pi(S^k(1/\sqrt{2}) \times S^k(1/\sqrt{2}))$ $(k = (n+1)/2)$ of some $CP^{(n+1)/2}$ in CP^m, where $S^k(r)$ is a k-dimensional sphere of radius r, π is the Hopf fibration and PX is the tangential part of JX (see also [7]). For a minimal real hypersurface M of CP^m $(m \geq 3)$, Maeda [9] studied the pinching problem in terms of Ricci curvatures of M. He proved that if the Ricci tensor S of a minimal real hypersurface satisfies $(2m-2)g(X,X) \leq S(X,X) \leq 2mg(X,X)$, then it is locally congruent to $\pi(S^m(1/\sqrt{2}) \times S^m(1/\sqrt{2}))$.

Received September 18, 2007; revised June 29, 2009; revised July 20, 2010
Mathematics Subject Classification: 53C40, 53C55
Key words and phrases: Ricci curvature, CR submanifold, complex projective space

On the other hand, Yamagata-Kon [14] proved that if the Ricci tensor S of a compact n-dimensional minimal generic submanifold M of CP^m, which is not totally real, satisfies $S(X, X) \geq (n-1)g(X, X)$, then M is a real hypersurface of CP^m, that is, $2m - n = 1$.

In this paper, we prove a reduction theorem for the codimension of a compact n-dimensional minimal proper CR submanifold M in CP^m. We prove that if the Ricci curvature of M is equal to or greater than $n - 1$, then M is a real hypersurface of some $CP^{(n+1)/2}$ in CP^m (Theorem 2). Using this result, we classify compact n-dimensional minimal CR submanifolds M immersed in CP^m whose Ricci tensors S satisfy $S(X, X) \geq (n-1)g(X, X) + kg(PX, PX)$, $k = 0, 1, 2$, for any vector field X tangent to M (Theorem 3, 4, 5).

The author would like to express her sincere gratitude to the referee for valuable suggestions.

2. Preliminaries

Let CP^m denote the complex projective space of complex dimension m with constant holomorphic sectional curvature 4. We denote by J the complex structure, and by g the metric of CP^m.

Let M be a real n-dimensional Riemannian manifold isometrically immersed in CP^m. We denote by the same g the Riemannian metric on M induced from g, and by p the codimension of M, that is, $p = 2m - n$.

We denote by $T_x(M)$ and $T_x(M)^\perp$ the tangent space and the normal space of M at x, respectively.

DEFINITION 1. A submanifold M of a Kähler manifold $\tilde{M}$ with complex structure J is called a CR submanifold of $\tilde{M}$ if there exists a differentiable distribution $H : x \mapsto H_x \subset T_x(M)$ on M satisfying the following conditions:

(i) H is holomorphic, i.e., $JH_x = H_x$ for each $x \in M$, and

(ii) the complementary orthogonal distribution $H^\perp : x \mapsto H_x^\perp \subset T_x(M)$ is anti-invariant, i.e. $JH_x^\perp \subset T_x(M)^\perp$ for each $x \in M$.

In the following, we put $\dim H_x = h$ and $\dim H_x^\perp = q$. If $q = 0$ (resp. $h = 0$), then the CR submanifold M is a complex submanifold (resp. totally real submanifold) of $\tilde{M}$. If $h > 0$ and $q > 0$, then a CR submanifold M is said to be *proper*.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in CP^m, and by ∇ that in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \tilde{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X and Y tangent to M and any vector field V normal to M, where D denotes the normal connection. We call both A and B the *second fundamental form* of M and are related by $g(B(X, Y), V) = g(A_V X, Y)$. The second fundamental forms A and B are symmetric with respect to X and Y.

The mean curvature vector of M is defined to be the trace of the second fundamental form B, that is, $\mathrm{tr}B = \sum_i B(e_i, e_i)$, $\{e_i\}$ being an orthonormal basis of $T_x(M)$. If the mean curvature vector vanishes identically, then M is said to be *minimal*.

The covariant derivative $(\nabla_X A)_V Y$ of A is defined by

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V \nabla_X Y\,.$$

If $(\nabla_X A)_V Y = 0$ for any vector fields X and Y tangent to M, then the second fundamental form of M is said to be *parallel in the direction of the normal vector* V. If the second fundamental form is parallel in any direction, it is said to be *parallel*. A vector field V normal to M is said to be *parallel* if $D_X V = 0$ for any vector field X tangent to M.

For $x \in M$, the *first normal space* $N_1(x)$ is the orthogonal complement in $T_x(M)^\perp$ of the set $N_0(x) = \{V \in T_x(M)^\perp : A_V = 0\}$. If $D_X V \in N_1(x)$ for any vector field V with $V_x \in N_1(x)$ and any vector field X of M at x, then the first normal space $N_1(x)$ is said to be parallel with respect to the normal connection.

In the sequel, we assume that M is a CR submanifold of CP^m. The tangent space $T_x(M)$ of M is decomposed as $T_x(M) = H_x + H_x^\perp$ at each point x of M. Similarly, we see that $T_x(M)^\perp = JH_x^\perp + N_x$, where N_x is the orthogonal complement of $JH_x^\perp$ in $T_x(M)^\perp$.

For any vector field X tangent to M, we put

$$JX = PX + FX\,,$$

where PX is the tangential part of JX and FX the normal part of JX. For any vector field V normal to M, we put

$$JV = tV + fV\,,$$

where tV is the tangential part of JV and fV the normal part of JV. Then we see that $FP = 0$, $fF = 0$, $tf = 0$ and $Pt = 0$.

We define the covariant derivatives of P, F, t and f by $(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y$, $(\nabla_X F)Y = D_X(FY) - F\nabla_X Y$, $(\nabla_X t)V = \nabla_X(tV) - tD_X V$ and $(\nabla_X f)V = D_X(fV) - fD_X V$, respectively. We then have

$$(\nabla_X P)Y = A_{FY}X + tB(X, Y)\,, \quad (\nabla_X F)Y = -B(X, PY) + fB(X, Y)\,,$$
$$(\nabla_X t)V = -PA_V X + A_{fV}X, \quad (\nabla_X f)V = -FA_V X - B(X, tV)\,.$$

For any vectors X and Y in $H_x^\perp = tT_x(M)^\perp$, we obtain $A_{FX}Y = A_{FY}X$.

The Riemannian curvature tensor $\tilde{R}$ of a complex projective space CP^m is given by

$$\begin{aligned}\tilde{R}(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ & - g(JX, Z)JY + 2g(X, JY)JZ\end{aligned}$$

for any vector fields X, Y and Z of CP^m. Then the *equation of Gauss* and the *equation of Codazzi* are given respectively by

$$\begin{aligned}R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY\\ -2g(PX,Y)PZ + A_{B(Y,Z)}X - A_{B(X,Z)}Y\end{aligned}$$

and

$$\begin{aligned}&g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z)\\ &\quad = g(Y,PZ)g(X,tV) - g(X,PZ)g(Y,tV) - 2g(X,PY)g(Z,tV)\end{aligned}$$

for any vector fields X, Y and Z tangent to M and V normal to M, where R is the Riemannian curvature tensor field of M.

We denote by S the Ricci tensor field of M. Then

$$\begin{aligned}S(X,Y) = (n-1)g(X,Y) + 3g(PX,PY)\\ + \sum_a \operatorname{tr} A_a g(A_a X, Y) - \sum_a g(A_a^2 X, Y)\,,\end{aligned}$$

where A_a is the second fundamental form in the direction of v_a, $\{v_1, \dots, v_p\}$ being an orthonormal basis of $T_x(M)^\perp$, and tr denotes the trace of an operator. If the Ricci tensor S satisfies $S(X,Y) = \alpha g(X,Y)$ for some constant α, then M is called an *Einstein manifold*. When M is a real hypersurface of CP^m with a unit normal vector field V, if the Ricci tensor S satisfies $S(X,Y) = \alpha g(X,Y) + \beta g(X,tV)g(Y,tV)$ for some constants α and β, then M is said to be *pseudo-Einstein*.

We define the curvature tensor $R^\perp$ of the normal bundle $T(M)^\perp$ of M by

$$R^\perp(X,Y)V = D_X D_Y V - D_Y D_X V - D_{[X,Y]}V\,.$$

Then we have the *equation of Ricci*:

$$\begin{aligned}&g(R^\perp(X,Y)V,U) + g([A_U, A_V]X, Y)\\ &\quad = g(FY,V)g(FX,U) - g(FX,V)g(FY,U) + 2g(X,PY)g(fV,U)\,,\end{aligned}$$

where [,] denotes the commutator and $[A_V, A_U] = A_V A_U - A_U A_V$.

We need the following examples of CR submanifolds in CP^m.

EXAMPLE 1 ([1]). An n-dimensional complete totally geodesic submanifold M of CP^m is either a complex projective space $CP^{n/2}$ or a real projective space RP^n of constant curvature 1. A real projective space RP^n is a totally real submanifold of CP^m.

EXAMPLE 2. Let $z^0, z^1, \dots, z^m$ be homogeneous coordinates of CP^m. The *complex quadric* Q^{m-1} is a complex hypersurface of CP^m defined by the equation

$$(z^0)^2 + (z^1)^2 + \cdots + (z^m)^2 = 0\,.$$

Then Q^{m-1} is a Kähler manifold. Moreover, Q^{m-1} is an Einstein manifold with Ricci curvature $2(m-1)$ (see [13]).

EXAMPLE 3. For an integer k and for $0 < r < \pi/2$, we define $M(k, r)$ in S^{2m+1} by

$$\sum_{j=0}^{k} |z_j|^2 = \cos^2 r, \qquad \sum_{j=k+1}^{m} |z_j|^2 = \sin^2 r\,.$$

$M(k, r)$ is a standard product $S^{2k+1}(\cos r) \times S^{2l+1}(\sin r)$, $l = m - k - 1$. We consider the Hopf fibration $\pi : S^{2m+1} \longrightarrow CP^m$, where S^{2m+1} denotes the unit sphere. Then $M^c(k, r) = \pi(M(k, r))$ is a real hypersurface in CP^m. For an integer $1 \le k \le m-2$, we see that $M^c(k, r)$ is the tube of radius r over CP^k (see [3]).

When r satisfies $\cos r = \sqrt{(2k+1)/(2m)}$ and $\sin r = \sqrt{(2l+1)/(2m)}$, $M^c(k, r)$ is a minimal real hypersurface of CP^m. Moreover, we see that $M^c(k, r)$ is a pseudo-Einstein real minimal hypersurface of CP^m if and only if $k = l = (m-1)/2$ and $r = \pi/4$. Then the Ricci tensor S satisfies $S(X, Y) = (2m-2)g(X, Y) + 2g(PX, PY)$.

3. Integral formula

In this section, for later use, we compute the Laplacian for the square of the length of the second fundamental form A of an n-dimensional minimal submanifold M immersed in a complex projective space CP^m. In the following, we put $\nabla_i = \nabla_{e_i}$ and $D_i = D_{e_i}$, where $\{e_i\}$ being an orthonormal basis of M, to simplify the notation. We use the following (see Simons [12])

LEMMA 1. *Let M be a submanifold of a locally symmetric Riemannian manifold $\bar{M}$. If the mean curvature vector field of M is parallel, then*

$$\begin{aligned} g((\nabla^2 B)(X, Y), V) &= \sum_i g((\nabla_i \nabla_i B)(X, Y), V) \\ &= \sum_i (2g(\bar{R}(e_i, Y)B(X, e_i), V) + 2g(\bar{R}(e_i, X)B(Y, e_i), V) \\ &\quad - g(A_V X, \bar{R}(e_i, Y)e_i) - g(A_V Y, \bar{R}(e_i, X)e_i) + g(\bar{R}(e_i, B(X, Y))e_i, V) \\ &\quad + g(\bar{R}(B(e_i, e_i), X)Y, V) - 2g(A_V e_i, \bar{R}(e_i, X)Y)) \\ &\quad + \sum_a (\operatorname{tr} A_a g(A_V A_a X, Y) - \operatorname{tr} A_a A_V g(A_a X, Y) + 2g(A_a A_V A_a X, Y) \\ &\quad - g(A_a^2 A_V X, Y) - g(A_V A_a^2 X, Y)) \end{aligned}$$

for any vectors X, Y tangent to M and any vector V normal to M.

We compute the equation of Lemma 1 for an n-dimensional minimal submanifold M in CP^m. We notice that CP^m is locally symmetric. Using the expression of the curvature tensor $\tilde{R}$ of CP^m, we have the equation of Lemma 1 in the following:

$$\begin{aligned} &g((\nabla^2 B)(X,Y),V)\\ &=\sum_i g((\nabla_i\nabla_i B)(X,Y),V)\\ &=-2g(A_{FY}X,tV)-2g(A_{FX}Y,tV)\\ &\quad+2\sum_i g(Y,tV)g(A_{Fe_i}e_i,X)+2\sum_i g(X,tV)g(A_{Fe_i}e_i,Y)\\ &\quad-4g(A_{fV}X,PY)-4g(A_{fV}Y,PX) \qquad\qquad (1)\\ &\quad+ng(A_VX,Y)-3g(A_VX,P^2Y)-3g(A_VY,P^2X)\\ &\quad+3g(A_{FtV}X,Y)-6g(A_VPX,PY)\\ &\quad+\sum_a(-\operatorname{tr}A_aA_V g(A_aX,Y)+2g(A_aA_VA_aX,Y)\\ &\qquad-g(A_a^2A_VX,Y)-g(A_VA_a^2X,Y)).\end{aligned}$$

We have $g((\nabla^2 B)(X,Y),V)=g((\nabla^2 A)_V X,Y)$. Hence

$$\begin{aligned} &g(\nabla^2 A,A)\\ &=n\sum_a\operatorname{tr}A_a^2-3\sum_{a,b}\operatorname{tr}A_aA_b g(tv_a,tv_b)-6\sum_a\operatorname{tr}P^2A_a^2+6\sum_a(\operatorname{tr}A_aP)^2\\ &\quad+4\sum_{a,b}(g(A_atv_b,A_btv_a)-g(A_atv_a,A_btv_b))-8\sum_a\operatorname{tr}A_aA_{fa}P\\ &\quad+\sum_{a,b}(-(\operatorname{tr}A_aA_b)^2+2\operatorname{tr}(A_aA_b)^2-2\operatorname{tr}A_a^2A_b^2),\end{aligned}$$

where we put $A_{fa}=A_{fv_a}$. Moreover we obtain

$$\begin{aligned}\sum_{a,b}\operatorname{tr}A_aA_b g(tv_a,tv_b)&=\sum_a\operatorname{tr}A_a^2-\sum_{a,b}\operatorname{tr}A_aA_b g(fv_a,fv_b)\\ &=\sum_a\operatorname{tr}A_a^2-\sum_{a,b,c}\operatorname{tr}A_aA_b g(fv_a,v_c)g(fv_b,v_c)\\ &=\sum_a\operatorname{tr}A_a^2-\sum_{a,b,c,i}g(A_ae_i,A_be_i)g(v_a,fv_c)g(v_b,fv_c)\\ &=\sum_a\operatorname{tr}A_a^2-\sum_a\operatorname{tr}A_{fa}^2,\end{aligned}$$

$$2\sum_{a,b}(\operatorname{tr}A_a^2A_b^2-\operatorname{tr}(A_aA_b)^2)=\sum_{a,b}|[A_a,A_b]|^2,$$

$$2\sum_a (\mathrm{tr}(A_aP)^2 - \mathrm{tr}\,A_a^2P^2) = \sum_a |[P, A_a]|^2\,,$$

where $|\cdot|$ denotes the length of the tensor. Therefore we have the following theorem.

THEOREM 1. *Let M be an n-dimensional minimal submanifold of a complex projective space CP^m. Then we have*

$$\begin{aligned} g(\nabla^2 A, A) &\\ = (n-3)\sum_a \mathrm{tr}\,A_a^2 + 3\sum_a \mathrm{tr}\,A_{fa}^2 &\\ +4\sum_{a,b}(g(A_atv_b, A_btv_a) - g(A_atv_a, A_btv_b)) - 8\sum_a \mathrm{tr}\,A_aA_{fa}P &\\ +3\sum_a |[P, A_a]|^2 - \sum_{a,b}|[A_a, A_b]|^2 - \sum_{a,b}(\mathrm{tr}\,A_aA_b)^2\,. \end{aligned}$$

4. Reduction of the codimension

In this section we prove the following reduction theorem for the codimension.

THEOREM 2. *Let M be a compact n-dimensional minimal proper CR submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X)$ for any vector field X tangent to M, then M is a real hypersurface of some $CP^{(n+1)/2}$ in CP^m.*

First of all, we prove

LEMMA 2. *Let M be a compact n-dimensional minimal CR submanifold of CP^m which is not a complex submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X)$, then M is a real projective space RP^n or $q = 1$, that is,* $\dim H_x^\perp = 1$.

PROOF. Since M is minimal, by the assumption, we have

$$S(X, X) - (n-1)g(X, X) = 3g(PX, PX) - \sum_a g(A_a^2X, X) \geq 0\,. \tag{2}$$

If $P = 0$, then M is a totally real submanifold of CP^m. Moreover the above inequality implies that $A_a = 0$ for all a. So M is totally geodesic in CP^m, and hence M is a real projective space RP^n by a theorem of Abe [1].

We next suppose that $P \neq 0$. For any normal vector fields U and V, we have $A_UtV = 0$. Using this,

$$0 = (\nabla_X A)_UtV - A_UPA_VX + A_UA_{fV}X\,,$$

from which

$$g((\nabla_X A)_UY, tV) = g((\nabla_X A)_UtV, Y)$$

$$= g(A_U PA_V X, Y) - g(A_U A_{fV} X, Y)\,.$$

So the equation of Codazzi implies

$$-2g(X, PY)g(tU, tV) = g(A_U PA_V X, Y) + g(A_V PA_U X, Y) \\ -g(A_U A_{fV} X, Y) + g(A_{fV} A_U X, Y)\,. \tag{3}$$

Since $\sum_a g(tv_a, tv_a) = q$, it follows that

$$2\sum_a g(A_a PA_a X, PX) - \sum_a g((A_a A_{fa} - A_{fa} A_a)X, PX) \\ = 2q g(PX, PX)\,.$$

On the other hand, we have

$$S(PX, PX) = (n+2)g(PX, PX) - \sum_a g(A_a PX, A_a PX)\,.$$

These equations imply

$$\sum_a g(A_a PX, A_a PX) = \sum_a g(A_a PA_a X, PX) - \frac{1}{2}\sum_a g((A_a A_{fa} - A_{fa} A_a)X, PX) \\ +(n+2-q)g(PX, PX) - S(PX, PX)\,.$$

Thus we have, for any orthonormal basis $\{e_i\}$ of $T_x(M)$,

$$\begin{aligned} &\frac{1}{2}\sum_a |[P, A_a]|^2 \\ &= (n+2-q)h - \sum_i S(Pe_i, Pe_i) + \frac{1}{2}\sum_a \mathrm{tr}P(A_a A_{fa} - A_{fa} A_a) \\ &= -hq + \sum_a \mathrm{tr}A_a^2 + \sum_a \mathrm{tr}PA_a A_{fa}\,. \end{aligned} \tag{4}$$

By (2), we obtain $\sum_a \mathrm{tr}A_a^2 \le 3h$. From these,

$$\frac{1}{2}\sum_a |[P, A_a]|^2 \le h(3-q) + \sum_a \mathrm{tr}PA_a A_{fa}\,.$$

We take a basis $\{v_1, \ldots, v_p\}$ of $T_x(M)^\perp$ such that $\{v_1, \ldots, v_q\}$ is an orthonormal basis of $FT_x(M)$ and $\{v_{q+1}, \ldots, v_p\}$ is that of N_x. We denote by the same $\{v_1, \ldots, v_p\}$ an orthonormal normal vector fields in a neighborhood of x. By (3), we have $\sum_{\lambda=q+1}^p \mathrm{tr}PA_\lambda A_{f\lambda} = \sum_{\lambda=q+1}^p \mathrm{tr}A_\lambda PA_\lambda P$. From these and

$$\frac{1}{2}\sum_{a=1}^p |[P, A_a]|^2 = \frac{1}{2}\sum_{y=1}^q |[P, A_y]|^2 + \sum_{\lambda=q+1}^p \mathrm{tr}A_\lambda PA_\lambda P - \sum_{\lambda=q+1}^p \mathrm{tr}P^2 A_\lambda^2\,,$$

we obtain

$$0 \leq \frac{1}{2}\sum_{y=1}^{q} |[P, A_y]|^2 + \sum_{i=1}^{n}\sum_{\lambda=q+1}^{p} g(A_\lambda Pe_i, A_\lambda Pe_i) \leq h(3-q)\,.$$

Thus we see that $q \leq 3$. Suppose $q = 3$. Then, $PA_y = A_yP$ for $y = 1, 2, 3$ and $A_\lambda P = 0$ for $\lambda = 4, \ldots, p$. Hence we have $A_{fV}PX$=0 for any normal vector V and tangent vector X. Then, it follows from (3) that

$$2g(PX, PY)g(tV, tU) = g(A_UA_VPX, PY) + g(A_VA_UPX, PY)$$

for any tangent vectors X, Y and normal vectors $U, V \in FT_x(M)$. So we see that if $g(U, V) = 0$, then $A_UA_V + A_VA_U = 0$. Moreover, $A_y^2X = X$ and $g(A_yX, A_zX) = g(X, X)g(tv_y, tv_z)$ for any $X \in H_x$ and $y, z = 1, 2, 3$. We denote by H_1 and H_2 the eigenspaces of A_1 corresponding to 1 and -1, respectively. If $X \in H_1$, then $A_1A_2X = -A_2A_1X = -A_2X$ and $A_1A_3X = -A_3A_1X = -A_3X$. So we have $A_2X \in H_2$ and $A_3X \in H_2$. Similarly, if $X \in H_2$, then $A_2X \in H_1$ and $A_3X \in H_1$. We can take an orthonormal basis $\{e_i\}$ of H_1 which satisfies $A_1e_i = e_i,\ i = 1, \ldots, s$, where $s = \dim H_1 = \dim H_2$ since M is minimal. Then A_1 can be diagonalized with respect to the orthonormal basis $\{e_1, \ldots, e_s, A_2e_1, \ldots, A_2e_s, e_{2s+1}, \ldots, e_n\}$. Then, for $e_i, e_j \in H_1$,

$$g(A_2e_i, e_j) = 0, \quad g(A_2^2e_i, A_2e_j) = 0, \quad g(A_2e_i, A_2e_j) = \delta_{ij}\,.$$

So A_2 can be represented by a matrix of the form

$$A_2 = \left(\begin{array}{c|c|c} 0 & I_s & 0 \\ \hline I_s & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

where I_s denotes the identity matrix of degree s. Similarly,

$$A_3 = \left(\begin{array}{c|c|c} 0 & * & 0 \\ \hline * & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right).$$

Thus we obtain $A_2A_3 = A_3A_2$. Since $A_2A_3 + A_3A_2 = 0$, we have $A_2A_3 = A_3A_2 = 0$. Hence $A_2 = A_3^2A_2 = 0$. This is a contradiction .

Suppose $q = 2$. We have $A_{fy} = 0$ for $y = 1, 2$. Then

$$\begin{aligned}
&\sum_{y,i,j} g(\nabla_j tv_y, e_i)g(e_j, \nabla_i tv_y) \\
&\quad= \sum_{y,i,j} g(-PA_ye_j + tD_jv_y, e_i)g(-PA_ye_i + tD_iv_y, e_j) \\
&\quad= -\sum_{y,j} g(PA_ye_j, A_yPe_j) + \sum_{y,i,j} g(tD_jv_y, e_i)g(tD_iv_y, e_j)
\end{aligned}$$

$$= \sum_{y} \mathrm{tr}(PA_y)^2 + \sum_{y,z,w} g(D_{tw}v_y, v_z)g(D_{tz}v_y, v_w)$$
$$= \sum_{y} \mathrm{tr}(PA_y)^2 + \sum_{y,z} g(D_{tz}v_y, v_z)^2 ,$$

where $y, z, w = 1, 2$ and $D_{ty} = D_{tv_y}$. On the other hand, we have

$$\begin{aligned}\sum_{y}(\mathrm{div}\, tv_y)^2 &= \sum_{y,i,j} g(\nabla_i tv_y, e_i)g(\nabla_j tv_y, e_j)\\ &= \sum_{y,i,j} g(-PA_y e_i + tD_i v_y, e_i)g(-PA_y e_j + tD_j v_y, e_j)\\ &= \sum_{y,i,j} g(tD_i v_y, e_i)g(tD_j v_y, e_j)\\ &= \sum_{y,z} g(D_{tz}v_y, v_z)^2 .\end{aligned}$$

Since S satisfies

$$\begin{aligned}&\mathrm{div}(\nabla_X X) - \mathrm{div}((\mathrm{div} X)X)\\ &\quad = S(X, X) + \sum_{i,j} g(\nabla_j X, e_i)g(e_j, \nabla_i X) - (\mathrm{div}\, \mathrm{X})^2\end{aligned}$$

for any tangent vector field X (cf. [15; p. 44]), it follows that

$$\begin{aligned}&\sum_{y}(\mathrm{div}(\nabla_{ty} tv_y) - \mathrm{div}((\mathrm{div} tv_y)tv_y))\\ &= \sum_{y} S(tv_y, tv_y) + \sum_{y} \mathrm{tr}(PA_y)^2\\ &= 2(n-1) + \frac{1}{2}\sum_{y} |[P, A_y]|^2 + \sum_{y} \mathrm{tr}(P^2 A_y^2)\\ &= 2(n-1) - 2h + \sum_{y} \mathrm{tr} A_y^2 + \sum_{y} \mathrm{tr} PA_y A_{fy} + \sum_{y} \mathrm{tr}(P^2 A_y^2)\\ &\geq 2 .\end{aligned}$$

Here we used (4) and $fv_y = 0$. However, since M is compact, this is a contradiction. So we have $q = 1$. □

If M is proper, then $h > 0$ and $q > 0$. Then Lemma 2 reduces to

LEMMA 3. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X)$, then $q = 1$, that is,* $\dim H_x^{\perp} = 1$.

In the following, we shall prove that the first normal space of M is just $FH_x^\perp$ and is of dimension 1 under the condition of Lemma 3. To prove this, we prepare some lemmas.

LEMMA 4. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then the following hold*:

(a) $\nabla f = 0$.

(b) *For any X tangent to M and any $V \in FH^\perp$, we have $D_X V \in FH^\perp$.*

(c) *For any X tangent to M and any $U \in N$, we have $D_X U \in N$.*

PROOF. By the proof of Lemma 2, if the Ricci tensor S of a minimal CR submanifold M satisfies $S(X,X) \geq (n-1)g(X,X)$ for any tangent vector field X, then $A_U tV = 0$ for any U and V normal to M. Thus we have

$$\begin{aligned} g((\nabla_X f)V, U) &= -g(FA_V X, U) - g(B(X, tV), U) \\ &= g(X, A_V tU) - g(A_U tV, X) \\ &= 0 \end{aligned}$$

for any X tangent to M and any U and V normal to M. This means that f is parallel.

Since M is proper, by Lemma 3, we have $\dim H_x^\perp = 1$. Let V be a vector field in $FH^\perp$. Then we see $g(D_X V, fU) = -g(V, (\nabla_X f)U) = 0$ for any vector field $U \in N$. This proves (b).

Next we prove (c). For any vector field U in N, there exists U' in N such that $U = fU'$. Therefore we have

$$D_X U = D_X(fU') = fD_X U'.$$

This shows $D_X U \in N$. □

LEMMA 5. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then the second fundamental form A satisfies the following*:

(a) $A_v P A_v = P$, *where v is a unit vector field in $FH^\perp$.*

(b) $|[P, A_v]|^2 = 2\,\mathrm{tr}\, A_v^2 - 2(n-1)$, *where v is a unit vector field in $FH^\perp$.*

(c) $A_V A_U = A_U A_V$ *for any $V \in FH^\perp$ and $U \in N$.*

(d) $PA_U = A_{fU}$ *and* $PA_U + A_U P = 0$ *for any $U \in N$.*

PROOF. By Lemma 3, we have $\dim H_x^\perp = 1$. Let $\{v_1, \dots, v_p\}$ be an orthonormal basis of $T_x(M)^\perp$ such that $v_1 = v \in FH_x^\perp$ and $v_2, \dots, v_p \in N_x$.

By (3) and $fv = 0$, we obtain

$$2g(A_v P A_v X, Y) = -2g(X, PY)g(tv, tv)$$

for any X and Y tangent to M. Thus we have (a). Using this, we can prove (b) by a straightforward computation.

Next we prove (c). From the equation of Ricci and Lemma 4 (b), we see

$$g([A_U, A_V]X, Y)$$

$$
\begin{aligned}
&= g(Y, tV)g(X, tU) - g(X, tV)g(Y, tU) - 2g(X, PY)g(V, fU) \\
&= 0
\end{aligned}
$$

for any X and Y tangent to M and $V \in FH^{\perp}$, $U \in N$. This shows (c).

From the Weingarten formula and Lemma 4 (a), we have

$$\tilde{\nabla}_X JU = \tilde{\nabla}_X fU = -A_{fU}X + D_X fU = -A_{fU}X + fD_X U \,.$$

On the other hand, it follows from $\tilde{\nabla} J = 0$ and Lemma 4 (c) that

$$\tilde{\nabla}_X JU = J\tilde{\nabla}_X U = -PA_U X - FA_U X + fD_X U \,,$$

from which $PA_U = A_{fU}$. Since A_{fU} is symmetric and P is skew-symmetric, we obtain $PA_U + A_U P$=0. This proves (d). □

Using Theorem 1 and Lemma 5, we next compute the Laplacian for the square of the length of the second fundamental form of the minimal submanifold in CP^m whose Ricci tensor satisfies $S(X, X) \geq (n-1)g(X, X)$ for any tangent vector field X.

LEMMA 6. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X)$, then*

$$
\begin{aligned}
g(\nabla^2 A, A) = (n+3)\operatorname{tr} A_v^2 + (n+4)\sum_a \operatorname{tr} A_{fa}^2 - 6(n-1) \\
- \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b} (\operatorname{tr} A_a A_b)^2 \,.
\end{aligned}
$$

PROOF. From Lemma 5, we have $\sum_a \operatorname{tr} A_a A_{fa} P = \sum_a \operatorname{tr} A_{fa}^2$. Next we compute $\sum_a |[P, A_a]|^2$. Using Lemma 5,

$$
\begin{aligned}
\sum_a |[P, A_a]|^2 &= |[P, A_v]|^2 + \sum_{a \geq 2} |[P, A_a]|^2 \\
&= -2(n-1) + 2\operatorname{tr} A_v^2 + 4\sum_a \operatorname{tr} A_{fa}^2 \,.
\end{aligned}
$$

From these equations and Theorem 1, we have our result. □

LEMMA 7. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X)$, then*

$$
\begin{aligned}
&\sum_j g((\nabla^2 A)_v e_j, A_v e_j) = (n+3)\operatorname{tr} A_v^2 - 6(n-1) - (\operatorname{tr} A_v^2)^2, \\
&\sum_{a \geq 2, j} g((\nabla^2 A)_a e_j, A_a e_j) = (n+4)\sum_a \operatorname{tr} A_{fa}^2 - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b \geq 2} (\operatorname{tr} A_a A_b)^2 \,.
\end{aligned}
$$

PROOF. By Lemma 5 (c) and (d), for any $v_a \in N$,

$$\begin{aligned}\operatorname{tr} A_a A_v &= -\operatorname{tr} A_{f^2 a} A_v = -\operatorname{tr} P A_{fa} A_v = -\operatorname{tr} A_{fa} A_v P = -\operatorname{tr} A_v A_{fa} P \\ &= \operatorname{tr} A_v P A_{fa} = \operatorname{tr} A_v A_{f^2 a} = -\operatorname{tr} A_v A_a = -\operatorname{tr} A_a A_v\,.\end{aligned}$$

Hence we have $\operatorname{tr} A_a A_v = 0$. Thus, using (1) and Lemma 5, we have

$$\begin{aligned}&\sum_j g((\nabla^2 A)_v e_j, A_v e_j) \\ &= \sum_j g((\nabla^2 B)(e_j, A_v e_j), v) \\ &= ng \sum_j g(A_v e_j, A_v e_j) - 3 \sum_j g(A_v e_j, P^2 A_v e_j) \\ &\quad -3 \sum_j g(A_v^2 e_j, P^2 e_j) - 3 \sum_j g(A_v e_j, A_v e_j) - 6 \sum_j g(A_v P e_j, P A_v e_j) \\ &\quad + \sum_{a,j} (-\operatorname{tr} A_a A_v g(A_a e_j, A_v e_j) + 2 g(A_a A_v A_a e_j, A_v e_j) \\ &\quad - g(A_a^2 A_v e_j, A_v e_j) - g(A_v A_a^2 e_j, A_v e_j)) \\ &= (n-3) \operatorname{tr} A_v^2 + 3 |[P, A_v]|^2 - \sum_a (\operatorname{tr} A_a A_v)^2 + \sum_a |[A_a, A_v]|^2 \\ &= (n+3) \operatorname{tr} A_v^2 - 6(n-1) - (\operatorname{tr} A_v^2)^2\,.\end{aligned}$$

From this equation and Lemma 6, we obtain

$$\begin{aligned}&\sum_{a \geq 2, j} g((\nabla^2 A)_a e_j, A_a e_j) \\ &\quad = g(\nabla^2 A, A) - \sum_j g((\nabla^2 A)_v e_j, A_v e_j) \\ &\quad = (n+4) \sum_a \operatorname{tr} A_{fa}^2 - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b} (\operatorname{tr} A_a A_b)^2 + (\operatorname{tr} A_v^2)^2 \\ &\quad = (n+4) \sum_a \operatorname{tr} A_{fa}^2 - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b \geq 2} (\operatorname{tr} A_a A_b)^2\,.\end{aligned}$$

Hence we have our equation. □

Next we give inequalities for $\sum_{a,b} |[A_a, A_b]|^2$ and $\sum_{a,b \geq 2} (\operatorname{tr} A_a A_b)^2$ in the equation in Lemma 7.

LEMMA 8. *Let M be a compact n-dimensional minimal proper CR submanifold of*

CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X)$, then

$$\sum_{a,b} |[A_a, A_b]|^2 \leq 4\sum_a \operatorname{tr} A_{fa}^2,$$

$$\sum_{a,b\geq 2} (\operatorname{tr} A_a A_b)^2 \leq \frac{1}{2}\Big(\sum_a \operatorname{tr} A_{fa}^2\Big)^2.$$

PROOF. From (2), we have $3g(PX, PX) \geq \sum_a g(A_a X, A_a X)$ for any X tangent to M. On the other hand, by Lemma 5,

$$\begin{aligned}
&\sum_{i,a} g(A_v^2 A_{fa} e_i, A_{fa} e_i) \\
&\quad = \sum_{i,a} g(A_v A_{fa} A_v e_i, A_{fa} e_i) = \sum_{i,a} g(A_v P A_a A_v e_i, A_{fa} e_i) \\
&\quad = \sum_{i,a} g(A_v P A_v A_a e_i, A_{fa} e_i) = \sum_{i,a\geq 2} g(P A_a e_i, P A_a e_i).
\end{aligned}$$

Using these and Lemma 5, we obtain

$$\begin{aligned}
3\sum_a \operatorname{tr} A_{fa}^2 &= 3\sum_{i,a} g(PA_{fa}e_i, PA_{fa}e_i) \\
&\geq \sum_{i,a,b} g(A_b A_{fa} e_i, A_b A_{fa} e_i) \\
&= \sum_{i,a} g(A_v A_{fa} e_i, A_v A_{fa} e_i) + \sum_{i,a,b} g(A_{fa}^2 A_{fb}^2 e_i, e_i) \\
&= \sum_{i,a\geq 2} g(PA_a e_i, PA_a e_i) + \frac{1}{2}\sum_{a,b} |[A_a, A_b]|^2 \\
&= \sum_a \operatorname{tr} A_{fa}^2 + \frac{1}{2}\sum_{a,b} |[A_a, A_b]|^2,
\end{aligned}$$

from which $4\sum_a \operatorname{tr} A_{fa}^2 \geq \sum_{a,b} |[A_a, A_b]|^2$. Hence our first inequality holds. In the next place, we take a basis $\{v, v_2, \dots, v_{p'}, v_{p'+1} = fv_2, \dots, v_p = fv_{p'}\}$ $(p = 2p'+1)$ of $T_x(M)^\perp$ such that $\sum_{a,b\geq 2} (\operatorname{tr} A_a A_b)^2 = \sum_{a=2}^p (\operatorname{tr} A_a^2)^2$. Since $\operatorname{tr} A_a^2 = \operatorname{tr} A_{fa}^2$ for $a \geq 2$, we have

$$\sum_{a=2}^{p} (\operatorname{tr} A_a^2)^2 = 2\sum_{a=2}^{p'} (\operatorname{tr} A_a^2)^2 = 2\bigg(\Big(\sum_{a=2}^{p'} \operatorname{tr} A_a^2\Big)^2 - \sum_{a,b\geq 2, a\neq b}^{p'} \operatorname{tr} A_a^2 \operatorname{tr} A_b^2\bigg).$$

On the other hand, we see

$$\Big(\sum_{a=2}^{p} \operatorname{tr} A_a^2\Big)^2 = \Big(2\sum_{a=2}^{p'} \operatorname{tr} A_a^2\Big)^2 = 4\Big(\sum_{a=2}^{p'} \operatorname{tr} A_a^2\Big)^2.$$

Therefore

$$\sum_{a=2}^{p}(\operatorname{tr} A_a^2)^2 = \frac{1}{2}\Big(\sum_{a=2}^{p}\operatorname{tr} A_a^2\Big)^2 - 2\sum_{a,b\geq 2,a\neq b}^{p'}\operatorname{tr} A_a^2 \operatorname{tr} A_b^2 \leq \frac{1}{2}\Big(\sum_{a=2}^{p}\operatorname{tr} A_a^2\Big)^2 ,$$

from which $\sum_{a,b\geq 2}^{p}(\operatorname{tr} A_a A_b)^2 \leq (1/2)(\sum_a \operatorname{tr} A_{fa}^2)^2$. Hence we have the second inequality. □

Using Lemma 3-Lemma 8, we prove the following lemma.

LEMMA 9. *Let M be a compact n-dimensional minimal proper CR submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X)$, then $A_{fa} = 0$ for all a.*

PROOF. From Lemma 7 and Lemma 8, we have

$$\begin{aligned}
&\frac{1}{2}\Delta\Big(\sum_a \operatorname{tr} A_{fa}^2\Big)\\
&= \sum_{a\geq 2,i} g((\nabla^2 A)_a e_i, A_a e_i) + \sum_{a\geq 2,i} g((\nabla A)_a e_i, (\nabla A)_a e_i)\\
&\geq \sum_{a\geq 2,i} g((\nabla^2 A)_a e_i, A_a e_i)\\
&= (n+4)\sum_a \operatorname{tr} A_{fa}^2 - \sum_{a,b} |[A_a, A_b]|^2 - \sum_{a,b\geq 2}(\operatorname{tr} A_a A_b)^2\\
&\geq \Big(\sum_a \operatorname{tr} A_{fa}^2\Big)\Big(n - \frac{1}{2}\sum_a \operatorname{tr} A_{fa}^2\Big).
\end{aligned}$$

On the other hand, by the assumption, the Ricci tensor S satisfies

$$\sum_i S(e_i, e_i) = (n+3)(n-1) - |A|^2 \geq (n-1)\sum_i g(e_i, e_i) ,$$

which reduces to $|A|^2 = \operatorname{tr} A_v^2 + \sum_a \operatorname{tr} A_{fa}^2 \leq 3(n-1)$. Moreover, Lemma 5 (b) implies $\operatorname{tr} A_v^2 \geq n-1$. Hence we have $\sum_a \operatorname{tr} A_{fa}^2 \leq 2(n-1) < 2n$. Therefore, by the Hopf's lemma, $\sum_a \operatorname{tr} A_{fa}^2$ is constant so that $\Delta(\sum_a \operatorname{tr} A_{fa}^2) = 0$ (cf. [5; p. 338]). Thus we have $A_{fa} = 0$ for all a. □

(Proof of Theorem 2)

From Lemma 4 and Lemma 9, the first normal space of M is of dimension 1 and parallel with respect to the normal connection.

Let S^{2m+1} be a $(2m+1)$-dimensional unit sphere. We consider the Hopf fibration $\pi : S^{2m+1} \to CP^m$. Then the first normal space of $\bar{M} = \pi^{-1}(M)$ in S^{2m+1} is of dimension 1 and is also parallel with respect to the normal connection. Therefore, there is a totally geodesic $(n+2)$-dimensional submanifold S^{n+2} of S^{2m+1} containing $\bar{M}$ (cf. [4]). Hence there is a totally geodesic $CP^{(n+1)/2}$ of CP^m containing M (cf. [15; p. 227]).

5. Pinching theorems for the Ricci curvature

To prove our theorems, we need some well-known results.

In the following, we take the unit normal vector field v of a real hypersurface M in CP^m, and we put $\xi = -Jv$. Then ξ is the unit tangent vector field of M and $P^2X = -X + g(X, \xi)\xi$, $P\xi = 0$. We also put $A_v = A$ to simplify the notation. Then $\nabla_X \xi = PAX$ for any vector field X tangent to M.

PROPOSITION A([3]). *Let M be a real hypersurface (with unit normal vector v) of a complex projective space CP^m on which ξ is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ and the focal map ϕ_r has constant rank on M. Then the following hold*:

(a) *M lies on a tube (in the direction $\eta = \gamma'(r)$, where $\gamma(r) = \exp_x(rv)$ and x is a base point of the normal vector v) of radius r over a certain Kähler submanifold N in CP^m.*

(b) *Let $\cot\theta$, $0 < \theta < \pi$, be a principal curvature of the second fundamental form A_η at $y = \gamma(r)$ of the Kähler submanifold N. Then the real hypersurface M has a principal curvature $\cot(r - \theta)$ at $x = \gamma(0)$.*

PROPOSITION B([10]). *Let M be a real hypersurface of a complex projective space CP^m. If $A\xi = 0$, except for the null set on which the focal map ϕ_r degenerates, M is locally congruent to one of the following*:

(a) *a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic CP^k $(1 \le k \le m-1)$,*

(b) *a nonhomogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kähler submanifold N with nonzero principal curvatures $\neq \pm 1$.*

Using these results, we prove the following

THEOREM 3. *Let M be a compact n-dimensional minimal CR submanifold of a complex projective space CP^m which is not a complex submanifold of CP^m. If the Ricci tensor S of M satisfies $S(X, X) \ge (n-1)g(X, X)$ for any vector field X tangent to M, then M is congruent to one of the following*:

(a) *a totally geodesic real projective space RP^n of CP^m,*

(b) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m,*

(c) *a real hypersurface of some $CP^{(n+1)/2}$ in CP^m which lies on a tube of radius $\pi/4$ over certain Kähler submanifold N with principal curvatures $\cot\theta$, $0 < \theta \le \pi/12$.*

PROOF. We suppose that M is proper. Then Theorem 2 implies that M is a real hypersurface of some totally geodesic complex projective space $CP^{(n+1)/2}$ in CP^m. By the proof of Lemma 2, we have $A\xi = 0$. On the other hand, from Lemma 5, we obtain $APAX = PX$ for any X tangent to M. Thus we see that if $AX = \lambda X$, then $APX = (1/\lambda)PX$. Since $3g(PX, PX) \geq g(A^2X, X)$, we have $\lambda^2 \leq 3$. We also have $\operatorname{rank} A \leq n - 1$ because $A\xi = 0$. A homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic CP^k is minimal if and only if $k = (n-1)/4$, that is, M is $M^c_{k,k}$. The principal curvatures of this real hypersurface is ± 1 (see [3; p. 493]).

For a nonhomogeneous real hypersurface M which lies on a tube of radius $\pi/4$ over a Kähler submanifold N, by the condition $\lambda^2 \leq 3$ and (b) of Proposition A, we see that $\cot^2(\pi/4 - \theta) \leq 3$. Thus we have $0 < \theta \leq \pi/12$. Consequently, using Proposition A and Proposition B, we have our theorem. □

REMARK 1. The author does not know any example of a Kähler submanifold N having the properties required in Case (c) in Theorem 3.

COROLLARY 1. *Let M be a compact n-dimensional minimal proper CR submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n - 1)g(X, X)$, then M is congruent to one of the following*:

(a) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m*,

(b) *a real hypersurface of some $CP^{(n+1)/2}$ in CP^m which lies on a tube of radius $\pi/4$ over certain Kähler submanifold N with principal curvatures* $\cot\theta$, $0 < \theta \leq \pi/12$.

Using the theorem in [9], we have

COROLLARY 2. *Let M be a compact n-dimensional minimal proper CR submanifold of a complex projective space CP^m, $n \geq 5$. If the Ricci tensor S satisfies $(n-1)g(X, X) \leq S(X, X) \leq (n+1)g(X, X)$, then M is congruent to a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m.*

Next we prove the following

THEOREM 4. *Let M be a compact n-dimensional minimal CR submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X, X) \geq (n-1)g(X, X) + g(PX, PX)$ for any vector field X tangent to M, then M is congruent to one of the following*:

(a) *a totally geodesic real projective space RP^n of CP^m*,

(b) *a totally geodesic complex projective space $CP^{n/2}$ of CP^m*,

(c) *a complex $(n/2)$ dimensional complex quadric $Q^{(n/2)}$ of some $CP^{n/2+1}$ of CP^m*,

(d) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m*,

(e) *a real hypersurface of some $CP^{(n+1)/2}$ in CP^m which lies on a tube of radius $\pi/4$ over certain Kähler submanifold N with principal curvatures* $\cot\theta$, *where θ satisfies* $0 < \sin 2\theta \leq 1/3$.

For the proof of the theorem, we prepare some lemmas for complex submanifolds. We take an orthonormal basis $\{v_1, \dots, v_p, v_{p+1} = fv_1, \dots, v_{2p} = fv_p\}$ of $T_x(M)^\perp$.

LEMMA 10 ([6]). *Let M be a complex k-dimensional Kähler submanifold of a complex m-dimensional Kähler manifold $\bar{M}$. Then*

$$\frac{1}{k}|A|^4 \le \sum_{a,b=1}^{2p} |[A_a, A_b]|^2 \le |A|^4 ,$$

$$\frac{1}{2p}|A|^4 \le \sum_{a,b=1}^{2p} (\mathrm{tr}\, A_a A_b)^2 \le \frac{1}{2}|A|^4 ,$$

where $p = m - k$. If $\bar{M}$ is of constant holomorphic sectional curvature c, then M is Einstein if and only if $\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 = |A|^4/k$.

From Theorem 1, we see

LEMMA 11. *Let M be a complex k-dimensional Kähler submanifold of CP^m. Then*

$$g(\nabla^2 A, A) = 2(k+2)|A|^2 - \sum_{a,b=1}^{2p} |[A_a, A_b]|^2 - \sum_{a,b=1}^{2p} (\mathrm{tr}\, A_a A_b)^2 .$$

In the following we prove Theorem 4. From Theorem 2, if M is proper, then it is a real hypersurface of some $CP^{(n+1)/2}$ in CP^m.

Next we suppose that M is a complex $(n/2)$ dimensional complex submanifold of CP^m. Since M is complex minimal submanifold of CP^m, we have

$$S(X, Y) = (n+2)g(X, Y) - \sum_{a=1}^{2p} g(A_a^2 X, Y) .$$

Thus we have $\sum_{a=1}^{2p} g(A_a^2 X, X) \le 2g(X, X)$, from which $|A|^2 \le 2n$. Moreover, we see that $2I - \sum_a A_a^2$ is a positive semi-definite operator. The symmetricity of A_a implies that $\sum_a A_a^2$ is positive semi-definite. The operators $\sum_a A_a^2$ and $2I - \sum_a A_a^2$ can be transformed simultaneously by an orthogonal matrix into diagonal forms at each point of M, thus we see that $(\sum_a A_a^2)(2I - \sum_a A_a^2)$ is positive semi-definite. Hence we have

$$\mathrm{tr}\Big(\sum_{a=1}^{2p} A_a^2\Big)^2 \le 2|A|^2 \le 4n . \tag{5}$$

On the other hand, we obtain

$$\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 = 2 \sum_{a,b=1}^{2p} \mathrm{tr}\, A_a^2 A_b^2 = 2\,\mathrm{tr}\Big(\sum_{a=1}^{2p} A_a^2\Big)^2 .$$

Therefore we get $\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 \le 4|A|^2$. From Lemma 10, Lemma 11 and these equations, we have,

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 &= g(\nabla^2 A, A) + |\nabla A|^2 \\ &\ge g(\nabla^2 A, A) \ge |A|^2\left(n - \frac{1}{2}|A|^2\right) \ge 0 .\end{aligned} \tag{6}$$

Hence, by the theorem of E. Hopf, $|A|^2$ is constant so that $\Delta|A|^2 = 0$ (cf. [5; p. 338]). Thus we have $|A| = 0$ or $|A|^2 = 2n$. When $|A| = 0$, M is totally geodesic.

Next we suppose $|A|^2 = 2n$. By (5), we have $\mathrm{tr}(\sum_{a=1}^{2p} A_a^2)^2 = 4n$, that is,

$$\sum_{a,b=1}^{2p} |[A_a, A_b]|^2 = 8n = \frac{2|A|^4}{n} .$$

From Lemma 10, M is an Einstein complex submanifold of CP^m.

For any normal vector field V with $V_x \in N_0(x) = \{V \in T_x(M)^\perp : A_V = 0\}$, we have

$$\nabla_Y(A_V X) = (\nabla_Y A)_V X + A_{D_Y V} X + A_V(\nabla_Y X) = 0$$

at $x \in M$. Hence $A_{D_Y V} X + (\nabla_Y A)_V X = 0$. Since the equality of (6) holds, we get $\nabla A = 0$, from which we see that N_0 is parallel with respect to the normal connection. Let $V \in N_0$ and $U \in N_1$. Then

$$Xg(U, V) = g(D_X U, V) + g(U, D_X V) = 0 .$$

Hence the first normal space is parallel with respect to the normal connection. On the other hand, since the equality of (6) holds, we have $\sum_{a,b=1}^{2p}(\mathrm{tr}\, A_a A_b)^2 = (1/2)|A|^4$. In the next place, we take a basis $\{v_1, \dots, v_p, v_{p+1} = fv_1, \dots, v_{2p} = fv_p\}$ of $T_x(M)^\perp$ such that $\sum_{a,b=1}^{2p}(\mathrm{tr}\, A_a A_b)^2 = \sum_{a=1}^{2p}(\mathrm{tr}\, A_a^2)^2$. Then

$$\sum_{a=1}^{2p}(\mathrm{tr}\, A_a^2)^2 = \frac{1}{2}|A|^4 - 2\sum_{a\neq b}^{p}(\mathrm{tr}\, A_a^2)(\mathrm{tr}\, A_b^2) ,$$

and therefore $\sum_{a\neq b}^{p}(\mathrm{tr}\, A_a^2)(\mathrm{tr}\, A_b^2) = 0$. This implies $\dim N_1 = 2$. Consequently, M is an Einstein complex hypersurface of some $CP^{n/2+1}$ in CP^m, that is, a complex quadric $Q^{n/2}$ of $CP^{n/2+1}$ (see [13]). From this and Theorem 3, we have our theorem.

REMARK 2. In 1974, Chen and Ogiue [2] proved that if the Ricci curvature of n-dimensional Kähler submanifold of CP^m is everywhere equal to $n/2$, then M is locally Q^n in some CP^{n+1} in CP^m (see also [11]).

We suppose that M is a compact n-dimensional minimal CR submanifold of a complex projective space CP^m. When the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X) + 2g(PX,PX)$ for any vector X tangent to M, the cases (c) and (e) in Theorem 4 do not occur. Thus we obtain

THEOREM 5 ([8]). *Let M be a compact n-dimensional minimal CR submanifold of a complex projective space CP^m. If the Ricci tensor S of M satisfies $S(X,X) \geq (n-1)g(X,X) + 2g(PX,PX)$ for any vector field X tangent to M, then M is congruent to one of the following*:

(a) *a totally geodesic real projective space RP^n of CP^m*,

(b) *a totally geodesic complex projective space $CP^{n/2}$ of CP^m*,

(c) *a pseudo-Einstein real hypersurface $M^c((n-1)/4, \pi/4)$ of some $CP^{(n+1)/2}$ in CP^m*.

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