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Estimates of the Eigenvalues of Hill's Operator with Distributional Coefficients

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Abstract. We give an optimal upper bound for the eigenvalues of the Hill operator with a distributional coefficient.

1. Introduction

In this paper, we consider the eigenvalues of the Hill operator which is formally expressed as

$$H = -\frac{d^2}{dx^2} + q'(x) \quad \text{in } \mathcal{H} = L^2((0,\pi))\,,$$

where $q \in L^2((0,\pi))$ is a real-valued function. We recall the precise definition of this operator from [1]. We define a symmetric quadratic form a in $\mathcal{H}$ by

$$a(\varphi,\psi) = \int_0^\pi \varphi'(x)\overline{\psi'(x)}\,dx - \int_0^\pi q(x)\varphi'(x)\overline{\psi(x)}\,dx - \int_0^\pi q(x)\varphi(x)\overline{\psi'(x)}\,dx\,,$$

$$Q(a) = \{y \in H^1((0,\pi)) \mid y(0) = y(\pi)\}\,.$$

It is useful to note that if $q' \in L^2((0,\pi))$, then

$$a(\varphi,\psi) = (-\varphi'' + q'\varphi, \psi)_{\mathcal{H}} \quad \text{for } \varphi, \psi \in C_0^\infty((0,\pi))\,.$$

We also note that there exists a constant $b > 0$ such that

$$\left| a(\varphi,\varphi) - \int_0^\pi |\varphi'(x)|^2\,dx \right| \le \frac{1}{2}\|\varphi'\|_{\mathcal{H}}^2 + b\|\varphi\|_{\mathcal{H}}^2 \quad \text{for every } \varphi \in Q(a)\,,$$

see [1, formula (2.12)]. This combined with the KLMN theorem (see e.g., [4, Theorem X.17]) implies that there is a unique self-adjoint operator H in $\mathcal{H}$ for which

$$a(\varphi,\psi) = (H\varphi,\psi)_{\mathcal{H}}$$

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for any $\varphi \in \mathrm{Dom}(H)$ and $\psi \in Q(a)$. The spectrum of H is discrete. For non-negative integers j, let λ_j stand for the $(j+1)$th eigenvalue of H counted with multiplicity.

Our main result is now stated as follows, which we prove in Section 2.

THEOREM. *Let n be a non-negative integer. Suppose that*

$$\int_0^\pi q(x)e^{2ikx}dx = 0 \quad for \quad k = 0, 1, 2, \dots, 2n\,.$$

Then we have

$$\lambda_{2n} \le 4n^2,$$

and the equality sign holds if and only if q is identically equal to 0.

We describe the background to our work here. In [2], Blumenson proved the above theorem in the case where $q \in C^2([0,\pi])$. His proof largely relies on the reduction of the Hill equation to the Riccati equation. It seems that such a method is not applicable to our problem, since the potential is distributional. In order to eliminate this difficulty, we use an abstract method; our proof is based on the min-max principle with suitably chosen trial functions (see (2.2), (2.3) and (2.4)). It is worth mentioning that our result considerably extends that of Blumenson.

2. Proof of Theorem

Let $g_j,\ j \in \mathbf{Z}$, be the Fourier coefficients of q:

$$q(x) = \sum_{j=-\infty}^{\infty} g_j e^{2ijx}\,.$$

Since q is real-valued, we have $g_j = \overline{g_{-j}}$. For $k \in \mathbf{Z}$, we define

$$\varphi_k(x) = \frac{1}{\sqrt{\pi}} e^{2ikx}\,.$$

We note that $\{\varphi_k\}_{k\in\mathbf{Z}}$ is a complete orthonormal system of $\mathcal{H}$.

First, we prove the assertion for $n = 0$. By the min-max principle (see e.g., [3, Theorem 4.5.1]), we have

$$\lambda_0 \le a(\varphi_0, \varphi_0) = 0\,.$$

Let us show that $\lambda_0 < 0$ if q is not identically equal to 0. Suppose that q is not identically equal to 0. Then, there exists $l \in \mathbf{N}$ such that $g_l \neq 0$. Let $\tilde{q} : \mathbf{R} \to \mathbf{R}$ be the periodic extension of q. We note that λ_j is invariant under the substitution of the potential $q(\cdot) \mapsto \tilde{q}(\cdot + t)$, $t \in \mathbf{R}$. Thus, we may assume without any loss of generality that $\mathrm{Im}\, g_l \neq 0$. For $\varepsilon \in \mathbf{R}\setminus\{0\}$, we define

$$\psi_\varepsilon = \frac{1}{\sqrt{(1+\varepsilon^2)\pi}}(1 + \varepsilon e^{2ilx})\,.$$

Then, $\|\psi_\varepsilon\|_{\mathcal{H}} = 1$, and

$$a(\psi_\varepsilon, \psi_\varepsilon) = \frac{4}{1+\varepsilon^2}[l(\mathrm{Im}\, g_l)\varepsilon + l^2\varepsilon^2]\,.$$

Therefore $a(\psi_\varepsilon, \psi_\varepsilon) < 0$, provided $\varepsilon \mathrm{Im}\, g_l < 0$ and $|\varepsilon|$ is sufficiently small. This combined with the min-max principle implies that $\lambda_0 < 0$. So, we have the assertion for $n = 0$.

Next, we prove the assertion for $n \in \mathbf{N}$. By assumption, we have $g_j = 0$ for $j = 0, 1, 2, \ldots, 2n$, from which

$$a(\varphi_k, \varphi_m) = 4k^2\delta_{k,m} \quad \text{for} \quad |k| \le n \quad \text{and} \quad |m| \le n\,, \tag{2.1}$$

where $\delta_{k,m}$ is Kronecker's symbol. Combining this with the min-max principle, we get $\lambda_{2n} \le 4n^2$.

Suppose that q is not identically equal to 0. Let us show that $\lambda_{2n} < 4n^2$. Let l be the smallest positive integer such that $g_{n+l} \ne 0$. We may assume, as above, that $\mathrm{Im}\, g_{n+l} \ne 0$. For $\varepsilon \in \mathbf{R} \setminus \{0\}$, we define

$$\psi_n(x) = \frac{1}{\sqrt{(1+\varepsilon^2)\pi}}(e^{2inx} + \varepsilon e^{-2ilx})\,, \tag{2.2}$$

$$\psi_{-n}(x) = \frac{1}{\sqrt{(1+\varepsilon^2)\pi}}(e^{-2inx} + \varepsilon e^{2ilx})\,. \tag{2.3}$$

We also put

$$\psi_j = \varphi_j \quad \text{for} \quad |j| \le n-1\,. \tag{2.4}$$

We note that $\{\psi_j\}_{j=-n}^{n}$ is an orthonormal system of $\mathcal{H}$. We get

$$\begin{aligned} a(\psi_n, \psi_n) &= \frac{1}{1+\varepsilon^2}(4n^2 + 4l^2\varepsilon^2 + 4(n+l)\varepsilon\, \mathrm{Im}\, g_{-n-l}) \\ &= a(\psi_{-n}, \psi_{-n}) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} a(\psi_n, \psi_{-n}) &= \frac{4il\varepsilon^2}{1+\varepsilon^2} g_{2l} \\ &= \overline{a(\psi_{-n}, \psi_n)}\,. \end{aligned} \tag{2.6}$$

For $m = -n+1, -n+2, \ldots, n-2, n-1$, we obtain

$$\begin{aligned} a(\psi_n, \psi_m) &= \frac{-1}{\pi\sqrt{1+\varepsilon^2}} \int_0^\pi q(x)[2i(n-m)e^{2i(n-m)x} - 2i(l+m)e^{-2i(l+m)x}]dx \\ &= 0\,, \end{aligned} \tag{2.7}$$

$$a(\psi_{-n}, \psi_m) = \frac{1}{\pi\sqrt{1+\varepsilon^2}} \int_0^{\pi} q(x)[2i(n+m)e^{-2i(n+m)x} - 2i(l-m)e^{2i(l-m)x} dx \\ = 0\,, \tag{2.8}$$

since $|n \pm m| \leq 2n-1$ and $|l \pm m| \leq n+l-1$. Now we put $a_n = a(\psi_n, \psi_n)$, $b_n = a(\psi_n, \psi_{-n})$. Let

$$A = (a(\psi_m, \psi_k))_{\substack{-n \leq m \leq n \\ -n \leq k \leq n}} .$$

It follows from (2.1), (2.7) and (2.8) that

$$\det(A - \lambda I) = -((a_n - \lambda)^2 - |b_n|^2)\lambda \prod_{j=1}^{n-1} (4j^2 - \lambda)^2$$

$$= -(a_n + |b_n| - \lambda)(a_n - |b_n| - \lambda)\lambda \prod_{j=1}^{n-1} (4j^2 - \lambda)^2 ,$$

where I is the $(2n+1) \times (2n+1)$ identity matrix. Combining this with (2.5) and (2.6), we infer that the largest eigenvalue of A is given by $a_n + |b_n|$ when $|\varepsilon|$ is sufficiently small. Noticing $|b_n| = \mathrm{O}(\varepsilon^2)$ as $\varepsilon \to 0$, we have $a_n + |b_n| < 4n^2$, provided $\varepsilon \mathrm{Im} g_{-n-l} < 0$ and $|\varepsilon|$ is sufficiently small. This together with the min-max principle implies that $\lambda_{2n} < 4n^2$, which concludes the proof.

References

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