TOKYO J. MATH.
VOL. 33, NO. 2, 2010

The Euler Adic Dynamical System and Path Counts in the Euler Graph

Karl PETERSEN and Alexander VARCHENKO

University of North Carolina at Chapel Hill

(Communicated by H. Nakada)

Abstract. We give a formula for generalized Eulerian numbers, prove monotonicity of sequences of certain ratios of the Eulerian numbers, and apply these results to obtain a new proof that the natural symmetric measure for the Bratteli-Vershik dynamical system based on the Euler graph is the unique fully supported invariant ergodic Borel probability measure. Key ingredients of the proof are a two-dimensional induction argument and a one-to-one correspondence between most paths from two vertices at the same level to another vertex.

1. Introduction

The Euler graph is the infinite directed graph with vertices (i, j), $i, j \geq 0$, with $j + 1$ edges from (i, j) to $(i + 1, j)$ and $i + 1$ edges from (i, j) to $(i, j + 1)$; see Figure 1. The number $i + j$ is called the level of (i, j).

A *generalized Eulerian number* $A_{p,q}(i, j)$ is the number of paths in the graph from (p, q) to $(p + i, q + j)$. We prove that

$$(1.1)\quad A_{p,q}(i, j) = \sum_{t=0}^{i} (-1)^{i-t} \binom{p+q+t+1}{t} \binom{p+q+i+j+2}{i-t} (p+1+t)^{i+j}$$

and

$$(1.2)\qquad \frac{A_{p,q}(i, j+1)}{A_{p,q-1}(i, j+1)} \leq \frac{A_{p,q}(i, j)}{A_{p,q-1}(i, j)} \leq \frac{q+j}{q+1+j} \frac{A_{p,q}(i+1, j)}{A_{p,q-1}(i+1, j)} .$$

As a corollary we show that

$$(1.3)\qquad \frac{A_{p,q}(i, j)}{A_{0,0}(p+i, q+j)} \quad \text{tends to} \quad \frac{1}{(p+q+1)!}$$

as both i, j tend to infinity.

Received March 19, 2009; revised August 25, 2009
1991 *Mathematics Subject Classification*: 37A05, 37A25, 05A10, 05A15 (Primary), 37A50, 37A55 (Secondary)
Key words and phrases: adic transformation, invariant measure, ergodicity, Eulerian numbers
The research of the second author was supported in part by NSF grant DMS-0555327.

These results are motivated by continuing study of the adic dynamical system associated with the Euler graph. They yield a new proof of the fact that the natural symmetric measure for the Bratteli-Vershik dynamical system based on the Euler graph is the unique fully supported invariant ergodic Borel probability measure.

The Euler adic system is a particularly interesting nonstationary Bratteli-Vershik (or adic) system based on an infinite directed graded graph with remarkable combinatorial properties. How such systems arise from reinforced walks on graphs is explained in [6]. For the viewpoint of urn models, see for example [4, p. 68 ff.]. A first step in studying the Euler adic system, or the associated C^* algebra, is the identification of the adic-invariant measures (sometimes called central measures or traces), namely those that give equal measure to each cylinder set determined by an initial path segment from the root vertex to another fixed vertex. In [1] it was proved by a supermartingale argument that the natural *symmetric measure*, which assigns equal measure to all cylinders of the same length, is ergodic. In [5] this result was strengthened by using a coding of paths by permutations to show that in fact the symmetric measure is the unique fully supported ergodic probability measure for this system. We found out recently that the paper [8] contains related results, arrived at by different arguments and including also identification of all the other (partially supported) ergodic measures, and that a version of Formula (1.1) appears in [2]. Although the connection with path counting is not made explicitly in [2], it seems that it is not too far from Formula (6.2) of that paper to our Formula (1.1). When one seeks to study higher-dimensional versions of the Euler adic system, the coding by permutations is no longer available; therefore we have developed a proof via a different approach, which we present here. This proof also yields a stronger result, namely identification of the generic points for the symmetric measure and indeed a "directional unique ergodicity" property such as was established in [10, 11] for the Pascal adic system.

For background on adic systems, we refer to [1, 7, 9–12, 15–17]. A Bratteli diagram is an infinite directed graded graph. At level 0 there is a single vertex, R, called the *root*. At each level $n \geq 1$ there are finitely many vertices. There are edges only from vertices at level n to vertices at level $n+1$, for all n. Each vertex has only finitely many edges leaving or entering it. For nontriviality we assume that each vertex has at least one edge leaving it and, except for R, at least one edge entering it. The set of edges entering each vertex is ordered. Often when the diagram is drawn we assume that the edges are ordered from left to right. The phase space X of the dynamical system based on the diagram is the set of infinite paths that begin at R. The space X is a compact metric space with the distance between two paths that agree on exactly the first n levels being $1/2^n$. A *cylinder set* is the set of all paths with a specified initial segment of finite length. That length is called the *length* of the cylinder. Cylinder sets are open and closed and form a base for the topology of X. We define a partial order on the set X. Two paths x and y are *comparable* if they coincide after some level. In this case we determine which of x and y is larger by comparing the last edges that differ. In this partial order there is a set $X_{\max}$ of maximal paths and a set $X_{\min}$ of minimal paths. The *adic transformation* $T : X \setminus X_{\max} \to X \setminus X_{\min}$ is defined by letting Tx be the smallest y such that $y > x$. Both T and T^{-1} are continuous where defined.

We recall quickly now how counting paths between vertices in a Bratteli diagram is related to the identification of ergodic invariant measures. For vertices P, Q of a directed graph, denote by $\dim(P, Q)$ the number of paths from P to Q. Cylinder sets determined by initial paths terminating at a common vertex are mapped to one another by powers of T and so they must be assigned equal measure by any invariant measure. For a path $x \in X$, denote by x_n the vertex of x at level n. In the case of the Pascal and Euler graphs, we give x_n the rectangular coordinates (i_n, j_n). It can be proved (see [13, 14]) by using either the Ergodic Theorem or Reverse Martingale Theorem that if μ is a T-invariant ergodic Borel probability measure on X and C is any cylinder set terminating at a vertex P, then

$$\mu(C) = \lim_{n\to\infty} \frac{\dim(P, x_n)}{\dim(R, x_n)} \quad \text{for } \mu\text{-almost every } x \in X\,. \tag{1.4}$$

In this paper we show that if μ is a fully supported ergodic measure for the adic system on the Euler graph, then for μ-almost every x the limit in (1.4) has the same value for any two cylinders of the same length. Consequently we show that there is only one fully supported ergodic T-invariant measure for the Euler adic system, namely, the symmetric measure, which assigns equal measure to all cylinder sets of a given length. In particular, if n is the length of a cylinder C, then $\mu(C) = 1/(n+1)!$.

THEOREM 1.1. *In the Euler graph, for each vertex P the limit*

$$\lim_{n\to\infty} \frac{\dim(P, (i_n, j_n))}{\dim(R, (i_n, j_n))} \tag{1.5}$$

exists for all infinite paths (i_n, j_n), $n \ge 0$, for which i_n and j_n are unbounded. Moreover, this limit is constant as P varies over the vertices at any fixed level.

Theorem 1.1, the new proof of which is the main point of this paper, is a corollary of Formula (1.3).

In Section 2 we introduce the recurrence relations for generalized Eulerian numbers. In Sections 3 and 4 we prove Formulas (1.1) and (1.2), respectively. In Section 5 we prove that

$$\frac{A_{p,q}(i,j)}{A_{p,q-1}(i,j)} \to \infty \quad \text{and} \quad \frac{A_{p,q}(i,j)}{A_{p-1,q}(i,j)} \to \infty \tag{1.6}$$

as both i, j tend to infinity.

In Section 6, we consider the set $\mathcal{A}_{p,q}(i,j)$ of all paths from (p,q) to $(p+i, q+j)$ and introduce a subset $\mathcal{G}_{p,q}(i,j) \subset \mathcal{A}_{p,q}(i,j)$ of "good" paths. Good paths are defined to be those which use each of a particular set of labels at least once (see Section 6). These are designed to substitute, in a way that will extend to higher-dimensional Euler adic systems, for the path-coding permutations in the two-dimensional case with only singleton "clusters" in [5] which are predominant. We show that almost all paths are good asymptotically as both i, j tend to infinity.

In Section 7 we show that the number of good paths from (p, q) to (i, j) is equal to the number of good paths from (p', q') to (i, j), if $p+q = p'+q'$ and $i, j \geq p+q+1$. This will complete the proof of Formula (1.3) and Theorem 1.1.

2. The Euler graph

The Euler graph is an infinite directed graph. The vertices of the graph are labeled by pairs of nonnegative integers (i, j). The list of edges is given by the rule: for any (i, j) there are $j+1$ edges from (i, j) to $(i+1, j)$ and $i+1$ edges from (i, j) to $(i, j+1)$; see Figure 1.

Define a *generalized Eulerian number* $A_{p,q}(i, j)$ to be the number of paths in the graph from (p, q) to $(p+i, q+j)$. For fixed p, q, the numbers satisfy the recurrence relation

$$A_{p,q}(i, j) = (j+q+1)A_{p,q}(i-1, j) \ + \ (i+p+1)A_{p,q}(i, j-1) \tag{2.1}$$

and initial conditions

$$A_{p,q}(i, 0) \ = \ (q+1)^i\,, \quad A_{p,q}(0, j) = (p+1)^j\,, \quad i, j > 0\,. \tag{2.2}$$

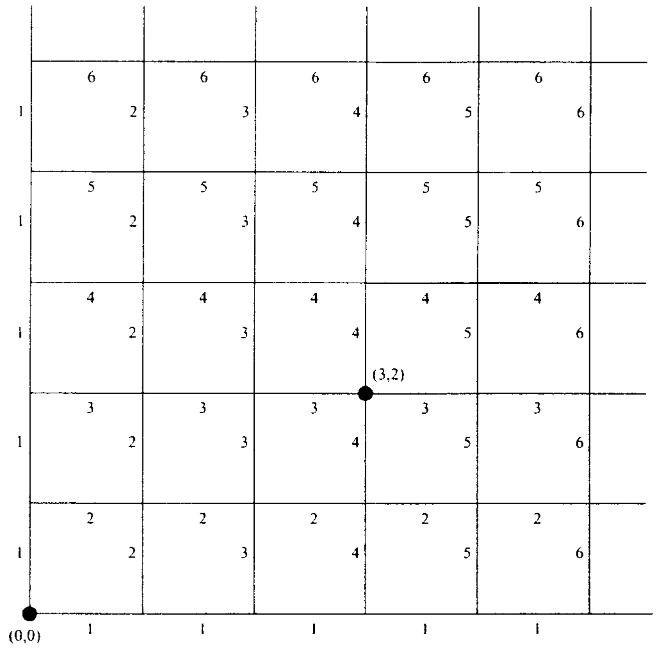

FIGURE 1. The first part of the Euler graph. Numbers along edges indicate multiple edges.

A closed form is known [3] for $A_{0,0}(i, j)$:

$$A_{0,0}(i, j) = \sum_{t=0}^{i} (-1)^{i-t} \binom{i+j+2}{i-t} (1+t)^{i+j+1} .$$

We develop a closed form for all $A_{p,q}(i, j)$ in the next section.

3. The recurrence relation

We say that a collection of numbers $\{A(i, j),\ i, j \geq 0,\ (i, j) \neq (0, 0)\}$ satisfies recurrence relation (2.1), if the relation holds for any $i, j > 0$. The numbers

$$A(i, 0),\ A(0, j) \quad \text{with} \quad i, j > 0$$

will be called the *initial conditions* of the collection.

It is clear that for any collection of numbers $\{a(i, 0),\ a(0, j),\ i, j \geq 0,\ (i, j) \neq (0, 0)\}$, there exists a unique collection $\{A(i, j),\ i, j \geq 0,\ (i, j) \neq (0, 0)\}$ satisfying the recurrence relation (2.1) and initial conditions

$$A(i, 0) = a(i, 0) , \quad A(0, j) = a(0, j) .$$

THEOREM 3.1. *The collection* $\{A_{p,q}(i, j),\ i, j \geq 0,\ (i, j) \neq (0, 0)\}$ *satisfying the recurrence relation* (2.1) *and initial conditions* (2.2) *is given by the following formula*:

$$\text{(3.1)} \quad A_{p,q}(i, j) \ = \ \sum_{t=0}^{i} (-1)^{i-t} \binom{p+q+t+1}{t} \binom{p+q+i+j+2}{i-t} (p+1+t)^{i+j} .$$

PROOF. Let $t = 0, 1, \ldots$. For $i, j \geq 0$, define the numbers

$$\begin{aligned} a_t(i, j) \ &= \ (-1)^{i-t} \binom{p+q+i+j+2}{i-t} (p+1+t)^{i+j} \\ &= \ (-1)^{i-t} \frac{\Gamma(p+q+i+j+3)}{\Gamma(i-t+1)\Gamma(p+q+j+t+3)} (p+1+t)^{i+j} , \end{aligned}$$

where $\Gamma(x)$ is Euler's gamma function. The last expression, in particular, says that

$$a_t(i, j) = 0 \quad \text{if} \quad j \geq 0 , \quad t > i \geq 0 .$$

LEMMA 3.2. *For any $t \geq 0$, the collection $\{a_t(i, j),\ i, j \geq 0,\ (i, j) \neq (0, 0)\}$ satisfies recurrence relation* (2.1) *with the initial conditions*

$$\begin{aligned} &a_t(0, j) = 0 \quad \text{if} \ t > 0 , \quad a_t(0, j) = (p+1)^j \quad \text{if} \ t = 0 , \\ &a_t(i, 0) = (-1)^{i-t} \binom{p+q+i+2}{i-t} (p+1+t)^i \end{aligned}$$

$$= (-1)^{i-t} \frac{\Gamma(p+q+i+3)}{\Gamma(i-t+1)\Gamma(p+q+t+3)} (p+1+t)^i ,$$

for all $i, j > 0$.

The lemma is proved by direct verification.

By Lemma 3.2 any linear combination $\{\sum_{t=0}^{\infty} C_t\, a_t(i, j),\ i, j \geq 0\}$ is well defined and satisfies relation (2.1). Therefore, let us look for $A_{p,q}(i, j)$ in the form

$$A_{p,q}(i, j) = \sum_{t=0}^{\infty} C_t\, (-1)^{i-t} \binom{p+q+j+i+2}{i-t} (p+1+t)^{i+j} .$$

The constants C_t can be found from the equations

$$\sum_{t=0}^{i} C_t\, (-1)^{i-t} \binom{p+q+i+2}{i-t} (p+1+t)^i = (q+1)^i .$$

LEMMA 3.3. *For all* $t \geq 0$,

$$C_t = \binom{p+q+t+1}{t} .$$

PROOF. We need to show that for any $i > 0$,

$$\sum_{t=0}^{i} (-1)^{i-t} \binom{p+q+t+1}{t} \binom{p+q+i+2}{i-t} (p+1+t)^i = (q+1)^i . \tag{3.2}$$

The right and left sides of (3.2) are polynomials in q of degree i. To see that the two polynomials are equal it is enough to check that they are equal at

$$q = -(p+2), -(p+3), \dots, -(p+2+i) .$$

If $q = -(p+2+t)$ for $0 \leq t \leq i$, then the left side has exactly one nonzero summand. Moreover, that nonzero summand equals the right-hand side polynomial at $q = -(p+2+t)$. More precisely, denote by $L(q)$ and $R(q)$ the left and right sides of (3.2). Then

$$L(q) = \left(\prod_{j=0}^{i} (p+q+j+2) \right) \sum_{t=0}^{i} (-1)^{i-t} \frac{1}{p+q+t+2} \frac{1}{t!\,(i-t)!} (p+1+t)^i .$$

Hence, for $0 \leq t \leq i$, we have

$$L(-p-2-t) = (-1)^t\, t!\,(i-t)!\, (-1)^{i-t} \frac{1}{t!\,(i-t)!} (p+1+t)^i = (-1)^i\, (p+1+t)^i .$$

But $R(-p-2-t) = (-p-1-t)^i$. □

Theorem 3.1 follows from Lemma 3.3. □

COROLLARY 3.4. *The collection* $\{A_{p,q}(i,j),\ i,j \geq 0\}$ *satisfying the recurrence relation* (2.1) *and initial conditions* (2.2) *is given also by the following formula*:

$$A_{p,q}(i,j) = \sum_{t=0}^{j} (-1)^{j-t} \binom{p+q+t+1}{t} \binom{p+q+i+j+2}{j-t} (q+1+t)^{i+j}. \tag{3.3}$$

PROOF. The corollary follows from Theorem 3.1 due to the symmetry of (2.1) and (2.2) with respect to the transformation $(p,q,i,j) \to (q,p,j,i)$. $\square$

4. Monotonicity of ratios

THEOREM 4.1. *For* $p,q \geq 0$, *assume that a collection of positive numbers* $\{a(i,j),\ i,j \geq 0,\ (i,j) \neq (0,0)\}$ *satisfies*

$$a(i,j) = (j+q+1)a(i-1,j) + (i+p+1)a(i,j-1)$$

and a collection of positive numbers $\{b(i,j),\ i,j \geq 0,\ (i,j) \neq (0,0)\}$ *satisfies*

$$b(i,j) = (j+q)b(i-1,j) + (i+p+1)b(i,j-1).$$

Assume that the initial conditions of these collections satisfy the inequalities:

$$\begin{aligned} \frac{a(0,j+1)}{b(0,j+1)} &\leq \frac{a(0,j)}{b(0,j)}, && j > 0, \\ \frac{a(0,1)}{b(0,1)} &\leq \frac{q}{q+1} \frac{a(1,0)}{b(1,0)}, \\ \frac{a(i,0)}{b(i,0)} &\leq \frac{q}{q+1} \frac{a(i+1,0)}{b(i+1,0)}, && i > 0. \end{aligned} \tag{4.1}$$

Then for any $i,j \geq 0$ *we have*

$$\frac{a(i,j+1)}{b(i,j+1)} \leq \frac{a(i,j)}{b(i,j)} \leq \frac{q+j}{q+1+j} \frac{a(i+1,j)}{b(i+1,j)}. \tag{4.2}$$

COROLLARY 4.2. *The generalized Eulerian numbers satisfy the inequalities*

$$\frac{A_{p,q}(i,j+1)}{A_{p,q-1}(i,j+1)} \leq \frac{A_{p,q}(i,j)}{A_{p,q-1}(i,j)} \leq \frac{q+j}{q+1+j} \frac{A_{p,q}(i+1,j)}{A_{p,q-1}(i+1,j)} \tag{4.3}$$

for all $p, q-1, i, j \geq 0$.

PROOF OF THEOREM 4.1. The proof is by induction. Denoting

$$\begin{aligned} x &= (i,j+1), & w &= (i+1,j+1), \\ y &= (i,j), & z &= (i+1,j), \end{aligned}$$

we assume that

$$\frac{a(x)}{b(x)} \leq \frac{a(y)}{b(y)} \leq \frac{q+j}{q+1+j} \frac{a(z)}{b(z)} \tag{4.4}$$

and prove

$$\frac{a(w)}{b(w)} \leq \frac{a(z)}{b(z)} \quad \text{and} \quad \frac{a(x)}{b(x)} \leq \frac{q+1+j}{q+2+j}\frac{a(w)}{b(w)} . \tag{4.5}$$

In fact, we will use not (4.4) but its corollary

$$\frac{b(z)}{b(x)} \leq \frac{q+j}{q+1+j}\frac{a(z)}{a(x)} . \tag{4.6}$$

We will use also the recurrence relations

$$\begin{aligned} a(w) &= (q+2+j)\, a(x) + (p+2+i)\, a(z) , \\ b(w) &= (q+1+j)\, b(x) + (p+2+i)\, b(z) . \end{aligned}$$

To prove the first inequality in (4.5), we need to prove that

$$\frac{a(w)}{a(z)} = (q+2+j)\frac{a(x)}{a(z)} + (p+2+i) \leq \frac{b(w)}{b(z)} = (q+1+j)\frac{b(x)}{b(z)} + (p+2+i)$$

or

$$\frac{a(x)}{a(z)} \leq \frac{q+1+j}{q+2+j} \cdot \frac{b(x)}{b(z)} . \tag{4.7}$$

Using (4.6) we write

$$\frac{a(x)}{a(z)} \leq \frac{q+j}{q+1+j} \cdot \frac{b(x)}{b(z)} \leq \frac{q+1+j}{q+2+j} \cdot \frac{b(x)}{b(z)} ,$$

and this gives (4.7).

To prove the second inequality in (4.5), we write

$$\frac{a(w)}{a(x)} = (q+2+j) + (p+2+i)\frac{a(z)}{a(x)}$$

and

$$\frac{q+2+j}{q+1+j} \cdot \frac{b(w)}{b(x)} = (q+2+j) + (p+2+i)\frac{q+2+j}{q+1+j} \cdot \frac{b(z)}{b(x)} .$$

Using inequality (4.6) we continue

$$\begin{aligned} &(q+2+j) + (p+2+i) \cdot \frac{q+2+j}{q+1+j} \cdot \frac{b(z)}{b(x)} \\ &\quad \leq (q+2+j) + (p+2+i) \cdot \frac{q+2+j}{q+1+j} \cdot \frac{q+j}{q+1+j} \cdot \frac{a(z)}{a(x)} \\ &\quad \leq (q+2+j) + (p+2+i) \cdot \frac{a(z)}{a(x)} = \frac{a(w)}{a(x)} . \end{aligned}$$

Thus, we get

$$\frac{q+2+j}{q+1+j}\cdot\frac{b(w)}{b(x)} \le \frac{a(w)}{a(x)}\,,$$

which is the second inequality in (4.5). □

REMARK 4.3. It is interesting that in order to prove monotonicity in each coordinate direction, we have to consider both directions simultaneously and we have to involve a speed in one of the two directions.

5. Limit theorems for generalized Eulerian numbers

To study asymptotics of the ratios discussed in the preceding section, we first determine their limits in the coordinate directions.

PROPOSITION 5.1. *For fixed* $i \ge 0$,

$$\frac{A_{p,q}(i,j)}{A_{p,q-1}(i,j)} \searrow \frac{p+q+i+1}{p+q+1} \quad \textit{as} \;\; j \to \infty; \tag{5.1}$$

while for fixed $j \ge 0$,

$$\frac{A_{p,q}(i,j)}{A_{p-1,q}(i,j)} \searrow \frac{p+q+j+1}{p+q+1} \quad \textit{as} \;\; i \to \infty. \tag{5.2}$$

PROOF. The dominant term in both the numerator and denominator of (5.1) occurs when $t = i$ in Formula (3.1), and the quotient of the two is $(p+q+i+1)/(p+q+1)$. The fact that this limit is a decreasing limit follows from Theorem 4.1 .

For Formula (5.2), we interchange the roles of p and q and use Corollary 3.4. □

THEOREM 5.2. *Both ratios*

$$\frac{A_{p,q}(i,j)}{A_{p,q-1}(i,j)} \quad \textit{and} \quad \frac{A_{p,q}(i,j)}{A_{p-1,q}(i,j)} \tag{5.3}$$

tend to ∞ *as both* i, j *tend to infinity*; *that is*, *given* M *there are* I, J *such that each ratio is greater than* M *whenever* $i \ge I$ *and* $j \ge J$.

PROOF. For the first ratio, given M choose I so that $(p+q+I-1)/(p+q-1) > M$. Then $A_{p,q}(i,j)/A_{p,q-1}(i,j) > M$ for all i, j with $i \ge I$ according to Proposition 5.1 and Corollary 4.2. A similar argument applies to the second ratio. □

6. Good paths

For any vertex of the Euler graph, we fix an order on the set of all horizontal edges exiting that vertex and an order on the set of all vertical edges exiting that vertex.

Let $\mathcal{A}_{p,q}(i,j)$ be the set of all paths from (p,q) to $(p+i,q+j)$. We will define a subset $\mathcal{G}_{p,q}(i,j) \subset \mathcal{A}_{p,q}(i,j)$ of good paths. The number of elements in $\mathcal{A}_{p,q}(i,j)$ is the Eulerian number $A_{p,q}(i,j)$. The number of elements in $\mathcal{G}_{p,q}(i,j)$ will be denoted by $G_{p,q}(i,j)$.

Fix (p,q) and call it a *base point*. For any $k,l \geq 0$, the vertex $(p+k,q+l)$ has $p+q+k+l+2$ edges leaving it, $q+l+1$ horizontally and $p+k+1$ vertically. We label the first $q+1$ horizontal edges by symbols $s_1,\dots,s_{q+1}$, respectively, and the first $p+1$ vertical edges by symbols $s_{q+2},\dots,s_{p+q+2}$, respectively.

A path $x \in \mathcal{A}_{p,q}(i,j)$ is a sequence of edges each of which is labeled or not. A path is called *good* if its edges have each of the labels $s_1,\dots,s_{p+q+2}$ at least once.

The subset $\mathcal{G}_{p,q}(i,j)$ of good paths is nonempty if and only if $i \geq q+1$ and $j \geq p+1$.

THEOREM 6.1. *For fixed* (p,q),

$$\frac{G_{p,q}(i,j)}{A_{p,q}(i,j)} \to 1 \tag{6.1}$$

as both i,j *tend to infinity.*

PROOF. It is clear that the number of elements of the set $\mathcal{A}_{p,q}(i,j) \setminus \mathcal{G}_{p,q}(i,j)$ of bad paths is not greater than $(q+1)A_{p,q-1}(i,j) + (p+1)A_{p-1,q}(i,j)$. By Theorem 5.2, the ratio

$$\frac{(q+1)A_{p,q-1}(i,j) + (p+1)A_{p-1,q}(i,j)}{A_{p,q}(i,j)} \to 0 \quad \text{as} \quad i,j \to \infty\,.$$

This implies the theorem. □

7. A one-to-one correspondence between two sets of good paths

THEOREM 7.1. *The number* $G_{p,q}(i-p,j-q)$ *of good paths from* (p,q) *to* (i,j) *is equal to the number* $G_{p',q'}(i-p',j-q')$ *of good paths from* (p',q') *to* (i,j), *if* $p+q = p'+q'$ *and* $i,j \geq p+q+2$.

PROOF. Denote $n = p+q = p'+q'$.

Take (p,q) as a base point. For any $k,l \geq 0$, using that base point define (as in Section 6) the labeled edges $s_1,\dots,s_{n+2}$ exiting any vertex $(p+k,q+l)$.

Similarly, take (p',q') as a base point. For any $k,l \geq 0$, using that base point define (as in Section 6 the labeled edges $s_1,\dots,s_{n+2}$ exiting any vertex $(p'+k,q'+l)$.

For any good path $x \in \mathcal{G}_{p,q}(i-p,j-q)$ from (p,q) to (i,j) (with respect to the first labels) we will construct a good path $y \in \mathcal{G}_{p',q'}(i-p',j-q')$ from (p',q') to (i,j) (with respect to the second labels). This construction will establish a bijection between the corresponding sets of good paths.

Let $x = (E_1,E_2,\dots,E_r)$, $r = i+j-p-q$, be a path from $x_0 = (p,q)$ to $x_r = (i,j)$, where E_m are edges and for every m, the edge E_m connects a vertex x_{m-1} of level $p+q+m-1$ to a vertex x_m of level $p+q+m$. We will encode the path x by a new sequence of symbols

$D(x) = (D_1, \dots, D_r)$, called the *encoding sequence* of x, with each label D_i coming from an alphabet

$$\mathcal{D} = \{s_1, s_2, \dots ; h_1, h_2, \dots ; v_1, v_2, \dots\}, \tag{7.1}$$

as follows.

For any m and any a, if one of the edges $E_1, \dots, E_m$ has label s_a, then we unlabel the edge exiting x_m with label s_a. This procedure decreases the number of labeled edges exiting x_m. The edges exiting x_m which remain labeled will be called the *marked* edges. The edges exiting x_m which lost a label or were initially unlabeled will be called the *unmarked* edges.

For any m, we set $D_m = D(E_m)$ to be s_a if E_m is a marked edge with label s_a. Now we label the unmarked edges. We set D_m to be h_a if E_m is the a'th horizontal unmarked edge among the set of horizontal unmarked edges. We set D_m to be v_a if E_m is the a'th vertical unmarked edge among the set of vertical unmarked edges. (Recall that the set of horizontal edges exiting each vertex is ordered and the set of vertical edges exiting each vertex is ordered, so that the notion of the a'th edge among the unmarked horizontal (or vertical) edges is well defined.)

LEMMA 7.2. *Let x be a path from (p, q) to (i, j). Let $D(x) = (D_1, \dots, D_r)$ be its encoding sequence. Then for any $m \geq 1$, we have the following two statements*:

(1) *The number of horizontal unmarked edges exiting x_m equals the sum of the number of marked edges among $E_1, \dots, E_m$ and the number of vertical unmarked edges among $E_1, \dots, E_m$.*

(2) *The number of the vertical unmarked edges exiting x_m equals the sum of the number of marked edges among $E_1, \dots, E_m$ and the number of horizontal unmarked edges among $E_1, \dots, E_m$.*

PROOF. For any m, denote by H_m and V_m the numbers of unmarked horizontal and unmarked vertical edges leaving x_m, respectively. If E_m is a marked edge, then $H_m = H_{m-1} + 1$ and $V_m = V_{m-1} + 1$. If E_m is a horizontal unmarked edge, then $H_m = H_{m-1}$ and $V_m = V_{m-1} + 1$. If E_m is a vertical unmarked edge, then $H_m = H_{m-1} + 1$ and $V_m = V_{m-1}$. $\square$

Similarly to the above construction, for any path y from (p', q') to (i, j) we can define its encoding sequence $D(y) = (D_1, \dots, D_r)$, using the labels with respect to the base point (p', q'). Again every D_m is s_a, h_a or v_a for a suitable a.

LEMMA 7.3. *There is a bijection*

$$B : \mathcal{G}_{p,q}(i - p, j - q) \to \mathcal{G}_{p',q'}(i - p', j - q'), \quad x \mapsto y,$$

that is well defined by choosing y to satisfy the condition $D(y) = D(x)$.

PROOF. Let $x = (E_1, \dots, E_r)$ be a path from $x_0 = (p, q)$ to $x_r = (i, j)$ with encoding sequence $D(x) = (D_1, \dots, D_r)$. We need to show that there exists a unique path $y =$

$(E'_1, \dots, E'_r)$ from $y_0 = (p', q')$ to $y_r = (i, j)$ with encoding sequence $D(y)$ such that $D(y) = D(x)$. We prove the existence of edges E'_m by induction on m.

All edges exiting x_0 and y_0 are marked. In both cases the marks are $s_1, \dots, s_{n+2}$ where $n = p + q = p' + q'$. If E_1 has a mark s_a, then E'_1 is chosen to be the edge exiting y_0 with mark s_a.

Assume that for some $m > 1$ a path $(E'_1, \dots, E'_{m-1})$ from y_0 to y_m is constructed so that $D(E'_1, \dots, E'_{m-1}) = D(E_1, \dots, E_{m-1})$. By Lemma 7.2, the vertices y_m and x_m have the same number of exiting unmarked horizontal edges, the same number of exiting unmarked vertical edges, and the same number of exiting marked edges. Moreover, the exiting edges from x_m and y_m have exactly the same set of labels. Hence, for any $D(E_m)$ there exists a unique edge E'_m exiting y_m with $D(E'_m) = D(E_m)$.

Thus, there exists a unique path $y = (E'_1, \dots, E'_r)$ from $y_0 = (p', q')$ such that $D(y) = D(x)$. It is easy to see that y ends at (i, j). □

Lemma 7.3 implies Theorem 7.1. □

THEOREM 7.4.

$$\frac{A_{p,q}(i, j)}{A_{0,0}(p + i, q + j)} \quad \text{tends to} \quad \frac{1}{(p + q + 1)!}$$

as both i, j tend to infinity.

This theorem is a direct corollary of Theorems 6.1 and 7.1. Theorem 7.4 implies Theorem 1.1.

REMARK 7.5. In this remark, we explain briefly the statements in the introduction about generic points and directional unique ergodicity. If $x = \{(i_n, j_n), n \geq 0\}$ is generic for a fully supported measure μ, then i_n, j_n must be unbounded, since otherwise using x in Formula (1.4) will assign measure 0 to many cylinders. Conversely, let x be any path with i_n, j_n unbounded. Let C be the cylinder determined by an initial path of length n_0. Then

$$\lim_{n\to\infty} \frac{\dim((i_{n_0}, j_{n_0}), (i_n, j_n))}{\dim(R, (i_n, j_n))} = \mu(C) = \frac{1}{(n_0 + 1)!}\,. \tag{7.2}$$

Thus, each path with unbounded i_n, j_n is generic for the symmetric measure.

Moreover, it is not necessary to speak of paths. If C is a cylinder set with terminal vertex P at level n_0, then given $\varepsilon > 0$ there is M such that

$$\left| \frac{\dim(P, (i, j))}{\dim(R, (i, j))} - \frac{1}{(n_0 + 1)!} \right| < \varepsilon \tag{7.3}$$

for all $i, j \geq M$.

REMARK 7.6. The approach presented here to prove ergodicity and unique fully supported ergodicity of the symmetric measure on the Euler adic system was developed so as to apply to the case of the higher-dimensional Euler adics. It seems that the correspondence of good paths and monotonicity arguments extend readily. An important step is to develop necessary formulas extending those for the $A_{p,q}(i, j)$.

ACKNOWLEDGMENT. The authors thank Sarah Bailey Frick and Xavier Méla for conversations on this topic and Thomas Prellberg for finding reference [2].

References

[1] S. BAILEY, M. KEANE, K. PETERSEN and I. SALAMA, Ergodicity of the adic transformation on the Euler graph, Math. Proc. Camb. Phil. Soc. **141** (2006), 231–238.

[2] L. CARLITZ and RICHARD SCOVILLE, Generalized Eulerian numbers: combinatorial applications, J. Reine Angew. Math. **265** (1974), 110–137. MR 0429575 (55 #2587)

[3] LOUIS COMTET, *Advanced Combinatorics*, enlarged, D. Reidel Publishing Co., Dordrecht, 1974, The Art of Finite and Infinite Expansions.

[4] PHILIPPE FLAJOLET, PHILIPPE DUMAS and VINCENT PUYHAUBERT, Some exactly solvable models of urn process theory, DMTCS Proceedings, Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities (2006), 59–118, http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/issue/view/84.

[5] SARAH BAILEY FRICK and KARL PETERSEN, Random permutations and unique fully supported ergodicity for the Euler adic transformation, Ann. Inst. Henri Poincaré Probab. Stat. **44** (2008), 876–885. MR **2453848** (**2009h**:37004) (English, with English and French summaries)

[6] SARAH BAILEY FRICK and KARL PETERSEN, Reinforced random walks and adic transformations, to appear in J. Theor. Prob.

[7] THIERRY GIORDANO, IAN F. PUTNAM and CHRISTIAN F. SKAU, Topological orbit equivalence and $C(X, \mathcal{B}, \mu)*$-crossed products, J. Reine Angew. Math. **469** (1995), 51–111.

[8] ALEXANDER GNEDIN and GRIGORI OLSHANSKI, The boundary of the Eulerian number triangle, Mosc. Math. J. **6** (2006), 461–475. MR **2274860** (**2007j**:05006)

[9] RICHARD H. HERMAN, IAN F. PUTNAM and CHRISTIAN F. SKAU, Ordered Bratteli diagrams, dimension groups and topological dynamics, Internat. J. Math. **3** (1992), 827–864.

[10] XAVIER MÉLA, Dynamical Properties of the Pascal Adic and Related Systems, Ph. D. Dissertation, Chapel Hill, 2002.

[11] XAVIER MÉLA and KARL PETERSEN, Dynamical properties of the Pascal adic transformation, Ergodic Theory Dynam. Systems **25** (2005), 227–256.

[12] KARL PETERSEN and KLAUS SCHMIDT, Symmetric Gibbs measures, Trans. Amer. Math. Soc. **349** (1997), 2775–2811.

[13] A. M. VERSHIK, Description of invariant measures for the actions of some infinite-dimensional groups, Dokl. Akad. Nauk SSSR **218** (1974), 749–752.

[14] A. M. VERSHIK and S. V. KEROV, Asymptotic theory of characters of the symmetric group, Funkts. Anal. Prilozhen. **15** (1981), 15–27.

[15] A. M. VERSHIK, A theorem on Markov periodic approximation *in ergodic theory*, **115** (1982), 72–82, 306, transl. in J. Sov. Math. 28 (1985), 667–673. MR **660072** (**84m**:28030)

[16] A. M. VERSHIK and S. V. KEROV, Locally semisimple algebras, combinatorial theory and the K_0-functor, J. Soviet Math. **38** (1987), 1701–1733.

[17] A. M. VERSHIK and A. N. LIVSHITS, Adic models of ergodic transformations, spectral theory, substitutions, and related topics, Adv. Soviet Math. **9** (1992), 185–204. MR **1166202** (**93i**:46131)

Present Addresses:
KARL PETERSEN
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL,
CHAPEL HILL, NC 27599–3250, USA.
e-mail: petersen@math.unc.edu

ALEXANDER VARCHENKO
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL,
CHAPEL HILL, NC 27599–3250, USA.
e-mail: anv@math.unc.edu