TOKYO J. MATH.
VOL. 33, NO. 2, 2010

The Stabilizer of 1 in Habegger-Lin's Action for String Links

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(Communicated by J. Murakami)

Abstract. Let $H(k)$ be the group of all homotopy classes of k-string links. It has been proved that $f, g \in H(k)$ have the same closure if and only if there is $\beta \in S_k(1)$ such that $\beta \cdot f = g$, where $S_k(1)$ is the stabilizer of 1 for a certain action of the group $H(2k)$ on the set $H(k)$. If $\beta \in S_k(1)$, Artin's automorphism $\overline{\beta}$, induced by β on $RF(2k)$, the reduced free group in $2k$ generators, induces an automorphism $\overline{\overline{\beta}} \in A(RF(k))$, the group of all automorphisms of $RF(k)$ that send each generator to one of its conjugates. This can be used to compare the homotopy classes of links obtained by closing f and g. The association $\beta \mapsto \overline{\overline{\beta}}$ is a homomorphism from $S_k(1)$ to $A(RF(k))$. In this paper we determine its kernel.

1. Introduction

Artin [A1] introduced the group of (isotopy classes of) braids in k strings and proposed to use them in order to study knots and links. He also introduced an isomorphism from this group onto a certain group of automorphisms of a free group $F(k)$ in k generators, what is called Artin's representation theorem. Every link can be obtained by closing a braid [Aℓ], however many different braids provide the same link and the condition for two braids to have the same closure, provided by Markov [Ma], is not an easy one to verify due to the fact that braids with a different number of strings may have the same closure.

Artin [A2] also proposed a notion of homotopy for braids. The group of braids up to homotopy was studied by Goldsmith [G1].

Milnor [M1] proposed a notion of homotopy for links. He also obtained a homotopy classification of links with 3 or fewer components. Levine [L2] provided such a classification for links with 4 components. Habegger and Lin [HL] studied the relation between braids and links up to homotopy, finally obtaining the classification of links up to homotopy.

In order to do that, a generalization of the concept of braid was useful. That was the notion of a (pure) string link (see Definition 1). Even though any string link is homotopic to a braid, the concept simplifies the study of the relation between braids and links. Although the set of (isotopy classes of) string links with k strings does not form a group, Habegger and Lin showed that the set of homotopy classes of k-string links forms a group and provided

Received January 19, 2009; revised April 17, 2010
2000 *Mathematics Subject Classification*: 57M25 (Primary)
Key words and phrases: string link, link-homotopy, braid

for this group an analogous of Artin's representation theorem, where the free group $F(k)$ is replaced by the reduced free group $RF(k)$ (see Definition 3). All homotopy classes of links of k components can be obtained by closing only the homotopy classes of string links of k strings. Nevertheless, there still are different homotopy classes of string links with the same closure. Habegger and Lin however obtained a necessary and sufficient condition for two homotopy classes of string links to have the same closure.

In order to obtain their condition, they introduced an action of the group $H(2k)$, of homotopy classes of $2k$-string links, on the set $H(k)$, of homotopy classes of k-string links, and showed that $f, g \in H(k)$ have the same closure if and only if there is $\beta \in S_k(1)$ such that $\beta \cdot f = g$, where $S_k(1) = \{\beta \in H(2k) \mid \beta \cdot 1 = 1 \in H(k)\}$ is the stabilizer of 1 for the action. Due to that, $S_k(1)$ plays a key role in the Habegger-Lin classification of links up to link-homotopy and in the relation between the set of homotopy classes of links of k components and the group of homotopy classes of k-string links. Habegger-Lin's approach however was similar to that used by Artin and Markov to study isotopy of links, and still left open the problem of finding a more purely algebraic classification of links up to homotopy, like that one started by Milnor and Levine.

In [C2] we observe that the symmetry of being an element β of $S_k(1)$ can be algebraically characterized by the fact that Artin's automorphism $\overline{\beta}$ of $RF(2k)$, associated to β, induces a certain automorphism $\overline{\overline{\beta}} \in A(RF(k))$, the group of all automorphisms of $RF(k) = RF(x_1, \ldots, x_k)$ that send each generator x_i to one of its conjugates (the so called

$$\begin{tikzcd}
RF(k) \arrow[r, "\overline{\overline{\beta}}"] & RF(k) \\
RF(2k) \arrow[u, "\xi'"] \arrow[r, "\overline{\beta}"] & RF(2k) \arrow[u, "\xi'"'] \\
RF(2k) \arrow[u, "\bar{g} * 1"] \arrow[r, "\overline{\gamma}"] \arrow[d, "\xi'"'] & RF(2k) \arrow[u, "\bar{f} * 1"'] \arrow[d, "\xi'"] \\
RF(k) \arrow[r, "\overline{\overline{\gamma}}"] & RF(k)
\end{tikzcd} \tag{i}$$

"special automorphisms" of $RF(k)$). The association $\beta \mapsto \overline{\overline{\beta}}$ is a homomorphism that we use there to provide an algebraic criterion to decide when two k-string links f , g have the same closure up to link-homotopy. That happens if and only if there exists a certain commutative diagram of group homomorphisms (see diagram (i) above) where $\beta, \gamma \in S_k(1)$.

In this paper we provide information about $S_k(1)$ and about the previous diagram by computing the kernel of the homomorphism$^{(*)}$ $\beta \longmapsto \overline{\overline{\beta}}$ (Theorem 10).

2. String Links Homotopy

In this paragraph we briefly review some results of [HL].

We shall use the following notation: I is the interval $[0, 1]$, D is the unit disk $\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, $k \geq 1$ is an integer number, $\boldsymbol{k}$ is the set $\{1, 2, ..., k\}$, a_i is the point $\left(-1 + \frac{2i}{k+1}\,,\, 0\right) \in D$ for $i \in \boldsymbol{k}$ and $j_0 : \boldsymbol{k} \times I \to D \times I$ is the map defined by $j_0(i, x) = (a_i, x)$.

DEFINITION 1. A ***k*-string link** is a smooth or piecewise linear proper imbedding $f : \boldsymbol{k} \times I \to D \times I$ such that $f\,|_{\boldsymbol{k} \times \partial I} = j_0|_{\boldsymbol{k} \times \partial I}$.

A k-string link, like a braid, has a **closure** $\hat{f}$, that is a link of k components $\hat{f}$: $\bigcup_{i=1}^{k} S_i^1 \to S^3$ formed by identifying points of $\partial(D \times I)$ with their images under the projection $D \times I \to D$.

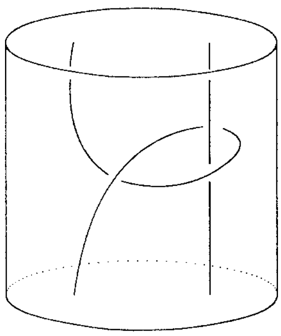

FIGURE 1. a 2-string link

DEFINITION 2. k-string links f and g are ***link-homotopic*** if there is a homotopy of the strings in $D \times I$, fixing the endpoints and deforming f to g, such that the images of different strings remain disjoint during the deformation.

$^{(*)}$In [C2] we affirmed that this homomorphism was an epimorphism but there was a mistake in our argument. This remains an open question.

We will denote by $H(k)$ the set of link-homotopy classes of k-string links.

The product of two k-string links f and g is given by stacking f on the top of g and reparametrizing.

With this product, $H(k)$ becomes a group. The homotopy class of j_0 is the neutral element.

DEFINITION 3. Let $F(k)$ be the free group in k generators $\alpha_1, \alpha_2, \ldots, \alpha_k$, and $RF(k)$ the quotient group obtained from $F(k)$ by adding relations which say that each α_i commutes with all of its conjugates. We call $RF(k)$ the ***reduced free group*** in k generators.

DEFINITION 4. Let f be a k-string link and let $X(f) = (D \times I) \setminus f(\boldsymbol{k} \times I)$ be the complement of the strings. The group $\pi_1(X(f), p)$, where $p = (0, -1, 0)$ is called the **group of** f and is denoted by $\pi(f)$.

Let f be a k-string link. We shall denote by $x_i = x_i(f)$, for $i \in \boldsymbol{k}$, the top meridians of f and by $y_i = y_i(f)$, for $i \in \boldsymbol{k}$, the bottom meridians of f (see [C1]).

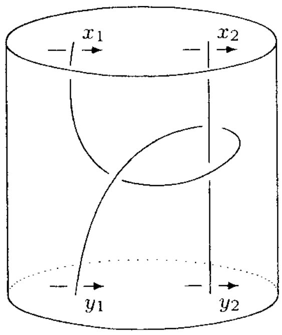

FIGURE 2. top and bottom meridians of a 2-string link

There are homomorphisms $\mu_0(f) : F(k) = F(\alpha_1, \alpha_2, \ldots, \alpha_k) \to \pi(f)$, $(\alpha_i)\mu_0(f) = x_i(f)$, and $\mu_1(f) : F(k) \to \pi(f)$, $(\alpha_i)\mu_1(f) = y_i(f)$, called respectively, the **top meridian map for** f and the **bottom meridian map for** f. Note that, as above, maps between groups will be written on the right of the argument.

Let $R\pi(f)$ be the quotient group obtained from $\pi(f)$ by adding relations which say that each $x_i(f)$ commutes with all of its conjugates.

Similarly to what Artin did for braids, only replacing $F(k)$ by $RF(k)$ and $\pi(f)$ by $R\pi(f)$, Habegger and Lin showed that if f is a k-string link, $\mu_0(f)$ and $\mu_1(f)$ induce isomorphisms

$$RF(k) \underset{\mu_0'(f)}{\overset{\cong}{\longrightarrow}} R\pi(f) \underset{\mu_1'(f)}{\overset{\cong}{\longleftarrow}} RF(k)$$

that provide an automorphism $\overline{f} = \mu'_0(f)\mu'_1(f)^{-1}$ of $RF(k)$ satisfying conditions

(1) $(\alpha_i)\overline{f}$ is a conjugate $u_i^{-1}\,\alpha_i\,u_i$ of α_i for any $i \in \mathbf{k}$, and

(2) $(\alpha_1\,\alpha_2\cdots\alpha_k)\overline{f} = \alpha_1\,\alpha_2\cdots\alpha_k$.

Furthermore, if f and g are link homotopic, $\overline{f} = \overline{g}$, and the association $f \mapsto \overline{f}$ is an isomorphism between the group $H(k)$ of link homotopic classes of k-string links and the group $A_0(RF(k))$ of all automorphisms of $RF(k)$ satisfying conditions (1) and (2) above.

Habegger and Lin also studied the problem of relating homotopy classes of string links to homotopy classes of links, obtained by closing them. They first introduced an action of the group $H(2k)$ on the set $H(k)$. Consider a three ball B decomposed in three subballs B_+ , B_0 and B_- .

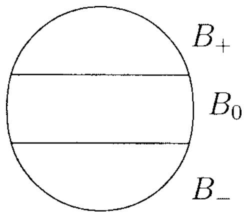

FIGURE 3. 3-ball B

$2k$ points $p_1, \ldots, p_k, \widetilde{p}_k, \ldots, \widetilde{p}_1$ are chosen in $B_+ \cap B_0$ and $2k$ points $p'_1, \ldots, p'_k, \widetilde{p}'_k, \ldots, \widetilde{p}'_1$ are chosen in $B_0 \cap B_-$.

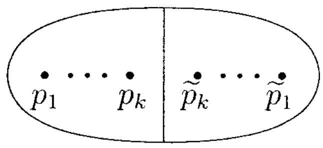

FIGURE 4. $B_+ \cap B_0$

We may think of a $2k$-string link β as a (proper smooth or piecewise linear) imbedding $\beta : \mathbf{2k} \times I \to B_0$ such that $\beta(i, 0) = p_i$, $\beta(i, 1) = p'_i$, $\beta(k+i, 0) = \widetilde{p}_{k-i+1}$ and $\beta(k+i, 1) = \widetilde{p}'_{k-i+1}$ for any $i \in \mathbf{k}$.

A k-string link f may be considered as an imbedding $f : \boldsymbol{k} \times I \to B_-$ such that $f(i, 0) = p'_i$ and $f(i, 1) = \widetilde{p}'_i$ for any $i \in \mathbf{k}$. The "union" of f and β gives rise to an imbedding $\boldsymbol{k} \times I \to B_0 \cup B_-$ that may be considered as a k-string link $\beta \cdot f$ if as a part of a string of $\beta \cdot f$, the orientation of $\beta(i \times I)$ for $i \in \{k+1, \ldots, 2k\}$ is reversed.

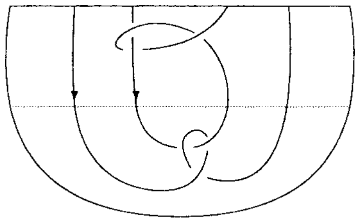

FIGURE 5. $B_0 \cup B_-$

What has been done for string links can be done for the homotopies between them so as to have an action of the group $H(2k)$ on the set $H(k)$ and the closure of a homotopy between string links.

Let $S_k(1)$ be the stabilizer of $1 \in H(k)$ for this action of the group $H(2k)$ on the set $H(k)$, that is, $S_k(1) = \{\beta \in H(2k) \mid \beta \cdot 1 = 1 \in H(k)\}$.

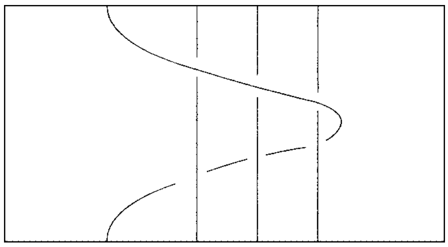

FIGURE 6. An element of $S_2(1)$.

THEOREM 1 (Habegger-Lin). *Suppose $f, g \in H(k)$, then $\widehat{f} = \widehat{g}$ (that is, the closures of their representatives are link-homotopic) if and only if there is $\beta \in S_k(1)$ such that $\beta \cdot f = g$.*

PROOF. see [HL]. ∎

3. The homomorphism Ψ_k

Let $F(2k)$ be the free group in $2k$-generators $x_1, x_2, \ldots, x_k, \widetilde{x_k}, \ldots, \widetilde{x_2}, \widetilde{x_1}$ and $\xi : F(2k) \to F(k)$ be the homomorphism defined by $(x_i)\xi = x_i$ and $(\widetilde{x_i})\xi = x_i^{-1}$ for any $i \in \mathbf{k}$.

Let $A(RF(k))$ be the group of all automorphisms of $RF(k)$ that satisfy condition

(1) $(x_i)f$ is a conjugate $u_i^{-1} x_i u_i$ of x_i for any $i \in \mathbf{k}$.

A proof for the next theorem can be found in [C2].

THEOREM 2. *For any $k \geq 1$, there is a homomorphism $\theta'_k : S_k(1) \to A(RF(k))$, $(\beta)\theta'_k = \overline{\overline{\beta}}$ such that the following diagram is commutative*

$$\begin{tikzcd} RF(2k) \arrow[r, "\overline{\beta}"] \arrow[d, "\xi'"'] & RF(2k) \arrow[d, "\xi'"] \\ RF(k) \arrow[r, "\overline{\overline{\beta}}"] & RF(k) \end{tikzcd}$$

where ξ' is induced by ξ. ∎

In [HL] it is observed that if $\Sigma \in H(2k)$ has strings $1, 2, \ldots, k, \widetilde{k}, \ldots, \widetilde{2}, \widetilde{1}$ and $\sigma \in H(k)$, for $i \in \boldsymbol{k}$, the i-th and $\widetilde{i}$-th strings of Σ, that we shall call **symmetric strings**, become part of the i-th string of $\Sigma \cdot \sigma$. Hence if Σ is changed by crossings of these pairs of strings, the link-homotopy class of $\Sigma \cdot \sigma$ remains unchanged. Thus the action of $H(2k)$ factors through a quotient, denoted by $H^*(2k)$, where two string links are equivalent if by crossing pairs of symmetric strings, if necessary, they are homotopic. The stabilizer of 1 for the new action is denoted by $S^*_k(1) = \{\beta \in H^*(2k) \mid \beta \cdot 1 = 1 \in H(k)\}$.

Let $RF^*(2k)$ be the quotient of $RF(2k) = RF\,(x_1, x_2, \ldots, x_k, \widetilde{x_k}, \ldots, \widetilde{x_2}, \widetilde{x_1})$ by the relations that say that conjugates of x_i commute with conjugates of $\widetilde{x_i}$ for any $i \in \mathbf{k}$. If $\beta \in H(2k)$, its automorphism $\overline{\beta} \in A(RF(2k))$ induces an automorphism of $RF^*(2k)$. If $\beta_1, \beta_2 \in H(2k)$ represent the same element in $H^*(2k)$ they induce the same automorphism in $RF^*(2k)$. It follows that if $\beta_1, \beta_2 \in S_k(1)$ represent the same element in $S^*_k(1)$, then $\overline{\overline{\beta}}_1 = \overline{\overline{\beta}}_2$. Therefore θ'_k induces a homomorphism $\Psi_k : S^*_k(1) \longrightarrow A(RF(k))$. We shall denote $(\beta)\Psi_k$ by $\overline{\overline{\beta}}$. Our main goal is to determine the kernel of Ψ_k.

4. Generators for $S^*_k(1)$

The following theorem is a consequence of [HL : Lemma 2.11.].

THEOREM 3. *$S^*_k(1)$ is generated by the class of the $2k$-string links a_{ij}, b_{ij}, c_{ij} $(1 \leq i < j \leq k)$ given in figure 7 below.*

PROOF. By omitting the external pair of strings (strings 1 and $\widetilde{1}$ in our notation), Habegger and Lin obtained a split short exact sequence

$$\text{(I)} \qquad 1 \longrightarrow K_{1k} \longrightarrow S^*_k(1) \longrightarrow S^*_{k-1}(1) \longrightarrow 1$$

and proved that K_{1k} is generated by the elements a_{1j}, β^{-1}_{1j}, γ^{-1}_{1j} $(2 \leq j \leq k)$ where β_{1j} and γ_{1j} can be represented, in our notation, as in figure 8 below.

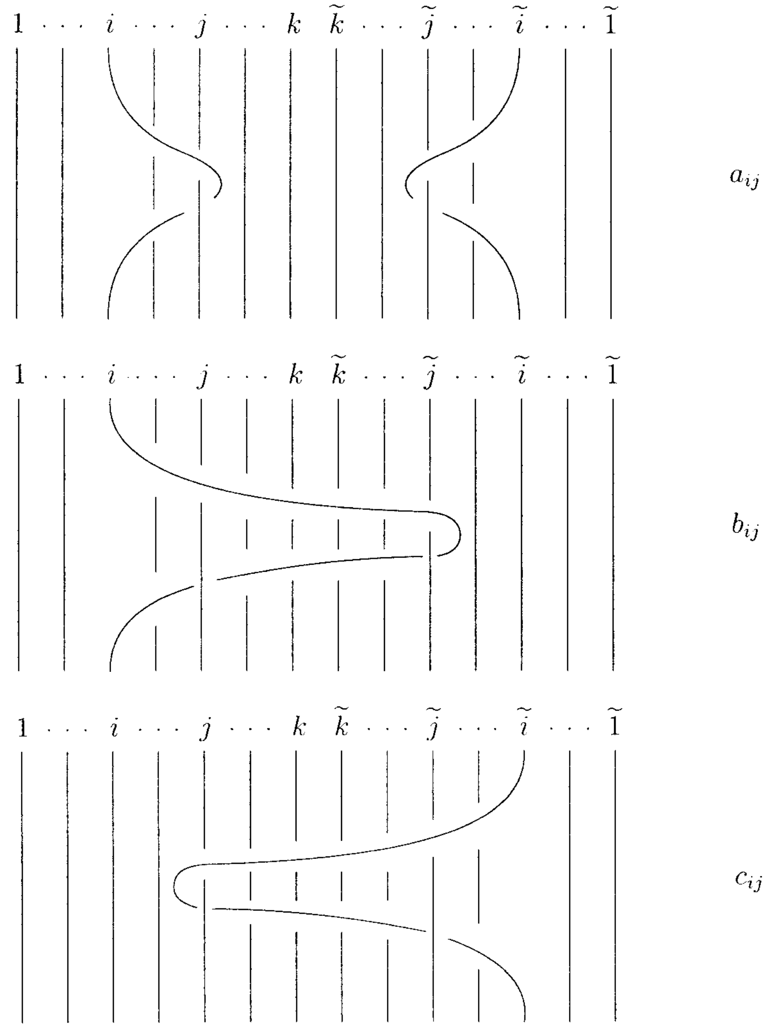

FIGURE 7

Now it is enough to observe that, for $j \in \{2, 3, \ldots, k\}$, $\beta_{1j} = b_{1j}\, a_{1j}^{-1}$ and $\gamma_{1j} = c_{1j}\, a_{1j}^{-1}$. ∎

Let us consider again the disk D with $k > 1$ aligned points $a_1, a_2, \ldots, a_k$. Any element $\lambda \in F(k-1)$ represents a path γ in $D \setminus \{a_2, a_3, \ldots, a_k\}$ based at a_1. A pure k-braid $\sigma(\lambda)$ in $D \times I$ can be associated to λ by letting the i-th string be given by $t \mapsto (\gamma(t), t)$

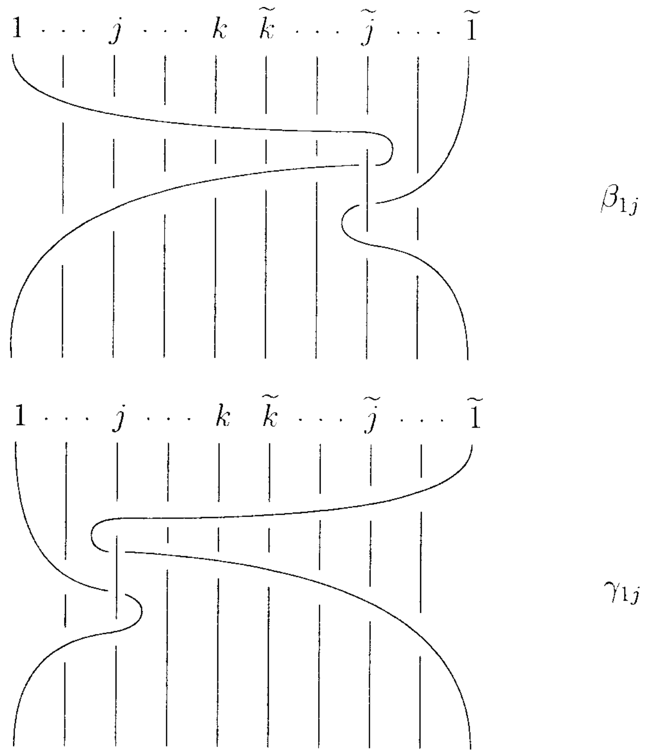

FIGURE 8

if $i = 1$ and $t \mapsto (a_i, t)$ if $i > 1$. This provides a split short exact sequence

$$1 \longrightarrow F(k-1) \xrightarrow{\sigma} PB(k) \longrightarrow PB(k-1) \longrightarrow 1\,, \tag{II}$$

where $PB(i),\ i \in \{k{-}1,\ k\}$, is the group of pure i-braids and the map $PB(k) \to PB(k-1)$ is given by omitting the first string.

The same reasoning taking into account the equivalence classes provides the split short exact sequences below (see [HL]).

$$1 \to RF(k-1) \to H(k) \to H(k-1) \to 1\,, \tag{III}$$

$$1 \to RF^*(2k-2) \times RF^*(2k-2) \to H^*(2k) \to H^*(2k-2) \to 1\,, \tag{IV}$$

where in (IV) the map $H^*(2k) \to H^*(2k-2)$ is given by omitting the external pair of strings as in the short exact sequence (I) . Comparing (I) and (IV) we see that each element β in K_{1k} is given by an ordered pair $(\ell_1\,,\ \widetilde{\ell}_1)$ of elements of $RF^*(2k{-}2) = RF^*(x_2, \ldots, x_k, \widetilde{x}_k, \ldots, \widetilde{x}_2)$. For example, a_{1j} is given by pair $(x_j^{-1}\,,\ \widetilde{x}_j)$, b_{1j} is given by pair $(\widetilde{x}_j^{-1} x_j^{-1}, 1)$ and c_{1j} is given by pair $(1, x_j\, \widetilde{x}_j)$. Furthermore, since β stabilizes 1 it follows that $[\ell_1]^{-1}[\widetilde{\ell_1}] = 1$ thus $[\ell_1] = [\widetilde{\ell_1}]$, where $[\ \ ]$ stands for the equivalence class in $RF(k-1) = RF(x_2, \ldots, x_k)$ obtained by identifying $\widetilde{x_i}$ with x_i^{-1} $(i \in \{2, 3, \ldots, k\})$.

5. The kernel of Ψ_k

Ψ_k is not injective, for example, $\overline{\overline{b_{ij}}} = \overline{\overline{c_{ij}}}$, for $1 \le i < j < k$. In order to get some of its geometric flavour, let us consider the following proposition, where β' is the k-string link obtained from a $2k$-string link β by deleting the last k strings.

PROPOSITION 4. *Given any $\gamma \in H(k)$, there is $\beta \in \ker \Psi_k$ with $\beta' = \gamma$.*

PROOF. Let us take a pure k-braid that represents γ (see [HL] for its existence). It can be written (see [B]) as a product of the k-braids a'_{ij}. Now it is enough to replace each occurrence of a'_{ij} by $c_{ij}^{-1}\, b_{ij}$. ■

Furthermore β' does not determine β'' for $\beta \in \ker \Psi_k$, where β'' is the k-string link obtained from β by deleting its first k strings. This immediately follows from the next proposition (with b_{1j} replaced by c_{1j}). Let us first observe that the association $a_{1j} \mapsto a'_{1j}$ is an isomorphism from $\langle a_{1j} \mid j = 2, \ldots, k \rangle$ to $\langle a'_{1j} \mid j = 2, \ldots, k \rangle$ and, if we identify these groups, then the restriction $\Psi_k \mid \langle a_{1j} \mid j = 2, \ldots, k \rangle$ becomes the isomorphism $a'_{1j} \mapsto \overline{a'_{1j}}$. The association $b_{1j} \mapsto b'_{1j}$ is also an isomorphism from $\langle b_{1j} \mid j = 2, \ldots, k \rangle$ to $\langle b'_{1j} \mid j = 2, \ldots, k \rangle = \langle a'_{1j} \mid j = 2, \ldots, k \rangle \cong RF(k-1)$, but in this case we have the following proposition, where $\mathbb{Z}$ is the abelian group of integers and $\langle b_{1j} \rangle = \langle b_{1j} \mid j = 2, \ldots, k \rangle$.

PROPOSITION 5. *There is a commutative diagram of group homomorphisms*

$$\begin{tikzcd}
\langle b_{1j} \rangle \arrow[r, "\Psi_k \mid \langle b_{1j} \rangle"] \arrow[d, "\cong"'] & im(\Psi_k \mid \langle b_{1j} \rangle) \arrow[d, "\cong"] \\
RF(k-1) \arrow[r] & \mathbb{Z}^{k-1}
\end{tikzcd}$$

where the vertical maps are isomorphisms and the map $RF(k-1) \to \mathbb{Z}^{k-1}$ is the abelianization.

PROOF. For any $s \in \{2, \ldots, k\}$, we have

$$(x_i)\,\overline{\overline{b_{1s}}} = \begin{cases} x_i & \text{if } \ i < s \\ x_1^{-1}\, x_s\, x_1 & \text{if } \ i = s \\ [x_1, x_s]\, x_i\, [x_s, x_1] & \text{if } \ i > s\,, \end{cases}$$

where $[a, b]$ is the commutator $a^{-1}\, b^{-1}\, ab$, hence if $s, t \in \{2, \dots, k\}$ with $s < t$, we have

$$(x_i)\,\overline{\overline{b_{1s}}}\;\overline{\overline{b_{1t}}} = \begin{cases} x_i & \text{if} \quad i < s \\ x_1^{-1}\, x_s\, x_1 & \text{if} \quad i = s \\ [x_1, x_s]\, x_i\, [x_s, x_1] & \text{if} \quad s < i < t \\ x_1^{-1}[x_1, x_s]\, x_t\, [x_s, x_1]x_1 & \text{if} \quad i = t \\ [x_1, x_t]\, [x_1, x_s]\, x_i\, [x_s, x_1]\, [x_t, x_1] & \text{if} \quad i > t\,, \end{cases}$$

and

$$(x_i)\,\overline{\overline{b_{1t}}}\;\overline{\overline{b_{1s}}} = \begin{cases} x_i & \text{if} \quad i < s \\ x_1^{-1}\, x_s\, x_1 & \text{if} \quad i = s \\ [x_1, x_s]\, x_i\, [x_s, x_1] & \text{if} \quad s < i < t \\ [x_1, x_s]\, x_1^{-1}\, x_t\, x_1[x_s, x_1] & \text{if} \quad i = t \\ [x_1, x_s]\, [x_1, x_t]\, x_i\, [x_t, x_1]\, [x_s, x_1] & \text{if} \quad i > t\,, \end{cases}$$

and taking into account that x_1 commutes with its conjugates we have $\overline{\overline{b_{1s}}}\;\overline{\overline{b_{1t}}} = \overline{\overline{b_{1t}}}\;\overline{\overline{b_{1s}}}$, for any $s, t \in \{2, \dots, k\}$. Therefore $im(\Psi_k \mid \langle\, b_{1j}\,\rangle)$ is an abelian group generated by $\overline{\overline{b_{1j}}}$, $j \in \{2, \dots, k\}$.

Let $\mathbb{Z}^{k-1}$ have generators $z_2\,, z_3\,, \dots, z_k$ and consider the epimorphism $z_j \mapsto \overline{\overline{b_{1j}}}$ from $\mathbb{Z}^{k-1}$ to $im(\Psi_k \mid \langle\, b_{1j}\,\rangle)$. The image of the element $z_2^{n_2} \cdots z_k^{n_k}$, $n_2\,, \dots, n_k \in \mathbb{Z}$, by this epimorphism is the element $\overline{\overline{b_{1k}^{n_k}}} \cdots \overline{\overline{b_{12}^{n_2}}}$ given by

$$(x_i)\,\overline{\overline{b_{1k}^{n_k}}} \cdots \overline{\overline{b_{12}^{n_2}}} =$$

$$= \begin{cases} x_1 & \text{if} \quad i = 1 \\ x_1^{-n_2}\, x_2\, x_1^{n_2} & \text{if} \quad i = 2 \\ x_1^{-n_3}[x_1, x_2]^{n_2}\, x_3\, [x_2, x_1]^{n_2}\, x_1^{n_3} & \text{if} \quad i = 3 \\ x_1^{-n_4}[x_1, x_3]^{n_3}\, [x_1, x_2]^{n_2}\, x_4\, [x_2, x_1]^{n_2}\, [x_3, x_1]^{n_3}\, x_1^{n_4} & \text{if} \quad i = 4 \\ \quad \dots & \\ x_1^{-n_k}[x_1, x_{k-1}]^{n_{k-1}} \cdots [x_1, x_2]^{n_2}\, x_k\, [x_2, x_1]^{n_2} \cdots [x_{k-1}, x_1]^{n_{k-1}}\, x_1^{n_k} & \text{if} \quad i = k\,, \end{cases}$$

and since powers of different variables do not commute, if $\overline{\overline{b_{1k}^{n_k}}} \cdots \overline{\overline{b_{12}^{n_2}}}$ is the identity map, then $n_2 = 0\,,\ n_3 = 0\,,\ \dots,\ n_k = 0\,.$ ∎

Of course a similar result is true if we replace b_{1j} by c_{1j}.

Let us recall from Section 4 that if $\beta \in K_{1k} = \langle\, a_{1j}\,, b_{1j}\,, c_{1j} \mid j = 2, \dots, k\,\rangle$ then β is given by an ordered pair $(\ell_1\,, \widetilde{\ell_1}) \in RF^*(x_2, \dots, x_k\,, \widetilde{x_k}, \dots, \widetilde{x_2})$ where $[\ell_1] = [\widetilde{\ell_1}] \in RF(x_2, \dots, x_k)$.

PROPOSITION 6. *If $\beta \in K_{1k}$ is given by the pair $(\ell_1\,, \widetilde{\ell_1})$ then, for $s \in \{2, 3, \dots, k\}$*

and $\delta \in \{-1, 1\}$,

$$(x_i)\,\overline{\overline{\beta}}\,\overline{\overline{b^{\delta}_{1s}}}\,\overline{\overline{\beta^{-1}}} = \begin{cases} x_i & \textit{if} \quad i < s \\ ([\ell_1]^{-1}\,x_1^{\delta}\,[\ell_1])^{-1}\,x_s\,([\ell_1]^{-1}\,x_1^{\delta}\,[\ell_1]) & \textit{if} \quad i = s \\ \big[[\ell_1]^{-1}\,x_1^{\delta}\,[\ell_1], x_s\big]\,x_i\,\,\big[x_s\,,\,[\ell_1]^{-1}\,x_1^{\delta}\,[\ell_1]\big] & \textit{if} \quad i > s\,. \end{cases}$$

PROOF. Let $\gamma \in \langle\, a_{1j}\,,\, b_{1j} \mid j = 2, \ldots, k\,\rangle$ be such that

$$(x_i)\,\overline{\overline{\gamma}} = \begin{cases} x_i & \text{if} \quad i < s \\ (w^{-1}\,x_1^{\delta}\,w)^{-1}\,x_s\,(w^{-1}\,x_1^{\delta}\,w) & \text{if} \quad i = s \\ \big[w^{-1}\,x_1^{\delta}\,w\,,\,x_s\big]\,x_i\,\big[x_s\,,\,w^{-1}\,x_1^{\delta}\,w\big] & \text{if} \quad i > s\,, \end{cases}$$

where $w \in RF(x_2, \ldots, x_k)$.

If $\varepsilon \in \{\pm 1\}$ we have

$$(x_i)\,\overline{\overline{b^{\varepsilon}_{1j}}} = \begin{cases} x_i & \text{if} \quad i < j \\ x_1^{-\varepsilon}\,x_j\,x_1^{\varepsilon} & \text{if} \quad i = j \\ \big[x_1^{\varepsilon}\,,\,x_j\big]\,x_i\,\big[x_j\,,\,x_1^{\varepsilon}\big] & \text{if} \quad i > j\,. \end{cases}$$

Taking into account that x_1 commutes with its conjugates and that in the composition $\overline{\overline{b^{\varepsilon}_{1j}}}\,\overline{\overline{\gamma}}$ each letter x_t of w is replaced by a conjugate of x_t by a product of conjugates of x_1, we obtain

$$(x_i)\,\overline{\overline{b^{\varepsilon}_{1j}}}\,\overline{\overline{\gamma}}\,\overline{\overline{b^{-\varepsilon}_{1j}}} = \begin{cases} x_i & \text{if} \quad i < s \\ (w^{-1}\,x_1^{\delta}\,w)^{-1}\,x_s\,(w^{-1}\,x_1^{\delta}\,w) & \text{if} \quad i = s \\ \big[w^{-1}\,x_1^{\delta}\,w\,,\,x_s\big]\,x_i\,\big[x_s\,,\,w^{-1}\,x_1^{\delta}\,w\big] & \text{if} \quad i > s\,. \end{cases}$$

On the other side,

$$(x_i)\,\overline{\overline{a^{\varepsilon}_{1j}}} = \begin{cases} x_j^{-\varepsilon}\,x_1\,x_j^{\varepsilon} & \text{if} \quad i = 1 \\ [x_j, x_1^{\varepsilon}]\,x_i\,[x_1^{\varepsilon}, x_j] & \text{if} \quad 1 < i < j \\ x_1^{-\varepsilon}\,x_j\,x_1^{\varepsilon} & \text{if} \quad i = j \\ x_i & \text{if} \quad i > j\,, \end{cases}$$

hence

$$(x_i)\,\overline{\overline{a^{\varepsilon}_{1j}}}\,\overline{\overline{\gamma}}\,\overline{\overline{a^{-\varepsilon}_{1j}}} = \begin{cases} x_i & \text{if} \quad i < s \\ (w^{-1}\,x_j^{-\varepsilon}\,x_1^{\delta}\,x_j^{\varepsilon}\,w)^{-1}\,x_s\,(w^{-1}\,x_j^{-\varepsilon}\,x_1^{\delta}\,x_j^{\varepsilon}\,w) & \text{if} \quad i = s \\ \Big[w^{-1}\,x_j^{-\varepsilon}\,x_1^{\delta}\,x_j^{\varepsilon}\,w\,,\,x_s\Big]\,x_i\,\Big[x_s\,,\,w^{-1}\,x_j^{-\varepsilon}\,x_1^{\delta}\,x_j^{\varepsilon}\,w\Big] & \text{if} \quad i > s\,. \end{cases}$$

The result now follows by induction on the length of β as a product of a^{ε}_{1j}, b^{ε}_{1j}, c^{ε}_{1j} taking into account that $\overline{\overline{c^{\varepsilon}_{1j}}} = \overline{\overline{b^{\varepsilon}_{1j}}}$. ■

Let $\langle\, b_{1j}\,\rangle^N = \langle\, b_{1j} \mid j = 2, \ldots, k\,\rangle^N$ be the normal subgroup of $\langle\, a_{1j}\,,\, b_{1j} \mid j = 2, \ldots, k\,\rangle$ generated by b_{1j}, $j = 2, \ldots, k$.

COROLLARY 7. $\big(\langle b_{1j} \rangle^N\big)\, \Psi_k$ *is an abelian group.*

PROOF. Let $s, t \in \{2, \ldots, k\}$ with $s < t$. If

$$(x_i)\, \overline{\overline{\beta}}\, \overline{\overline{b_{1s}}}\, \overline{\overline{\beta^{-1}}} = \begin{cases} x_i & \text{if} \quad i < s \\ (w_1^{-1}\, x_1\, w_1)^{-1}\, x_s\, (w_1^{-1}\, x_1\, w_1) & \text{if} \quad i = s \\ \Big[w_1^{-1}\, x_1\, w_1\,,\, x_s\Big]\, x_i\, \Big[x_s\,,\, w_1^{-1}\, x_1\, w_1\Big] & \text{if} \quad i > s\,, \end{cases}$$

and

$$(x_i)\, \overline{\overline{\gamma}}\, \overline{\overline{b_{1t}}}\, \overline{\overline{\gamma^{-1}}} = \begin{cases} x_i & \text{if} \quad i < t \\ (w_2^{-1}\, x_1\, w_2)^{-1}\, x_t\, (w_2^{-1}\, x_1\, w_2) & \text{if} \quad i = t \\ \Big[w_2^{-1}\, x_1\, w_2\,,\, x_t\Big]\, x_i\, \Big[x_t\,,\, w_2^{-1}\, x_1\, w_2\Big] & \text{if} \quad i > t\,, \end{cases}$$

then both $(\overline{\overline{\beta}}\, \overline{\overline{b_{1s}}}\, \overline{\overline{\beta^{-1}}})\, (\overline{\overline{\gamma}}\, \overline{\overline{b_{1t}}}\, \overline{\overline{\gamma^{-1}}})$ and $(\overline{\overline{\gamma}}\, \overline{\overline{b_{1t}}}\, \overline{\overline{\gamma^{-1}}})\, (\overline{\overline{\beta}}\, \overline{\overline{b_{1s}}}\, \overline{\overline{\beta^{1}}})$ are given by sending x_i to

$$\begin{cases} x_i & \text{if} \quad i < s \\ (w_1^{-1}\, x_1\, w_1)^{-1}\, x_s\, (w_1^{-1}\, x_1\, w_1) & \text{if} \quad i = s \\ \Big[w_1^{-1}\, x_1\, w_1\,,\, x_s\Big]\, x_i\, \Big[x_s\,,\, w_1^{-1}\, x_1\, w_1\Big] & \text{if} \quad s < i < t \\ (w_2^{-1}\, x_1\, w_2)^{-1}\, \Big[w_1^{-1}\, x_1\, w_1\,,\, x_s\Big]\, x_t\, \Big[x_s\,,\, w_1^{-1}\, x_1\, w_1\Big]\, (w_2^{-1}\, x_1\, w_2) & \text{if} \quad i = t \\ \Big[w_2^{-1} x_1 w_2\,,\, x_t\Big] \Big[w_1^{-1} x_1\, w_1, x_s\Big] x_i \Big[x_s, w_1^{-1} x_1 w_1\Big] \Big[x_t, w_2^{-1}\, x_1 w_2\Big] & \text{if} \quad i > t\,. \end{cases} \quad \blacksquare$$

THEOREM 8. $\langle a_{1j}\,,\, b_{1j} \mid j = 2, \ldots, k \rangle$ *is isomorphic to* $RF^*(2k-2) = RF^*(x_2, \ldots, x_k, z_k, \ldots, z_2)$ *through the association* $a_{1j} \mapsto x_j^{-1}$, $b_{1j} \mapsto z_j^{-1}$ *and the restriction* $\Psi_k \mid \langle a_{1j}, b_{1j} \mid j = 2, \ldots, k \rangle$ *has kernel* $\big[\langle b_{1j} \rangle^N\,,\, \langle b_{1j} \rangle^N\big]$.

PROOF. Let $\beta \in K_{1k}$. We have seen that the association $\beta \mapsto (\ell_1, \widetilde{\ell_1}) \in RF^*(2k-2) \times RF^*(2k-2)$, where $RF^*(2k-2) = RF^*(x_2, \ldots, x_k\,,\, \widetilde{x_k}, \ldots, \widetilde{x_2})$, sends a_{1j} to $(x_j^{-1}\,,\, \widetilde{x_j})$ and b_{1j} to $(\widetilde{x}_j^{-1} x_j^{-1}, 1)$ Noting that $RF^*(x_2, \ldots, x_k\,,\, \widetilde{x_k}, \ldots, \widetilde{x_2}) = RF^*(x_2, \ldots, x_k\,,\, z_k\,,\, \ldots\,,\, z_2)$, where $z_j = x_j\, \widetilde{x_j}$, the association $\beta \mapsto \ell_1$ provides the required isomorphism.

We have seen that since β stabilizes 1, $[\ell_1] = [\widetilde{\ell_1}]$, where [ ] stands for the equivalence class in $RF(k-1) = RF(x_2, \ldots, x_k)$. If $\overline{\overline{\beta}} = \mathrm{id}$, since $(x_1)\, \overline{\overline{\beta}} = [\ell_1]^{-1} x_1 [\ell_1]$, we obtain $[\ell_1] = [\widetilde{\ell_1}] = 1$, thus ℓ_1 and $\widetilde{\ell_1}$ belong to the normal subgroup of $RF^*(2k-2)$ generated by $x_2\, \widetilde{x_2}\,,\, x_3\, \widetilde{x_3}\,,\, \ldots\,,\, x_k\, \widetilde{x_k}$. In particular, if $\beta \in \langle a_{1j}\,,\, b_{1j} \mid j = 2, \ldots, k \rangle$, with $\overline{\overline{\beta}} = \mathrm{id}$, we have that $\widetilde{\ell_1} = 1$ and $\beta \in \langle b_{1j} \rangle^N$.

Corollary 7 shows that $\big[\langle b_{1j} \rangle^N\,,\, \langle b_{1j} \rangle^N\big] \subset \ker\big(\Psi_k \mid \langle a_{1j}\,,\, b_{1j} \mid j = 2, \ldots, k \rangle\big)$. We will show the other inclusion by showing that the induced homomorphism on

$\dfrac{\langle\, a_{1j}\,,\, b_{1j} \mid j = 2, \ldots, k\,\rangle}{\big[\langle\, b_{1j}\,\rangle^N\,,\, \langle\, b_{1j}\,\rangle^N\big]}$ is injective.

Since a_{1j} commutes with b_{1j}, any element $\beta \in \langle\, b_{1j}\,\rangle^N$ can be written as a product of $\beta_t^{-1}\, b_{1t}^{\varepsilon_t}\, \beta_t$, where $\beta_t \in \langle\, a_{1j} \mid j = 2, \ldots, k$ and $j \neq t\,\rangle$ and $\varepsilon_t \in \{\pm 1\}$. Since, by Corollary 7, $\big(\langle\, b_{1j}\,\rangle^N\big)\, \Psi_k$ is abelian, we have

$$(\beta)\, \Psi_k = \left(\prod_{s=1}^{n_k} \overline{\overline{\beta_{sk}\, b_{1k}^{\varepsilon_{sk}}\, \beta_{sk}^{-1}}}\right) \cdots \left(\prod_{s=1}^{n_2} \overline{\overline{\beta_{s2}\, b_{12}^{\varepsilon_{s2}}\, \beta_{s2}^{-1}}}\right),$$

where $\beta_{st} \in \langle\, a_{1j} \mid j = 2, \ldots, k$ and $j \neq t\,\rangle$ and $\varepsilon_{st} \in \{\pm 1\}$.

To each β_{st} there corresponds an element $\ell_1 = \ell_1(\beta_{st}) \in RF(x_2, \ldots, x_{t-1}\,,\, x_{t+1}, \ldots, x_k)$. Let us denote $\ell_1(\beta_{s2})$ by w_s, ε_{s2} by ε_s and n_2 by n.

From Proposition 6, we have

$$(x_2)\,(\beta)\,\Psi_k = (x_2) \prod_{s=1}^{n_2} \overline{\overline{\beta_{s2}\, b_{12}^{\varepsilon_{s2}}\, \beta_{s2}^{-1}}} =$$

$$= \big([w_n]^{-1} x_1^{\varepsilon_n} [w_n]\big)^{-1} \cdots \big([w_1]^{-1} x_1^{\varepsilon_1} [w_1]\big)^{-1}\, x_2\, \big([w_1]^{-1} x_1^{\varepsilon_1} [w_1]\big) \cdots \big([w_n]^{-1} x_1^{\varepsilon_n} [w_n]\big).$$

If $(x_2)\,(\beta)\,\Psi_k = x_2$, since $w_s = \ell_1(\beta_{s2}) \in RF(x_3, \ldots, x_k)$, it follows that there is an even number of factors x_1 (since $\sum_{j=1}^{n} \varepsilon_j = 0$) and, commuting the factors $[w_j]^{-1} x_1^{\varepsilon_j} [w_j]$ if necessary, $w_2 = w_1^{-1}$, $\varepsilon_2 = -\varepsilon_1$, $w_4 = w_3^{-1}$, $\varepsilon_4 = \varepsilon_3^{-1}$, $\ldots$, $w_n = w_{n-1}^{-1}$, $\varepsilon_n = \varepsilon_{n-1}^{-1}$.

If we allow the factors $\beta_{sj}\, b_{1j}^{\varepsilon_{sj}}\, \beta_{sj}^{-1}$ of β to commute among themselves, it follows by induction that if $(\beta)\Psi_k = \mathrm{id}$, then β is trivial. ∎

COROLLARY 9. *$\Psi_k \mid_{K_{1k}}$ has as its kernel the normal subgroup of K_{1k} generated by $b_{1j}\, c_{1j}^{-1}$ $(j = 2, \ldots, k)$ and $\big[\langle\, b_{1j} \mid j = 2, \ldots, k\,\rangle^N\,,\, \langle\, b_{1j} \mid j = 2, \ldots, k\,\rangle^N\big]$.*

Let $B_{ik} = \langle\, b_{ij} \mid j = i+1, \ldots, k\,\rangle^N$ be the normal subgroup of $\langle\, a_{ij}\,,\, b_{ij} \mid j = i+1, \ldots, k\rangle$ generated by b_{ij} $(j = i+1, \ldots, k)$.

THEOREM 10. *The kernel of the homomorphism $\Psi_k : S^*(1) \to A(RF(k))$ is $\langle\, b_{ij}\, c_{ij}^{-1}, [\, B_{ik}\,,\, B_{ik}\,] \mid 1 \leq i < j \leq k\rangle^N$.*

PROOF. Split short exact sequence (I) provides the split short exact sequence

$$1 \to K_{1k} \cap \ker \Psi_k \to \ker \Psi_k \to \ker \Psi_{k-1} \to 1\,.$$

The result follows from Theorem 8 by induction. ∎

ACKNOWLEDGMENT. This research was done while the author was visiting Universidad Complutense de Madrid, Spain. I would like to thank Professor J.M. Montesinos-Amilibia for his encouragement. This work was financially supported by CAPES - Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brazil.

References

[A1] E. ARTIN, Theorie der Zöpfe, Abh. Math. Sem. Univ. Hamburg **4** (1925), 47–72.

[A2] E. ARTIN, Theory of braids, Ann. of Math. **48** (1947), 101–126.

[Aℓ] J. W. ALEXANDER, A lemma on systems of knotted curves, Proc. Nat. Acad. Science USA **9** (1923), 93–95.

[B] J.S. BIRMAN, *Braids, links and mapping class groups*, Annals of Mathematics Studies 82, Princeton Univ. Press, 1975.

[C1] J.E.P.P. DE CAMPOS, *Boundary string links*, Journal of Knot Theory and Its Ramifications **8**, no. 7 (1999), 855–878.

[C2] J.E.P.P. DE CAMPOS, Distinguishing links up to link-homotopy by algebraic methods, Topology and its Applications **157** (2010), 605–614.

[D] J. Y. LE DIMET, Cobordism d'enlacements de disques, Memoire de la Société Mathématique de France, 32, Supplément au Bulletin de la S.M.F. **116**, fascicule 2 (1988).

[G1] D. GOLDSMITH, *Homotopy of braids*: *in answer to a question of E. Artin*, Topology Conference, Lecture Notes in Math. **375**, Springer Verlag, Berlin, Heidelberg, New York, 1973, 91–96.

[G2] D. GOLDSMITH, Concordance implies homotopy for classical links in M^3, Comment. Math. Helv. **54** (1979), 347–355.

[H1] J. R. HUGHES, Structured group and link-homotopy, Journal of Knot Theory and Its Ramifications **2**, no. 1 (1993), 37–63.

[H3] J. R. HUGHES, Distinguishing link-homotopy classes by pre-peripheral structures, Journal of Knot Theory and Its Ramifications **7**, no. 7 (1998), 925–944.

[HL] N. HABEGGER and X. S. LIN, The classification of links up to link-homotopy, Journal of the American Mathematical Society **3**, no. 2 (1990), 389–419.

[K] K. H. KO, Seifert matrices and boundary link cobordism, Transactions of the American Mathematical Society **299**, no. 2 (1987).

[L1] J. P. LEVINE, The $\overline{\mu}$-invariants of based links, Proceedings of the Second Topology Symposion, Ed. A. DOLD and B. ECKMANN (1987), 87–103, Springer-Verlag.

[L2] J. P. LEVINE, An approach to homotopy classification of links, Trans. Amer. Math. Soc. **306** (1988), 361–387.

[M1] J. MILNOR, Link group, Ann. of Math. **59** (1954), 177–195.

[M2] J. MILNOR, Isotopy of links, *Algebraic Geometry and Topology*, Princeton Univ. Press, Princeton, NJ, 1957.

[Ma] A. A. MARKOV, Über die freie Äquivalenz der geschlossenen Zöpfe, Recueil Mathematique Moscou **1** (1935), 73–78.

[MKS] W. MAGNUS, A. KARRAS, D. SOLITAR, *Combinatorial group theory*, Pure and Appl. Math. **XIII**, Interscience, NY, 1966.

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