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Stability of the Basic Solution of Kolmogorov Flow with a Bottom Friction

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Abstract. We consider stability of a stationary solution of Kolmogorov flow with a bottom friction. Any solution of nonstationary problem which is periodic with respect to x, y with periods $2\pi/\alpha$, 2π is shown to tend to the stationary solution as time tends to infinity when aspect ratio α is equal to or greater than 1.

1. Introduction

We shall treat a modified model of Kolmogorov flow, a plane periodic flow of an incompressible fluid under the action of a spatially periodic external force in a thin layer (see [1], [2]).

The corresponding Navier-Stokes equations in a stationary case take the form:

$$\begin{cases} uu_x + vu_y = -P_x + \nu\Delta u - \kappa u + \gamma \sin y\,, \\ uv_x + vv_y = -P_y + \nu\Delta v - \kappa v\,, \\ u_x + v_y = 0\,, \end{cases} \tag{1}$$

where the unknowns are velocity vector $V(x, y) = {}^t(u(x, y), v(x, y))$ and pressure $P(x, y)$, and positive numbers ν, γ, κ mean the kinematic viscosity, the amplitude of the external force $F = {}^t(\gamma \sin y, 0)$ and the coefficient of the bottom friction which can be defined by $\kappa \equiv 2\nu/h^2$ with h, depth of the fluid layer.

We consider the problem under the conditions that V has periods $2\pi/\alpha$, 2π with respect to x, y respectively. In addition, we shall require the following condition

$$\iint_D V(x, y)dxdy = 0\,, \quad D \equiv \left\{(x, y)\ :\ |x| \le \pi/\alpha,\ |y| \le \pi\right\} \tag{2}$$

be satisfied.

Introducing a stream function $\psi(x, y)$, we reproduce the velocity as $(u, v) = (\psi_y, -\psi_x)$. As the pressure is known to be determined by the velocity, the problem is reduced to seeking solutions of the equation

$$J(\Delta\psi, \psi) = \nu\Delta^2\psi - \kappa\Delta\psi + \gamma \cos y\,, \qquad J(f, g) \equiv f_x g_y - f_y g_x \tag{3}$$

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which are periodic with respect to x, y with periods $2\pi/\alpha$, 2π under the condition

$$\iint_D \psi dxdy = 0\,, \tag{4}$$

without loss of generality.

The problem (3) and (4) has a solution

$$\psi_0(x, y) \equiv -\gamma(\nu + \kappa)^{-1} \cos y \tag{5}$$

for any positive numbers γ, ν and κ. We call this ψ_0 a basic solution.

We denote by X the completion of the pre-Hilbert space

$$X_0 = \{\psi \in C^\infty(R^2) \;:\; \psi \text{ is periodic and satisfies (4).}\}\,,$$

provided the inner product

$$(\psi_1, \psi_2) \equiv \iint_D \Delta\psi_1 \Delta\psi_2 dxdy\,.$$

Our result is stated as follows:

THEOREM 1. *Let $\alpha \geq 1$. Then, for any positive ν, γ and κ, any solution of the following nonstationary equation*

$$\frac{\partial}{\partial t}\Delta\psi + J(\Delta\psi, \psi) = \nu\Delta^2\psi - \kappa\Delta\psi + \gamma \cos y \tag{6}$$

which has periods $2\pi/\alpha$, 2π with respect to x, y and satisfy the condition (4) *tends to the basic solution ψ_0 in X as $t \to \infty$.*

Iudovich [3] has given the same result when there is no friction parameter κ. Following his method, we make the proof more precise.

2. Proof of the theorem

We shall seek solutions of (6) in the form $\psi = \psi_0 + \Phi$, where ψ_0 is the basic solution (5). $\Phi = \Phi(t, x, y)$ satisfies the equation

$$\frac{\partial}{\partial t}\Delta\Phi + \frac{\gamma}{\nu + \kappa} \sin y \frac{\partial}{\partial x}(\Delta\Phi + \Phi) + J(\Delta\Phi, \Phi) - \nu\Delta^2\Phi + \kappa\Delta\Phi = 0\,. \tag{7}$$

Multiplying (7) by $\Delta\Phi + \Phi$ and integrating over D, we obtain

$$\frac{1}{2}\frac{d}{dt}(J_2^2(t) - J_1^2(t)) + \nu(J_3^2(t) - J_2^2(t)) + \kappa(J_2^2(t) - J_1^2(t)) = 0\,, \tag{8}$$

where $J_i(t)$ $(i = 1, 2, 3)$ are defined as follows:

$$J_1(t) \equiv \left(\iint_D (\nabla\Phi)^2 dxdy\right)^{\frac{1}{2}}\,,$$

$$J_2(t) \equiv \left(\iint_D (\Delta \Phi)^2 dx dy \right)^{\frac{1}{2}} ,$$

$$J_3(t) \equiv \left(\iint_D (\nabla \Delta \Phi)^2 dx dy \right)^{\frac{1}{2}} .$$

Theorem means that $J_2(t) = \|\Phi\|_X \to 0$ as $t \to \infty$. We consider (8) separately in two cases, $\alpha > 1$ and $\alpha = 1$, because the estimate for the case $\alpha = 1$ is quite different from that for $\alpha > 1$.

2.1. The case where $\alpha > 1$. Let $\Phi \in X$ expand in the Fourier series as

$$\Phi(t, x, y) = \sum_{k,\ell} c_{k,\ell}(t) e^{i(k\alpha x + \ell y)} ,$$

where the summation is taken over all the pairs of integers but $(k, \ell) = (0, 0)$. Since we assume that Φ satisfies the condition (4), we can put $c_{0,0}(t) \equiv 0$.

We rewrite (8) as

$$\frac{1}{2} \frac{d}{dt} K_1^2(t) + \nu K_2^2(t) + \kappa K_1^2(t) = 0 , \tag{9}$$

where $K_1^2(t) \equiv J_2^2(t) - J_1^2(t)$ and $K_2^2(t) \equiv J_3^2(t) - J_2^2(t)$. We see $K_1^2(t) \leq K_2^2(t)$, since we have

$$K_2^2(t) - K_1^2(t) = 4\pi^2 \alpha^{-1} \sum_{k,\ell} (k^2\alpha^2 + \ell^2)(k^2\alpha^2 + \ell^2 - 1)^2 c_{k,\ell}^2(t) \geq 0 \tag{10}$$

for $\alpha \geq 1$. Then, from (9), we have

$$\frac{1}{2} \frac{d}{dt} K_1^2(t) + (\nu + \kappa) K_1^2(t) \leq 0 .$$

The inequality above leads the following relation:

$$K_1^2(t) \leq e^{-2(\nu+\kappa)t} K_1^2(0) . \tag{11}$$

Therefore, we see that $K_1^2(t) \to 0$ as $t \to \infty$.

Next we shall seek a positive number A which satisfies $K_1^2(t) \geq A J_2^2(t)$ in order to show that $J_2^2(t) \to 0$ as $t \to \infty$.

We denote Φ by $\Phi = \Phi_1 + \Phi_2 + \Phi_3$, where

$$\Phi_1(t, x, y) \equiv c_{0,1}(t) e^{iy} + c_{0,-1}(t) e^{-iy} ,$$

$$\Phi_2(t, x, y) \equiv \sum_{\ell = \pm 2, \pm 3, \ldots, \pm L} c_{0,\ell}(t) e^{i\ell y} , \qquad L \equiv [\alpha] + 1 ,$$

$$\Phi_3(t, x, y) \equiv \Phi - (\Phi_1 + \Phi_2) .$$

Here $[\alpha]$ means the largest positive integer which remains $(0,\alpha]$. Note $(\Phi_i,\Phi_j)=0$ for $i\neq j$. So we can denote $J_i^2(t)$ of (8) by $J_i^2(t)=J_{i,1}^2(t)+J_{i,2}^2(t)+J_{i,3}^2(t)$ $(i=1,2,3)$, where

$$J_{1,j}(t)\equiv\left(\iint_D(\nabla\Phi_j)^2dxdy\right)^{\frac{1}{2}},$$

$$J_{2,j}(t)\equiv\left(\iint_D(\Delta\Phi_j)^2dxdy\right)^{\frac{1}{2}},$$

$$J_{3,j}(t)\equiv\left(\iint_D(\nabla\Delta\Phi_j)^2dxdy\right)^{\frac{1}{2}}\quad\text{for } j=1,2,3.$$

Also $K_i^2(t)$ of (9) can be written as $K_i^2(t)=K_{i,1}^2(t)+K_{i,2}^2(t)+K_{i,3}^2(t)$ $(i=1,2)$, where $K_{1,j}^2(t)=J_{2,j}^2(t)-J_{1,j}^2(t)$ and $K_{2,j}^2(t)=J_{3,j}^2(t)-J_{2,j}^2(t)$ for $j=1,2,3$.

We first note that $K_{1,1}^2(t)=K_{2,1}^2(t)\equiv 0$, since it holds

$$J_{1,1}^2(t)=J_{2,1}^2(t)=J_{3,1}^2(t)=4\pi^2\alpha^{-1}(c_{0,1}^2(t)+c_{0,-1}^2(t)).$$

And we have

$$K_{1,2}^2(t)=4\pi^2\alpha^{-1}\sum_{\ell=\pm2,\pm3,\dots,\pm L}\ell^2(\ell^2-1)c_{0,\ell}^2(t)\geq 0,$$

$$K_{1,3}^2(t)=4\pi^2\alpha^{-1}\sum_{(k,\ell)\neq(0,\pm1),\cdots,(0,\pm L)}(k^2\alpha^2+\ell^2)(k^2\alpha^2+\ell^2-1)c_{k,\ell}^2(t)\geq 0,$$

$$K_{2,2}^2(t)=4\pi^2\alpha^{-1}\sum_{\ell=\pm2,\pm3,\dots,\pm L}\ell^4(\ell^2-1)c_{0,\ell}^2(t)\geq 0,$$

$$K_{2,3}^2(t)=4\pi^2\alpha^{-1}\sum_{(k,\ell)\neq(0,\pm1),\dots,(0,\pm L)}(k^2\alpha^2+\ell^2)^2(k^2\alpha^2+\ell^2-1)c_{k,\ell}^2(t)\geq 0.$$

Since $K_{1,j}^2(t)\to 0$ as $t\to\infty$ for $j=2,3$ holds from (11), we seek a positive A which satisfies $K_{1,j}^2(t)\geq AJ_{2,j}^2(t)$ for $j=2,3$.

First we obtain

$$K_{1,2}^2(t)\geq 2^{-1}J_{2,2}^2(t),\tag{12}$$

which follows from the fact

$$K_{1,2}^2(t)-2^{-1}J_{2,2}^2(t)=2\pi^2\alpha^{-1}\sum_{\ell=\pm2,\pm3,\dots,\pm L}\ell^2(\ell^2-2)c_{0,\ell}^2(t)\geq 0.$$

And we also obtain the following inequality

$$K_{1,3}^2(t)\geq(1-\alpha^{-2})J_{2,3}^2(t),\tag{13}$$

since it holds that for $\alpha > 1$

$$K_{1,3}^2(t) - (1-\alpha^{-2})J_{2,3}^2(t) = 4\pi^2\alpha^{-1} \sum_{(k,\ell)\neq(0,\pm1),\dots,(0,\pm L)} (k^2\alpha^2+\ell^2)(k^2+\ell^2\alpha^{-2}-1)c_{k,\ell}^2(t)$$

is equal to or greater than 0. From (12) and (13), we obtain $J_{2,j}(t) \to 0$ as $t \to \infty$ for $j = 2, 3$.

If $J_{2,1}(t)$ also tends to 0 as $t \to \infty$, then we can estimate $J_2(t)$. Multiplying (7) by $\Delta\Phi_1(= -\Phi_1)$ and integrating over D, we have

$$\frac{1}{2}\frac{d}{dt}J_{2,1}^2(t) + (\nu+\kappa)J_{2,1}^2(t) = \iint_D \Delta\Phi_3 \frac{\partial\Phi_3}{\partial x}\frac{\partial\Phi_1}{\partial y}dxdy\,. \tag{14}$$

We treat (14) from now, since (9) is insufficient to estimate $J_{2,1}(t)$.

The right-hand side of (14) can be estimated as

$$\begin{aligned}\iint_D \Delta\Phi_3 \frac{\partial\Phi_3}{\partial x}\frac{\partial\Phi_1}{\partial y}dxdy &\le CJ_{2,3}(t)J_{1,3}(t)J_{1,1}(t)\\ &\le CJ_{2,3}^2(t)J_{2,1}(t) \qquad C\text{ ; const.}\,,\end{aligned}$$

since we see $\left|\frac{\partial\Phi_1}{\partial y}\right| \le |c_{0,1}(t)|+|c_{0,-1}(t)| \le CJ_{1,1}(t)$, $J_{1,3}(t) \le J_{2,3}(t)$ and $J_{1,1}(t) = J_{2,1}(t)$. Then, from (14), we have

$$\frac{1}{2}\frac{d}{dt}J_{2,1}^2(t) + (\nu+\kappa)J_{2,1}^2(t) \le CJ_{2,3}^2(t)J_{2,1}(t)\,. \tag{15}$$

(13) and $K_{1,3}^2(t) \le e^{-2(\nu+\kappa)t}K_1^2(0)$ which follows from (11) lead

$$J_{2,3}^2(t) \le M(\alpha)e^{-2(\nu+\kappa)t}\,,$$

where $M(\alpha) \equiv \alpha^2(\alpha^2-1)^{-1}K_1^2(0)$. Then, from (15), it holds that

$$\frac{1}{2}\frac{d}{dt}J_{2,1}^2(t) + (\nu+\kappa)J_{2,1}^2(t) \le M(\alpha)Ce^{-2(\nu+\kappa)t}J_{2,1}(t)\,,$$

which is also written as

$$\frac{d}{dt}\{e^{(\nu+\kappa)t}J_{2,1}(t)\} \le M(\alpha)Ce^{-(\nu+\kappa)t}\,.$$

From the inequality above, we have

$$J_{2,1}(t) \le J_{2,1}(0)e^{-(\nu+\kappa)t} + (\nu+\kappa)^{-1}M(\alpha)C\{e^{-(\nu+\kappa)t} - e^{-2(\nu+\kappa)t}\}\,.$$

Therefore, $J_{2,1}(t)$ goes to 0 as $t \to \infty$ and the theorem has proved in the case of $\alpha > 1$.

2.2. The case where $\alpha = 1$. Since (10) and (11) hold also for $\alpha = 1$, we see that $K_1^2(t) \to 0$ as $t \to \infty$.

We denote Φ by $\Phi = \Phi_4 + \Phi_5$, where

$$\Phi_4 \equiv \sum_{k^2+\ell^2=1} c_{k,\ell}(t) e^{i(kx+\ell y)},$$
$$\Phi_5 \equiv \Phi - \Phi_4 .$$

Note $(\Phi_4, \Phi_5) = 0$. Then, we also denote $J_i^2(t)$ of (8) by $J_i^2(t) = J_{i,4}^2(t) + J_{i,5}^2(t)$ $(i = 1, 2, 3)$, where

$$J_{1,j}(t) \equiv \left(\iint_D (\nabla \Phi_j)^2 dxdy \right)^{\frac{1}{2}},$$
$$J_{2,j}(t) \equiv \left(\iint_D (\Delta \Phi_j)^2 dxdy \right)^{\frac{1}{2}},$$
$$J_{3,j}(t) \equiv \left(\iint_D (\nabla \Delta \Phi_j)^2 dxdy \right)^{\frac{1}{2}} \quad \text{for } j = 4, 5,$$

and $K_i^2(t)$ of (9) by $K_i^2(t) = K_{i,4}^2(t) + K_{i,5}^2(t)$ $(i = 1, 2)$, where $K_{1,j}^2(t) = J_{2,j}^2(t) - J_{1,j}^2(t)$ and $K_{2,j}^2(t) = J_{3,j}^2(t) - J_{2,j}^2(t)$ for $j = 4, 5$.

We first note that $K_{1,4}^2(t) = K_{2,4}^2(t) \equiv 0$ follows from

$$J_{1,4}^2(t) = J_{2,4}^2(t) = J_{3,4}^2(t) = 4\pi^2 \{c_{1,0}^2(t) + c_{-1,0}^2(t) + c_{0,1}^2(t) + c_{0,-1}^2(t)\} . \tag{16}$$

We also have

$$K_{1,5}^2(t) = 4\pi^2 \sum_{k^2+\ell^2 \geq 2} (k^2+\ell^2)(k^2+\ell^2-1) c_{k,\ell}^2(t) \geq 0 ,$$
$$K_{2,5}^2(t) = 4\pi^2 \sum_{k^2+\ell^2 \geq 2} (k^2+\ell^2)^2(k^2+\ell^2-1) c_{k,\ell}^2(t) \geq 0 .$$

We find that $K_1^2(t) \to 0$ as $t \to \infty$ implies $K_{1,5}^2(t) = J_{2,5}^2(t) - J_{1,5}^2(t) \to 0$ as $t \to \infty$. $J_{2,5}^2(t)$ is estimated as

$$K_{1,5}^2(t) \geq 2^{-1} J_{2,5}^2(t) , \tag{17}$$

since it holds that

$$K_{1,5}^2(t) - 2^{-1} J_{2,5}^2(t) = 2\pi^2 \sum_{k^2+\ell^2 \geq 2} (k^2+\ell^2)(k^2+\ell^2-2) c_{k,\ell}^2(t) \geq 0 .$$

Therefore, we obtain $J_{2,5}^2(t) \to 0$ as $t \to \infty$.

Our next purpose is to show that $J_{2,4}(t) \to 0$ as $t \to \infty$. Multiplying (7) by $\Delta\Phi_4$ $(= -\Phi_4)$ and integrating it over D, we have

$$\frac{1}{2}\frac{d}{dt}J_{2,4}^2(t) + (\nu+\kappa)J_{2,4}^2(t) = G_1(t) + G_2(t)\,, \tag{18}$$

where

$$G_1(t) \equiv \gamma(\nu+\kappa)^{-1}\iint_D (\Delta\Phi_5 + \Phi_5)\sin y \frac{\partial \Delta\Phi_4}{\partial x} dxdy\,,$$
$$G_2(t) \equiv \iint_D \left(\Delta\Phi_5 \frac{\partial\Phi_5}{\partial y}\frac{\partial\Delta\Phi_4}{\partial x} + \Delta\Phi_5\frac{\partial\Phi_5}{\partial x}\frac{\partial\Phi_4}{\partial y}\right) dxdy\,.$$

Each $G_i(t)$ $(i = 1, 2)$ can be estimated as

$$\begin{aligned} G_1(t) &\le \gamma(\nu+\kappa)^{-1}\{J_{2,5}(t)J_{3,4}(t) + J_{1,5}(t)J_{3,4}(t)\} \\ &\le 2\gamma(\nu+\kappa)^{-1}J_{2,5}(t)J_{3,4}(t) = 2\gamma(\nu+\kappa)^{-1}J_{2,5}(t)J_{2,4}(t)\,, \\ G_2(t) &\le CJ_{2,5}(t)J_{1,5}(t)J_{1,4}(t) \\ &\le CJ_{2,5}^2(t)J_{2,4}(t) \qquad C\text{ ; const.}\,, \end{aligned}$$

because of (16), $J_{2,5}(t) \ge J_{1,5}(t)$ and

$$\left|\frac{\partial\Delta\Phi_4}{\partial x}\right| + \left|\frac{\partial\Phi_4}{\partial y}\right| \le |c_{1,0}(t)| + |c_{-1,0}(t)| + |c_{0,1}(t)| + |c_{0,-1}(t)| \le CJ_{1,4}(t)\,.$$

Then, from (18), we have

$$\frac{d}{dt}\{e^{(\nu+\kappa)t}J_{2,4}(t)\} \le \{2\gamma(\nu+\kappa)^{-1}J_{2,5}(t) + CJ_{2,5}^2(t)\}e^{(\nu+\kappa)t}\,. \tag{19}$$

$J_{2,5}(t)$ has been estimated by (17), and (11) leads

$$K_{1,5}^2(t) \le e^{-2(\nu+\kappa)t}K_1^2(0)\,. \tag{20}$$

Then, from (17) and (20), we have

$$J_{2,5}(t) \le \sqrt{2}e^{-(\nu+\kappa)t}K_1(0)\,.$$

So the right-hand side of (19) is estimated as follows:

$$\{2\gamma(\nu+\kappa)^{-1}J_{2,5}(t) + CJ_{2,5}^2(t)\}e^{(\nu+\kappa)t} \le N_1 + N_2e^{-(\nu+\kappa)t}\,,$$

where $N_1 \equiv 2\sqrt{2}\gamma(\nu+\kappa)^{-1}K_1(0)$ and $N_2 \equiv 2CK_1^2(0)$.

Thus, we obtain the following inequality from (19)

$$J_{2,4}(t) \le \{J_{2,4}(0) + N_2(\nu+\kappa)^{-1} + N_1 t\}e^{-(\nu+\kappa)t} - N_2(\nu+\kappa)^{-1}e^{-2(\nu+\kappa)t}\,,$$

which implies that $J_{2,4}(t) \to 0$ as $t \to \infty$. The theorem has proved in the case of $\alpha = 1$.

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