TOKYO J. MATH.
VOL. 2, NO. 1, 1979

Some Results on Additive Number Theory IV

Minoru TANAKA

Gakushuin University

§1. The main theorem.

Let $\omega(n)$ denote the number of distinct prime factors of a positive integer n.

THEOREM. *Let $\alpha<\beta$. Let $A(N;\alpha,\beta)$ denote, for sufficiently large positive integer N, the number of representations of N as the sum of the form $N=p+n$, where p is prime, and n is a positive integer such that*

$$\log\log N+\alpha\sqrt{\log\log N}<\omega(n)<\log\log N+\beta\sqrt{\log\log N}\ ,$$

then, as $N\to\infty$, we have

$$A(N;\alpha,\beta)\sim\frac{N}{\log N}\cdot\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}dx\ .$$

We shall give a proof of this theorem in section 2. Our proof runs in the same lines as in my paper [6], but it uses also Bombieri's mean value theorem and Brun-Titchmarsh's inequality. It is to be noticed that somewhat analogous theorem was proved in Halberstam [3] using Siegel-Walfisz's theorem. It might perhaps be possible to prove our theorem in a similar style as in [3], but I hope that it would be of interest to prove the theorem in our way.

As was shown in Gallagher [2], Bombieri's theorem can be deduced rather simply from Siegel-Walfisz's theorem, and is far more conveniently applicable in our situation. For Bombieri's theorem cf. Bombieri [1], Gallagher [2], Halberstam-Richert [4], p. 111, Mitsui [5], Chap. 8.

We shall shorten the paper by omitting the similar parts of the proof as in Tanaka [6].

The author expresses his thanks to Prof. S. Iyanaga for his kind advices.

Received October 24, 1978

§ 2. **Proof of the main theorem.**

Let a and b be non-negative integers. Then

$$(1)\qquad \sum_{c=0}^{b}(-1)^{c}\binom{a}{c}\begin{cases} =1\,, & \text{when } a=0\,,\\ \geqq 0\,, & \text{when } a>0 \text{ and } b \text{ is even}\,,\\ \leqq 0\,, & \text{when } a>0 \text{ and } b \text{ is odd}\,.\end{cases}$$

This is the same as Lemma 2 in [6].

Now we define some functions and sets which will be used in the sequel. The positive integer N will be assumed to be sufficiently large as occasion demands.

We define the set Q_N consisting of primes as

$$Q_N=\{p\colon p\nmid N,\ e^{(\log\log N)^2}<p<N^{(\log\log N)^{-2}}\}$$

and put

$$y(N)=\sum_{p\in Q_N}\frac{1}{p}\,.$$

Then we have

LEMMA 1. $y(N)=\log\log N+O(\log\log\log N)$.

PROOF. We can easily see that $\omega(N)=O(\log N)$, and hence

$$\sum_{p\mid N}\frac{1}{p}\leqq\sum_{p\leqq\omega(N)}\frac{1}{p}=O(\log\log\log N)\,.$$

The lemma can be obtained similarly as Lemma 4 in [6].

We denote by $\omega_N(n)$ the number of distinct prime factors of a positive integer n, which belong to the set Q_N:

$$\omega_N(n)=\sum_{p\mid n,\,p\in Q_N}1\,.$$

For any positive integer t, we define the set $M_N(t)$ consisting of positive integers as

$$M_N(t)=\{m\colon m \text{ is squarefree},\ m \text{ has } t \text{ prime factors},\ m \text{ is composed only of primes} \in Q_N\}\,.$$

We put for convenience $M_N(0)=\{1\}$.

For any positive integer t, we denote by $F(N;t)$ the number of representations of N as the sum of the form $N=p+n$, where p is prime, and n is a positive integer such that $\omega_N(n)=t$.

For any positive integer m such that $m\in M_N(t)$ with some positive integer t, we denote by $G(N;m)$ the number of representations of N as the sum of the form $N=p+n$, where p is prime, and n is a positive integer such that

$$(2)\qquad \prod_{p\mid n,\, p\in Q_N} p=m\ .$$

We obviously have

$$F(N;t)=\sum_{m\in M_N(t)} G(N;m)\ .$$

For any positive integers t and T, we put

$$\mathscr{H}^{(0)}(N;t,T)=\sum_{m\in M_N(t)}\mathscr{K}^{(0)}(N;m,T)\ ,$$

$$\mathscr{K}^{(0)}(N;m,T)=\sum_{\tau=0}^{2T}(-1)^{\tau}\mathscr{L}(N;m,\tau)\ ,$$

$$\mathscr{H}^{(1)}(N;t,T)=\sum_{m\in M_N(t)}\mathscr{K}^{(1)}(N;m,T)\ ,$$

$$\mathscr{K}^{(1)}(N;m,T)=\sum_{\tau=0}^{2T+1}(-1)^{\tau}\mathscr{L}(N;m,\tau)\ ,$$

$$\mathscr{L}(N;m,\tau)=\sum_{\substack{\mu\in M_N(\tau)\\ (\mu,m)=1}}\ \sum_{\substack{p+n=N\\ m\mu\mid n}}1\ .$$

LEMMA 2. $\mathscr{H}^{(1)}(N;t,T)\leqq F(N;t)\leqq\mathscr{H}^{(0)}(N;t,T)$.

PROOF. We can write

$$\mathscr{L}(N;m,\tau)=\sum_{\substack{p+n=N\\ m\mid n}}\binom{\omega_N(n)-t}{\tau}\ ,$$

so that

$$\mathscr{K}^{(0)}(N;m,T)=\sum_{\substack{p+n=N\\ m\mid n}}\sum_{\tau=0}^{2T}(-1)^{\tau}\binom{\omega_N(n)-t}{\tau}\ ,$$

$$\mathscr{K}^{(1)}(N;m,T)=\sum_{\substack{p+n=N\\ m\mid n}}\sum_{\tau=0}^{2T+1}(-1)^{\tau}\binom{\omega_N(n)-t}{\tau}\ .$$

Now, since $m\in M_N(t)$ and $m\mid n$, (2) is equivalent to the equality $\omega_N(n)=t$. Hence, by (1), we have

$$\mathscr{K}^{(1)}(N;m,T)\leqq G(N;m)\leqq\mathscr{K}^{(0)}(N;m,T)\ .$$

The lemma follows from this and the definitions of $F(N; t)$, $\mathscr{H}^{(0)}(N; t, T)$ and $\mathscr{H}^{(1)}(N; t, T)$.

We further put

$$H^{(0)}(N; t, T)=\sum_{m\in M_N(t)} K^{(0)}(N; m, T) ,$$

$$K^{(0)}(N; m, T)=\sum_{\tau=0}^{2T}(-1)^{\tau}L(N; m, \tau) ,$$

$$H^{(1)}(N; t, T)=\sum_{m\in M_N(t)} K^{(1)}(N; m, T) ,$$

$$K^{(1)}(N; m, T)=\sum_{\tau=0}^{2T+1}(-1)^{\tau}L(N; m, \tau) ,$$

$$L(N; m, \tau)=\sum_{\substack{\mu\in M_N(\tau)\\ (\mu, m)=1}} \frac{1}{\varphi(m\mu)} ,$$

where $\varphi(m\mu)$ is Euler's function of $m\mu$.

LEMMA 3. *Let $T=[5y(N)]$. Then, as $N\rightarrow\infty$, we have*

$$H^{(0)}(N; t, T)=\frac{\{y(N)\}^t e^{-y(N)}}{t!}\{1+o(1)\} ,$$

$$H^{(1)}(N; t, T)=\frac{\{y(N)\}^t e^{-y(N)}}{t!}\{1+o(1)\}$$

uniformly in t with $t<2y(N)$.

PROOF. The formulas in the lemma can be proved quite similarly as Lemma 6 in [6], if we replace the $L(N; m, \tau)$'s contained in the definitions of $H^{(0)}(N; t, T)$ and $H^{(1)}(N; t, T)$ by

$$L^*(N; m, \tau)=\sum_{\substack{\mu\in M_N(\tau)\\ (\mu, m)=1}} \frac{1}{m\mu} .$$

Hence it will suffice for the proof of the lemma to show that

$$L^*(N; m, \tau)=L(N; m, \tau)\{1+o(1)\} \tag{3}$$

uniformly in the relevant $L(N; m, \tau)$'s.

Now, for each summand of $L(N; m, \tau)$, the pair of positive integers m and μ is such that $(m, \mu)=1$, $m\in M_N(t)$, $t<2y(N)$, $\mu\in M_N(\tau)$, $\tau\leqq 10y(N)+1$, so that, by the definitions of the sets $Q_N(t)$, $M_N(t)$, and Lemma 1, $m\mu$ is squarefree, $\omega(m\mu)<c\log\log N$, $c>0$, and each of the prime factors of $m\mu$ is greater than $e^{(\log\log N)^2}$. Hence

$$\frac{1}{m\mu}<\frac{1}{\varphi(m\mu)}=\frac{1}{m\mu}\prod_{p|m\mu}\left(1-\frac{1}{p}\right)^{-1}<\frac{1}{m\mu}\prod_{p|m\mu}\left(1+\frac{2}{p}\right)$$

$$<\frac{1}{m\mu}(1+2e^{-(\log\log N)^2})^{c\log\log N}=\frac{1+o(1)}{m\mu},$$

or

$$\frac{1}{m\mu}=\frac{1+o(1)}{\varphi(m\mu)},$$

from which we see that (3) holds with the required uniformity.

LEMMA 4. *Let T be an increasing function of N such that $T=O(\log\log N)$. Then, as $N\to\infty$, we have*

$$\mathscr{H}^{(0)}(N;\,t,\,T)-H^0(N;\,t,\,T)\operatorname{li}N=o\left(\frac{N\{y(N)\}^t e^{-y(N)}}{t!\log N}\right),$$

$$\mathscr{H}^{(1)}(N;\,t,\,T)-H^0(N;\,t,\,T)\operatorname{li}N=o\left(\frac{N\{y(N)\}^t e^{-y(N)}}{t!\log N}\right)$$

uniformly in t with $t<2y(N)$, where li N *is the logarithmic integral of N.*

PROOF. The definition of $\mathscr{L}(N;\,m,\,\tau)$ can be rewritten as

$$\mathscr{L}(N;\,m,\,\tau)=\sum_{\substack{\mu\in M_N(\tau)\\(\mu,m)=1}}\pi(N;\,m\mu,\,N)$$

where $\pi(N;\,m\mu,\,N)$ is the number of primes p such that $p<N$ and $p\equiv m\mu(\operatorname{mod}N)$. Hence, by the definitions of $\mathscr{H}^{(0)}(N;\,t,\,T)$ and $H^0(N;\,t,\,T)$, we can write

$$|\mathscr{H}^{(0)}(N;\,t,\,T)-H^0(N;\,t,\,T)\operatorname{li}N|$$

$$\leqq\sum_{m\in M_N(t)}\sum_{\tau=0}^{2T}\sum_{\substack{\mu\in M_N(\tau)\\(\mu,m)=1}}\left|\pi(N;\,m\mu,\,N)-\frac{\operatorname{li}N}{\varphi(m\mu)}\right|.$$

Put here $m\mu=\nu$, then the same value of ν occurs at most $d(\nu)$ times, where $d(\nu)$ is the number of divisors of ν; by our assumptions, ν is squarefree and $\nu\in M_N(\tau)$, $\tau<c\log\log N$, so that $\omega(\nu)<c\log\log N$ and $d(\nu)<e^{c\log\log N}=\log^c N$, where c is a suitable positive constant; by the definition of the set Q_N, each prime factor of ν is less than $N^{(\log\log N)^{-2}}$, and so $\nu<N^{c(\log\log N)^{-1}}$. Hence we have

$$\left|\mathscr{H}^{(0)}(N;\,t,\,T)-H^0(N;\,t,\,T)\operatorname{li}N\right|<\log^c N\sum_{(\nu,N)=1}^{N^{c(\log\log N)^{-1}}}\left|\pi(N;\,\nu,\,N)-\frac{\operatorname{li}N}{\varphi(\nu)}\right|.$$

Now it follows from Bombieri's theorem that

$$\mathscr{H}^{(0)}(N;\, t,\, T)-H^{0}(N;\, t,\, T)\,\mathrm{li}\, N=O(N\log^{-\alpha}N)$$

with arbitrary positive constant α. (For our purpose somewhat weaker result than Bombieri's would suffice.) Again, since we assume $t<2y(N)$,

$$\frac{\{y(N)\}^{t}}{t!}>\left(\frac{t}{2}\right)^{t}\cdot\frac{1}{t^{t}}=2^{-t}>e^{-2y(N)}\,.$$

Hence we have

$$\mathscr{H}^{(0)}(N;\, t,\, T)-H^{0}(N;\, t,\, T)\,\mathrm{li}\, N=O\left(\frac{N\{y(N)\}^{t}e^{2y(N)}}{t!\log^{\alpha}N}\right)\,.$$

Similar result can be obtained for $\mathscr{H}^{(1)}(N;\, t,\, T)$, and, since $y(N)\sim \log\log N$ by Lemma 1, the lemma follows when we take α sufficiently large.

LEMMA 5. *Let $T=[5y(N)]$. Then, as $N\to\infty$,*

$$\mathscr{H}^{(0)}(N;\, t,\, T)=\frac{N\{y(N)\}^{t}e^{-y(N)}}{t!\log N}\{1+o(1)\}\,,$$

$$\mathscr{H}^{(1)}(N;\, t,\, T)=\frac{N\{y(N)\}^{t}e^{-y(N)}}{t!\log N}\{1+o(1)\}$$

uniformly in t with $t<2y(N)$.

PROOF. The lemma follows from Lemmas 3 and 4.

LEMMA 6. *As $N\to\infty$,*

$$F(N;\, t)=\frac{N\{y(N)\}^{t}e^{-y(N)}}{t!\log N}\{1+o(1)\}$$

uniformly in t with $t<2y(N)$.

PROOF. The lemma follows from Lemmas 2 and 5.

LEMMA 7. *Let $\alpha<\beta$. Let t be a positive integer such that $t=y(N)+u\sqrt{y(N)}$ with $\alpha<u<\beta$. Then, as $N\to\infty$,*

$$F(N;\, t)=\frac{N}{\sqrt{2\pi y(N)}\log N}e^{-u^{2}/2}\{1+o(1)\}$$

uniformly in t with above-mentioned restrictions.

PROOF. This lemma corresponds to Lemma 13 in [6], and can be proved similarly. The Stirling formula plays an important role in the proof.

LEMMA 8. *Let* $\alpha<\beta$, *and let* $A^{**}(N;\alpha,\beta)$ *denote the number of representations of* N *as the sum of the form* $N=p+n$, *where* p *is prime, and* n *is a positive integer such that*

$$y(N)+\alpha\sqrt{y(N)}<\omega_N(n)<y(N)+\beta\sqrt{y(N)} .$$

Then, as $N\to\infty$, *we have*

$$A^{**}(N;\alpha,\beta)\sim\frac{N}{\log N}\cdot\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}dx .$$

PROOF. This lemma corresponds to Lemma 14 in [6], and can be proved similarly.

LEMMA 9. *Let* $\alpha<\beta$, *and let* $A^{*}(N;\alpha,\beta)$ *denote the number of representations of* N *as the sum of the form* $N=p+n$, *where* p *is prime, and* n *is a positive integer such that*

$$y(N)+\alpha\sqrt{y(N)}<\omega(n)<y(N)+\beta\sqrt{y(N)} .$$

Then, as $N\to\infty$, *we have*

$$A^{*}(N;\alpha,\beta)\sim\frac{N}{\log N}\cdot\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-x^2/2}dx .$$

PROOF. We shall estimate the sum

$$S(N)=\sum_{p<N}\{\omega(N-p)-\omega_N(N-p)\}$$

in utilizing Brun-Titchmarsh's inequality. For this inequality, cf. Halberstam-Richert [4], p. 110, Mitsui [5], p. 154. Now, noting the fact that a positive integer has at most one prime factor greater than the square root of itself, we argue as

$$S(N)=\sum_{p<N}\sum_{\substack{q\mid(N-p)\\ q\notin Q_N}}1=\sum_{p<N}\sum_{\substack{q\mid(N-p)\\ q\notin Q_N,\,q\le\sqrt{N}}}1+O\Big(\sum_{p<N}1\Big)$$

$$=\sum_{\substack{q\le\sqrt{N}\\ q\notin Q_N}}\sum_{\substack{p<N\\ p\equiv N\,(\mathrm{mod}\,q)}}1+O\Big(\frac{N}{\log N}\Big)=\sum_{\substack{q\le\sqrt{N}\\ q\notin Q_N}}\pi(N;q,N)+O\Big(\frac{N}{\log N}\Big)$$

where q runs through the primes satisfying the specified conditions. On applying Brun-Titchmarsh's inequality to the last sum, we have

$$\sum_{\substack{q\leq\sqrt{N}\\ q\notin Q_N}}\pi(N;\,q,\,N)=O\Big(\sum_{\substack{q\leq\sqrt{N}\\ q\notin Q_N}}\frac{N}{q\log(N/q)}\Big)=O\Big(\frac{N}{\log N}\sum_{\substack{q\leq\sqrt{N}\\ q\notin Q_N}}\frac{1}{q}\Big)\,.$$

Again, similarly as in the proof of Lemma 4 in [6], we obtain

$$\sum_{\substack{q\leq\sqrt{N}\\ q\notin Q_N}}\frac{1}{q}=O(\log\log\log N)\,.$$

Thus it has been proved that

$$S(N)=O\Big(\frac{N}{\log N}\log\log\log N\Big)\,.$$

Now we can prove the lemma similarly as in the proof of Lemma 15 in [6], using this result in the form

$$\sum_{p+n=N}\{\omega(n)-\omega_N(n)\}=o\Big(\frac{N}{\log N}\sqrt{y(N)}\Big)\,.$$

It follows from this that, for any given $\varepsilon>0$, we can take $N_1=N_1(\varepsilon)$ so large that, when $N>N_1$, the number of representations of N as the sum of the form $N=p+n$ such that the inequality $\omega(n)-\omega_N(n)>\varepsilon\sqrt{y(N)}$ holds, is less thau $\varepsilon N/\log N$. Hence, for $N>N_1$,

$$A^{**}(N;\,\alpha,\,\beta-\varepsilon)-\varepsilon\frac{N}{\log N}<A^{*}(N;\,\alpha,\,\beta)<A^{**}(N;\,\alpha-\varepsilon,\,\beta)+\varepsilon\frac{N}{\log N}\,.$$

From this and Lemma 8, we conclude that

$$\frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta-\varepsilon}e^{-x^2/2}dx-\varepsilon\leq\liminf_{N\to\infty}\frac{A^{*}(N;\,\alpha,\,\beta)\log N}{N}$$
$$\leq\limsup_{N\to\infty}\frac{A^{*}(N;\,\alpha,\,\beta)\log N}{N}\leq\frac{1}{\sqrt{2\pi}}\int_{\alpha-\varepsilon}^{\beta}e^{-x^2/2}dx+\varepsilon\,,$$

which gives the lemma.

THE LAST STEP OF THE PROOF OF THE THEOREM. The remaining task is to replace $y(N)$ by $\log\log N$. This can be carried out quite similarly as in the proof of Lemma 16 in [6]. We avoid the repetition.

References

[1] E. BOMBIERI, On the large sieve, Mathematika, **12** (1965), 201–225.
[2] P. X. GALLAGHER, Bombieri's mean value theorem, Mathematika, **15** (1968), 1–6.
[3] H. HALBERSTAM, On the distribution of additive number-theoretic functions (III), J. London Math. Soc., **31** (1956), 14–27.
[4] H. HALBERSTAM and H.-E. RICHERT, Sieve Methods, Academic Press, London, 1974.
[5] T. MITSUI, The Theory of Numbers, Shibundo, Tokyo, 1970 (in Japanese).
[6] M. TANAKA, Some results on additive number theory III, TOKYO J. Math., **2** (1979), 159–182.

Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
Gakushuin UNIVERSITY
MEJIRO TOSHIMA-KU
TOKYO 171