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Analytic Continuation of Arithmetic Holomorphic Functions on a Half Plane

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Introduction

The entire arithmetic functions of one variable have been studied by many mathematicians. For example, see R. Boas [3] and R. Buck [4]. Recently V. Avanissian and R. Gay [1] studied entire arithmetic functions of exponential type of n variables using the theory of analytic functionals. In this paper we consider the arithmetic holomorphic functions on a half plane using the theory of analytic functional with non-compact carrier. We will obtain a sufficient condition for an arithmetic holomorphic function to be entire.

§1. Analytic functionals with non-compact carrier.

In this section we recall the definition of analytic functional with non-compact carrier. Let L be the closed half strip in the complex plane:

$$L=\{z=x+iy;\, x\geqq a,\, |y|\leqq k\}\,,\quad i=\sqrt{-1}\,.$$

By L_ε we denote the ε-neighborhood of L:

$$L_\varepsilon=L+[-\varepsilon,\varepsilon]+i[-\varepsilon,\varepsilon]\,.$$

For $\varepsilon>0$, $\varepsilon'>0$ and $0\leqq k'<1$, we define the function space $Q_b(L_\varepsilon; k'+\varepsilon')$ as follows:

$$Q_b(L_\varepsilon; k'+\varepsilon')=\left\{f\in\mathcal{O}(\mathrm{int}\, L_\varepsilon)\cap C(L_\varepsilon);\ \sup_{z\in L_\varepsilon}|f(z)\exp((k'+\varepsilon')z)|<+\infty\right\}$$

where $\mathcal{O}(\mathrm{int}\, L_\varepsilon)$ denotes the space of holomorphic functions on the interior

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of L_ε and $C(L_\varepsilon)$ denotes the space of continuous functions on L_ε. Now we define the function space $Q(L; k')$ as follows:

$$Q(L; k') = \lim_{\varepsilon \downarrow 0, \varepsilon' \downarrow 0} \operatorname{ind} Q_b(L_\varepsilon; k'+\varepsilon') .$$

Endowed with the natural inductive limit topology, $Q(L; k')$ becomes a DFS space. We denote the dual space of $Q(L; k')$ by $Q'(L; k')$ and an element of $Q'(L; k')$ is called an analytic functional with non-compact carrier in L and of type k'. Next we define the space of holomorphic functions of exponential type on the half plane $(-\infty, -k')+iR$ as follows:

$$\begin{aligned}&\operatorname{Exp}((-\infty, -k')+iR; L)\\ &\quad = \Big\{ f \in \mathcal{O}((-\infty, -k')+iR); \sup_{\operatorname{Re} t \leq -k'-\varepsilon'} |f(t) \exp(-(a-\varepsilon)\operatorname{Re} t - (k+\varepsilon)|\operatorname{Im} t|)| \\ &\quad < +\infty \text{ for every } \varepsilon > 0,\ \varepsilon' > 0 \Big\} .\end{aligned}$$

We define the Fourier-Borel transformation of an analytic functional $\mu \in Q'(L; k')$ as follows:

$$\hat{\mu}(t) = \langle \mu_z, \exp(zt) \rangle . \tag{1.1}$$

Remark that (1.1) is defined for t in the half plane $(-\infty, -k')+iR$. The following theorem of Paley-Wiener type characterizes the Fourier Borel transformation of the space $Q'(L; k')$.

THEOREM 1 (Morimoto [6], [7], [8]). *The Fourier-Borel transformation is a topological linear isomorphism of $Q'(L; k')$ onto* $\operatorname{Exp}((-\infty, -k')+iR; L)$.

§2. The Avanissian-Gay transformation.

If $0 \leq k' < 1$ and $w \notin \exp(-L)$, then the function of z, $(1-we^z)^{-1}$, belongs to $Q(L; k')$. Following Avanissian and Gay [1], we define the transformation $G_\mu(w)$ of an analytic functional $\mu \in Q'(L; k')$ as follows:

$$G_\mu(w) = \langle \mu_z, (1-we^z)^{-1} \rangle .$$

$G_\mu(w)$ is a function of $w \notin \exp(-L)$ and has the following properties.

PROPOSITION 1 (Morimoto-Yoshino [9]).

(i) *$G_\mu(w)$ is holomorphic in the complement of* $\exp(-L)$.

(ii) *The following Laurent expansion is valid:*

$$G_\mu(w) = -\sum_{n=1}^{\infty} \hat{\mu}(-n) w^{-n} \quad (|w| > e^{-a}) .$$

(iii) $\lim_{|w|\to\infty} G_\mu(w)=0$.

(iv) *For every* $\varepsilon>0$ *and* $\varepsilon'>0$, *there exists a positive number* C *such that*

$$|G_\mu(w)|\leqq C|w|^{-(k'+\varepsilon')} \quad (k+\varepsilon\leqq|\arg w|\leqq\pi) . \tag{2.1}$$

And we have the following inversion formula.

PROPOSITION 2 (Morimoto-Yoshino [9]). *If* $\mu\in Q'(L; k')$, $0\leqq k'<1$, $0\leqq k<\pi$ *and* $h(z)\in Q_b(L_\varepsilon; k'+\varepsilon')$, *then we have*

$$\langle\mu, h\rangle=(2\pi i)^{-1}\int_{\partial L_\varepsilon} G_\mu(e^{-z})h(z)dz . \tag{2.2}$$

§3. Transfinite diameter of $\exp(-L)$.

In this section we estimate the transfinite diameter of $\exp(-L)$. Let F be a compact set in the complex plane. We denote by $\gamma(F)$ the transfinite diameter of F. For the details of transfinite diameters, see Ahlfors [2] and Zalcman [11]. First we begin with two lemmas.

LEMMA 1 (Zalcman [11]). *Suppose F is a compact set in the complex plane. Then we have the following estimate*:

(i) $\gamma(F)\leqq(2\pi)^{-1}\inf_C$ *length* (C),

where C is a rectifiable curve of winding number 1 for each point of F.

(ii) *If F is a segment, we have the following equality*:

$$\gamma(F)=\frac{1}{4}m(F) ,$$

where $m(F)$ denotes the Lebesgue measure of F.

LEMMA 2 (Martineau [5]-Šeinov [10]). *Let F be a polynomially convex compact set in the complex plane. Suppose $\gamma(F)$ is less than 1 and $g(w)$ is a holomorphic function on the complement of F and* $\lim_{|w|\to\infty} g(w)=0$. *If the Laurent coefficients of $g(w)$ at infinity are all integers, then*

$$g(w)=A(w)B(w)^{-1}$$

where $A(w)$ and $B(w)$ are polynomials whose coefficients are all integers and moreover $B(w)$ is monic.

Using Lemma 1, we can estimate the transfinite diameter of $\exp(-L)$. The result is as follows.

PROPOSITION 3. *Suppose* $L=[a, \infty)+i[-k, k]$. *Then we have the following estimates*:

(i) $\gamma(\exp(-L))=(1/4)e^{-a}$ *if* $k=0$

(ii) $\gamma(\exp(-L))\leqq\pi^{-1}(k+1)e^{-a}$ *if* $0<k\leqq(1/2)\pi$

(iii) $\gamma(\exp(-L))\leqq\pi^{-1}(k+\sin k)e^{-a}$ *if* $(1/2)\pi\leqq k<\pi$.

PROOF. (i) In this case $\exp(-L)$ is the segment, whose Lebesgue measure is e^{-a}. Hence we have the above estimate.

(ii) In this case the length of boundary of $\exp(-L)$ is $2(k+1)e^{-a}$. Hence we obtain the above result by Lemma 1.

(iii) In this case $\exp(-L)$ is surrounded by the curve whose length is $2(k+\sin k)e^{-a}$. See Figure 1. q.e.d.

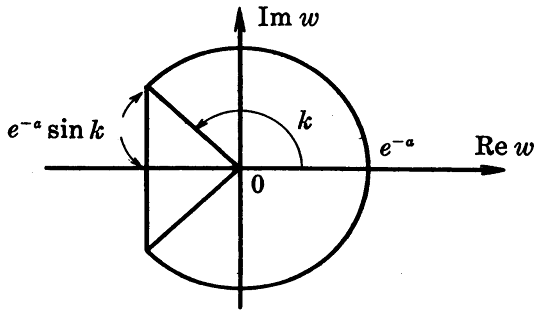

FIGURE 1

From Proposition 3, we obtain the following corollary.

COROLLARY. *If the pair* (a, k) *satisfies one of the following three conditions*:

(3.1) $$k=0 \quad \text{and} \quad a>-2\log 2\,,$$

(3.2) $$0<k\leqq\frac{\pi}{2} \quad \text{and} \quad a>\log \pi^{-1}(k+1)\,,$$

(3.3) $$\frac{\pi}{2}\leqq k<\pi \quad \text{and} \quad a>\log \pi^{-1}(k+\sin k)\,,$$

then $\gamma(\exp(-L))$ *is less than* 1.

§4. Analytic continuation of arithmetic holomorphic functions of exponential type on a half plane.

Let $f(t)$ be a holomorphic function defined in the half plane:

$(-\infty, -k')+iR$, where $0 \leqq k' < 1$. We call $f(t)$ arithmetic if $f(-n)$ are all integer for $n=1, 2, 3, \cdots$.

THEOREM 2. *Suppose $f(t)$ belongs to* $\mathrm{Exp}((-\infty, -k')+iR; L)$ *and that $f(t)$ is arithmetic. If the pair (a, k) satisfies one of the three conditions* (3.1), (3.2), (3.3), *then $f(t)$ is an entire function. Moreover, $f(t)$ has following form*:

$$f(t) = \sum_{i=1}^{l} P_i(t) \exp(\beta_i t)$$

where $P_i(t)$ are polynomials and $\mathrm{Re}\,\beta_i \geqq a$, $|\mathrm{Im}\,\beta_i| \leqq k$ *and* $\exp(-\beta_i)$ *are algebraic integers.*

PROOF. By Theorem 1, there exists $\mu \in Q'(L; k')$ such that

$$f(t) = \langle \mu_z, \exp(zt) \rangle = \hat{\mu}(t) .$$

By Proposition 1, we have

$$G_\mu(w) = -\sum_{n=1}^{\infty} \hat{\mu}(-n) w^{-n} = -\sum_{n=1}^{\infty} f(-n) w^{-n}$$

and

$$\lim_{w \to \infty} G_\mu(w) = 0 .$$

By the assumption and Proposition 3, $\gamma(\exp(-L))$ is less than 1 and $f(-n)$ are all integers. Therefore by Lemma 2, we can find polynomials $A(w)$ and $B(w)$ such that

$$G_\mu(w) = A(w) B(w)^{-1} . \tag{4.0}$$

From Proposition 1 (iv), we must have $B(0) \neq 0$ and $G_\mu(w)$ is holomorphic at $w=0$. Therefore there exists a positive number R such that $G_\mu(e^{-z})$ is holomorphic for $\mathrm{Re}\, z > R$. From the inversion formula (2.2), we have

$$f(t) = (2\pi i)^{-1} \int_{\partial L_\varepsilon} G_\mu(e^{-z}) \exp(zt) dz \quad (\mathrm{Re}\, t < -k') . \tag{4.1}$$

Now we consider the integral of the right hand side of (4.1). Put

$$L_+ = \{z \in L_\varepsilon;\ \mathrm{Re}\, z \geqq R\} , \quad L_- = \{z \in L_\varepsilon;\ \mathrm{Re}\, z \leqq R\}$$

and we have

$$L_\varepsilon = L_+ + L_- .$$

We divide path of integration ∂L_ε into the following two parts:

$$\partial L_\varepsilon = \partial L_+ + \partial L_- .$$

See Figure 2.

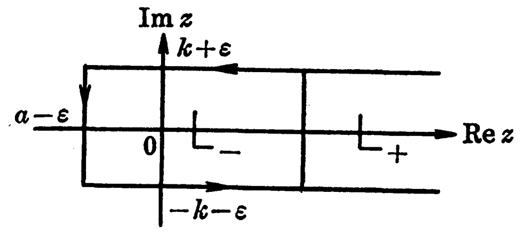

FIGURE 2

Since $\lim_{s\to\infty} \sup_{\mathrm{Re}\, z=s} |G_\mu(e^{-z}) \exp(zt)| = 0$, we obtain

$$\int_{\partial L_+} G_\mu(e^{-z}) \exp(zt) dz = 0$$

by Cauchy's theorem. Hence we have

$$f(t) = (2\pi i)^{-1} \int_{\partial L_-} G_\mu(e^{-z}) \exp(zt) dz . \tag{4.2}$$

Since the right hand side of (4.2) is an integration over a compact set, $f(t)$ is an entire function of t. As $B(w)$ is, by the Lemma 2, a monic polynomial with integer coefficients, we have

$$B(w) = \prod_{i=1}^{l} (w - b_i)^{n_i}$$

where b_i are algebraic integers and we obtain from (4.0),

$$G_\mu(e^{-z}) = A(e^{-z}) \prod_{i=1}^{l} (e^{-z} - b_i)^{-n_i} . \tag{4.3}$$

By Proposition 1-(i), $G_\mu(w)$ belongs to $\mathcal{O}(C \backslash \exp(-L))$, so b_i are in $\mathrm{Exp}(-L)$. Since every b_i belongs to $\exp(-L)$, there exists a unique point β_i of L such that

$$b_i = \exp(-\beta_i) \quad \text{where} \quad \mathrm{Re}\, \beta_i \geqq a , \quad |\mathrm{Im}\, \beta_i| \leqq k .$$

From (4.3), we obtain

$$G_\mu(e^{-z}) = A(e^{-z}) \prod_{i=1}^{l} (1 - \exp(z - \beta_i))^{-n_i} \exp\left(\sum_{i=1}^{l} n_i z\right) . \tag{4.4}$$

Inserting (4.4) into (4.2), we have the desired result by the residue theorem:

$$f(t)=\sum_{i=1}^{l} P_i(t)\exp(\beta_i t) .$$

q.e.d.

§5. Some examples and remarks.

(i) $f(t)=2^{-t}$ is arithmetic and belongs to $\mathrm{Exp}((-\infty, k')+iR; L)$ with $a=-\log 2$, $k=0$.

(ii) $f(t)=\sin(\pi/2)t$ is arithmetic and belongs to $\mathrm{Exp}((-\infty, k')+iR; L)$ with $a=0$, $k=\pi/2$.

(iii) $f(t)=2\cos(2/3)\pi t$ is arithmetic and belongs to $\mathrm{Exp}((-\infty, -k')+iR; L)$ with $a=0$, $k=(2/3)\pi$.

(iv) If $f(t)$ is arithmetic and belongs to $\mathrm{Exp}((-\infty, k')+iR; L)$ with $a>0$, then $f(t)$ vanishes identically. In fact, as we have $\lim_{n\to\infty} f(-n)=0$ and $f(-n)$ are all integers, there exists a positive integer N such that

$$f(-n)=0 \quad \text{for} \quad n>N .$$

By Carlson theorem (Boas [3], Morimoto-Yoshino [9]), we have

$$f(t)=0 .$$

(v) Let $\Gamma(t)$ be the Gamma function, then $\Gamma(t)^{-1}$ is arithmetic. But $\Gamma(t)^{-1}$ does not belong to $\mathrm{Exp}((-\infty, k')+iR; L)$. In fact, if $\Gamma(t)^{-1}$ belongs to $\mathrm{Exp}((-\infty, k')+iR; L)$, then $\Gamma(t)^{-1}$ vanishes identically by Carlson's theorem. This is impossible.

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