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Minimal Immersions of Riemannian Products into Real Space Forms

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Introduction

Let M_1(resp. M_2) be an m(resp. $n-m$)-dimensional Riemannian manifold. An isometric immersion f of a Riemannian product $M_1\times M_2$ into an $(n+p)$-dimensional Euclidean space R^{n+p} is called a product immersion if there is an orthogonal product decomposition $R^{n+p}=R^{n_1}\times R^{n_2}$ together with isometric immersions f_1: $M_1\to R^{n_1}$ and f_2: $M_2\to R^{n_2}$ such that $f=f_1\times f_2$. Furthermore an isometric immersion g of a Riemannian product $M_1\times M_2$ into an $(n+p)$-dimensional sphere $S^{n+p}(r)$ with radius r in R^{n+p+1} is called a product immersion if g is a product immersion of $M_1\times M_2$ into R^{n+p+1}. S. B. Alexander [1] and J. Moore [4] obtained some conditions for an immersion of a Riemannian product into Euclidean space to be a product immersion. On the other hand, K. Yano and S. Ishihara [7] determined compact orientable submanifolds with nonnegative sectional curvature immersed into a unit sphere whose mean curvature vectors are parallel and normal connections are trivial. These are products of spheres and these immersions are product immersions into the unit sphere. In this note, we shall investigate Riemannian products minimally immersed into a real space form and prove some theorems.

THEOREM. *A minimal submanifold of a hyperbolic space is irreducible. A minimal immersion of a Riemannian product into Euclidean space is a product of minimal immersions.*

THEOREM. *Let M_1(resp. M_2) be an m(resp. $n-m$)-dimensional compact orientable Riemannian manifold and M the Riemannian product of M_1 and M_2 minimally immersed into $(n+p)$-dimensional unit sphere. Then we have an integral inequality*

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$$\int_M \left(nK-\lambda K-\mu K-\frac{2}{p}K^2\right)*1 \geqq 0$$

where $*1$ *is the volume element of* M *and* λ *(resp.* μ*) is the minimum of the Ricci curvature of* M_1*(resp.* M_2*). The definition for* K *is given in the section* 5.

By the latter theorem, we shall characterize a Riemannian product $S^m(1)\times S^{n-m}(1)$ minimally immersed into $S^{n+m(n-m)}(1)$.

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§ 1. Preliminaries.

We denote by $M^k(c)$ a k-dimensional space form of constant curvature c. Let M_1(resp. M_2) be an m(resp. $n-m$)-dimensional Riemannian manifold and M the Riemannian product of M_1 and M_2 isometrically immersed into $M^{n+p}(c)$. Then the second fundamental form σ of the immersion is given by $\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$ and it satisfies $\sigma(X, Y)=\sigma(Y, X)$. We choose a local field of orthonormal frames $e_1, \cdots, e_m$, (resp. $e_{m+1}, \cdots, e_n$) of M_1(resp. M_2), then we may consider $\{e_a\}, \{e_s\}$ as a local field of orthonormal frames of M*. By an extension, we choose a local field of orthonormal frames $e_1, \cdots, e_m, e_{m+1}, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ in $M^{n+p}(c)$. With respect to the frame field of $M^{n+p}(c)$ chosen above, let $\omega^1, \cdots, \omega^m, \omega^{m+1}, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^{n+p}$ be the field of dual frames. Then the structure equations of $M^{n+p}(c)$ are given by

$$d\omega^A=-\sum \omega_B^A\wedge\omega^B\ , \qquad \omega_B^A+\omega_A^B=0\ , \tag{1.1}$$

$$d\omega_B^A=-\sum \omega_C^A\wedge\omega_B^C+c\omega^A\wedge\omega^B\ , \tag{1.2}$$

Restricting these forms to M, we obtain the structure equations of the immersion:

$$\omega^\alpha=0\ . \tag{1.3}$$

$$d\omega^i=-\sum\omega_j^i\wedge\omega^j\ , \qquad \omega_j^i+\omega_i^j=0\ . \tag{1.4}$$

$$d\omega_j^i=-\sum\omega_k^i\wedge\omega_j^k+\Omega_j^i\ , \qquad \Omega_j^i=\sum(1/2)R_{jkl}^i\omega^k\wedge\omega^l\ . \tag{1.5}$$

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* We use the following convention on the range of indices unless otherwise stated: $1\leqq A, B, C, \leqq n+p : 1\leqq a, b, c, \leqq m<r, s, t, u, \leqq n : 1\leqq i, j, k, l, \leqq n<\alpha, \beta, \gamma, \leqq n+p$, and agree that repeated indices under summation sign without any indication are summed over respective ranges.

(1.6) $$d\omega^\alpha_\beta = -\sum \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + \Omega^\alpha_\beta\ , \qquad \Omega^\alpha_\beta = \sum (1/2) R^\alpha_{\beta kl} \omega^k \wedge \omega^l\ .$$

Since M is the Riemannian product, we have

(1.7) $$\omega^a_s = 0 \quad \text{for all } a \text{ and } s\ .$$

From (1.1) and (1.3) it follows that $\sum \omega^\alpha_i \wedge \omega^i = 0$. Therefore, by Cartan's lemma, we may write

(1.8) $$\omega^\alpha_i = \sum h^\alpha_{ji} \omega^j\ , \qquad h^\alpha_{ij} = h^\alpha_{ji}\ .$$

The second fundamental form σ and h^α_{ij} are related by $\sigma(e_i, e_j) = \sum h^\alpha_{ij} e_\alpha$. The equations of Gauss and Ricci are given respectively by

(1.9) $$R^i_{jkl} = c(\delta^i_k \delta_{jl} - \delta^i_l \delta_{jk}) + \sum (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk})\ ,$$

(1.10) $$R^\alpha_{\beta kl} = \sum (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik})\ .$$

If we define h^α_{ijk} by

(1.11) $$\sum h^\alpha_{ijk} \omega^k = dh^\alpha_{ij} - \sum h^\alpha_{ki} \omega^k_j - \sum h^\alpha_{kj} \omega^k_i - \sum h^\beta_{ij} \omega^\alpha_\beta\ .$$

Then from (1.2), (1.3), and (1.8) we have $h^\alpha_{ijk} = h^\alpha_{ikj}$. If we define h^α_{ijkl} by

(1.12) $$\sum h^\alpha_{ijkl} \omega^l = dh^\alpha_{ijk} - \sum h^\alpha_{lij} \omega^l_k - \sum h^\alpha_{lkj} \omega^l_i - \sum h^\alpha_{lik} \omega^l_j + \sum h^\beta_{ijk} \omega^\alpha_\beta\ ,$$

and Δh^α_{ij} by $\sum h^\alpha_{ijkk}$, then we have the following.

LEMMA 1.1 ([2], [5]). *If M is minimal,*

$$\Delta h^\alpha_{ij} = \sum \left(\sum h^\alpha_{kl} R^l_{ijk} + \sum h^\alpha_{li} R^l_{kjk} - \sum h^\beta_{ki} R^\alpha_{\beta jk}\right)\ .$$

§ 2. Lemmas.

LEMMA 2.1 ([4]). *When $c=0$, an immersion of $M = M_1 \times M_2$ into $M^{n+p}(0)$ is a product immersion if and only if $\sigma(X, Y) = 0$, where X(resp. Y) is tangent to M_1(resp. M_2).*

LEMMA 2.2 ([8]). *An immersion of $M = M_1 \times M_2$ into $M^{n+p}(c)$ with $c > 0$, is a product immersion if and only if $\sigma(X, Y) = 0$, where X(resp. Y) is tangent to M_1(resp. M_2).*

PROOF. We may regard $M^{n+p}(c)$ as the sphere $S^{n+p}(\sqrt{1/c})$ of radius $\sqrt{1/c}$ in R^{n+p+1} and hence M as a submanifold of R^{n+p+1}. Let $\tilde{\sigma}$(resp. σ') be the second fundamental form of the immersion of M(resp. $S^{n+p}(\sqrt{1/c})$) into R^{n+p+1}. If X(resp. Y) is tangent to M_1(resp. M_2), then we have

$\tilde{\sigma}(X, Y)=\sigma(X, Y)$ since σ' is umbilical. From Lemma 2.1, we obtain the result. Q.E.D.

LEMMA 2.3. *When $M=M_1\times M_2$ is minimally immersed into $M^{n+p}(c)$, we have $m(n-m)c-\sum_\alpha(\sum_a h^\alpha_{aa})^2-\sum h^\alpha_{as}h^\alpha_{as}=0$.*

PROOF. Since M is the Riemannian product of M_1 and M_2, we have $R^a_{sas}=0$. From (1.9), we obtain $0=c+\sum_\alpha h^\alpha_{aa}h^\alpha_{ss}-\sum_\alpha h^\alpha_{as}h^\alpha_{as}$ and hence $0=m(n-m)c+\sum_\alpha(\sum_a h^\alpha_{aa})(\sum_s h^\alpha_{ss})-\sum h^\alpha_{as}h^\alpha_{as}$. Since M is minimally immersed into $M^{n+p}(c)$, we obtain $\sum_a h_{aa}+\sum_s h_{ss}=0$. Q.E.D.

§ 3. In the case $c\leqq 0$.

THEOREM 3.1. (1) *A minimal submanifold of $M^{n+p}(c)$ with $c<0$ is irreducible.* (2) *A minimal immersion of a Riemannian product into $M^{n+p}(0)$ is a product of minimal immersions.*

PROOF. (1) Lemma 2.3 implies that a submanifold minimally immersed into $M^{n+p}(c)$ with $c<0$ is irreducible. (2) We consider a Riemannian product of M_1 and M_2 minimally immersed into $M^{n+p}(0)$. From Lemma 2.3, we have $h^\alpha_{as}=0$ which, together with Lemma 2.1, implies that the immersion is a product immersion. Moreover, by $\sum h^\alpha_{aa}=\sum h^\alpha_{ss}=0$, we see that each immersion is minimal. Q.E.D.

REMARK. (1) More generally, a minimal submanifold immersed into a Riemannian manifold of negative curvature is irreducible. (2) Theorem 3.1 (2) holds for a Riemannian product of any number of Riemannian manifolds. (3) A Kähler immersion of a Riemannian product of Kähler manifolds into complex Euclidean space is a product of Kähler immersions.

§ 4. In the case $c>0$.

We may assume without loss of generality that $c=1$ and $M^{n+p}(1)=S^{n+p}(1)$. When an immersion of $M=M_1\times M_2$ into $S^{n+p}(1)$ is a product immersion, we have some positive numbers r_1, r_2 such that $(r_1)^2+(r_2)^2=1$, positive integers k_1, k_2 such that $k_1+k_2+1=n+p$, and isometric immersions f_1, f_2 such that f_i $(i=1, 2)$ is an immersion of M_i into $S^{k_i}(r_i)$ and $I\circ(f_1\times f_2)$ is the immersion of M into $S^{n+p}(1)$, where $I(x, y)=(x, y)$, where x(resp. y) is the position vector of $S^{k_1}(r_1)$(resp. $S^{k_2}(r_2)$) in R^{k_1+1}(resp. R^{k_2+1}) and (x, y) is the position vector of $S^{n+p}(1)$ in R^{n+p+1}. Immediately we have the following.

THEOREM 4.1. *When an immersion of $M=M_1\times M_2$ into $S^{n+p}(1)$ is a* *product immersion, it is minimal if and only if* $r_1=\sqrt{m/n}$, $r_2=\sqrt{(n-m)/n}$ *and* f_1, f_2 *are minimal immersions.*

In [2] and [6], we have some examples of Riemannian products minimally immersed into a unit sphere which are not product immersions. Some examples are:

EXAMPLE 1. Let $S^m(1)=\{(x_1,\cdots,x_{m+1})\in R^{m+1}; (x_1)^2+\cdots+(x_{m+1})^2=1\}$ and $S^{n-m}(1)=\{(y_1,\cdots,y_{n-m+1})\in R^{n-m+1}; (y_1)^2+\cdots+(y_{n-m+1})^2=1\}$. We define the immersion f of $S^m(1)\times S^{n-m}(1)$ into $S^{n+m(n-m)}(1)$ by $f((x_1,\cdots,x_{m+1}),(y_1,\cdots,y_{n-m+1}))=(\cdots,x_a y_s,\cdots)$, where $1\leqq a\leqq m$, $1\leqq s\leqq n-m$.

EXAMPLE 2. Let M_1(resp. M_2) be an m(resp. $n-m$)-dimensional Riemannian manifold minimally immersed into $S^{k_1}(1)$(resp. $S^{k_2}(1)$). Then, by Example 1, the Riemannian product $M_1\times M_2$ can be immersed minimally into $S^{k_1+k_2+k_1k_2}(1)$.

We characterize $S^m(1)\times S^{n-m}(1)$ in Example 1. We set $K=\sum h^\alpha_{as}h^\alpha_{as}$, which measures the deviation of the immersion from being product.

THEOREM 4.2. *Let* M_1*(resp.* M_2*) be an* m*(resp.* $n-m$*)-dimensional compact orientable Riemannian manifold and* M *be a Riemannian product of* M_1 *and* M_2 *minimally immersed into* $S^{n+p}(1)$. *Then we have an integral inequality,*

$$\int_M\left(n-\lambda-\mu-\frac{2}{p}K\right)K*1\geqq 0$$

where $*1$ *is the volume element of* M *and* λ*(resp.* μ*) is the minimum of the Ricci curvature of* M_1*(resp.* M_2*).*

COROLLARY 1. *Under the same assumption as Theorem 4.2, if the immersion is full and* $(n-\lambda-\mu-(2/p)K)K\leqq 0$, *then* K *is constant and equal to* 0 *or* $(p/2)(n-\lambda-\mu)$. *When* $K=(p/2)(n-\lambda-\mu)\neq 0$, M *is* $S^m(1)\times S^{n-m}(1)$ *in Example* 1 *and the immersion is rigid.*

COROLLARY 2. *Under the same assumption as Theorem 4.2, if* $\lambda+\mu\geqq n$, *then the immersion is a product immersion.*

PROOF OF THEOREM 4.2. K is a function on M. Let Δ be the Laplacian of M. Then we obtain $(1/2)\Delta K=\sum h^\alpha_{asi}h^\alpha_{asi}+\sum(\Delta h^\alpha_{as})h^\alpha_{as}$. From Lemma 1.1 we have

$$\sum(\Delta h^\alpha_{as})h^\alpha_{as}=\sum h^\alpha_{as}\left(\sum h^\alpha_{ai}R^i_{jsj}+\sum h^\alpha_{ji}R^j_{asi}+\sum h^\beta_{ai}R^\beta_{\alpha si}\right).$$

Since M is a Riemannian product, $R^j_{asi}=0$ and hence we have

$$\sum (\Delta h^\alpha_{as})h^\alpha_{as}=\sum h^\alpha_{as}h^\alpha_{at}R^i_{jsj}+\sum h^\alpha_{as}h^\beta_{at}R^\beta_{\alpha si}\ .$$

Since the Ricci curvature of $M_2 \geqq \mu$, we obtain

$$\sum h^\alpha_{as}h^\alpha_{at}R^t_{jsj}=\sum h^\alpha_{as}h^\alpha_{at}R^t_{usu}\geqq \mu\sum h^\alpha_{as}h^\alpha_{as}=\mu K\ ,$$

and hence

$$\begin{aligned}\sum (\Delta h^\alpha_{as})h^\alpha_{as}&\geqq \mu K+\sum h^\alpha_{as}h^\beta_{at}(\sum h^\beta_{sj}h^\alpha_{tj}-\sum h^\beta_{tj}h^\alpha_{sj})\\ &=\mu K+\sum h^\alpha_{as}h^\alpha_{tj}h^\beta_{at}h^\beta_{sj}-\sum h^\alpha_{as}h^\alpha_{sj}h^\beta_{at}h^\beta_{tj}\ .\end{aligned}$$

From (1.9), we have

$$\begin{aligned}\sum (\Delta h^\alpha_{as})h^\alpha_{as}&\geqq \mu K+\sum h^\alpha_{as}h^\alpha_{ij}(-R^\alpha_{jsi}+\delta^\alpha_s\delta_{ji}-\delta^\alpha_i\delta_{js}+\sum h^\beta_{as}h^\beta_{ji})\\ &\quad -\sum h^\alpha_{as}h^\alpha_{sj}(-R^\alpha_{iji}+\delta^\alpha_j\delta_{ii}-\delta^\alpha_j\delta_{ij}+\sum h^\beta_{aj}h^\beta_{ii})\\ &=\mu K+\sum h^\alpha_{as}h^\alpha_{ij}h^\beta_{as}h^\beta_{ji}+\sum h^\alpha_{as}h^\alpha_{sj}R^\alpha_{iji}-nK\\ &\geqq -nK+\lambda K+\mu K+\sum h^\alpha_{as}h^\alpha_{ij}h^\beta_{as}h^\beta_{ji}\\ &=(\lambda+\mu-n)K+\sum h^\alpha_{as}h^\alpha_{bc}h^\beta_{as}h^\beta_{bc}+\sum 2h^\alpha_{as}h^\alpha_{bt}h^\beta_{as}h^\beta_{bt}\\ &\quad +\sum h^\alpha_{as}h^\alpha_{tu}h^\beta_{as}h^\beta_{tu}\geqq (\lambda+\mu-n)K+2\sum h^\alpha_{as}h^\alpha_{bt}h^\beta_{as}h^\beta_{bt}\ .\end{aligned}$$

We set $S_{\alpha\beta}=h^\alpha_{as}h^\beta_{as}$. Then $(S_{\alpha\beta})$ is a symmetric matrix. Therefore we have $\sum h^\alpha_{as}h^\alpha_{bt}h^\beta_{as}h^\beta_{bt}=\sum S_{\alpha\beta}S_{\alpha\beta}\geqq (1/p)(\sum S_{\alpha\alpha})^2=(1/p)(\sum h^\alpha_{as}h^\alpha_{as})^2=(1/p)K^2$. The equality holds if and only if $(S_{\alpha\beta})$ is proportional to the identity matrix. Since M is compact and orientable, we obtain the integral inequality

$$\int_M\Big(n-\lambda-\mu-\frac{2}{p}K\Big)K*1\geqq 0\ .$$

PROOF OF COROLLARIES 1 and 2. If $(n-\lambda-\mu-(2/p)K)K\leqq 0$, we obtain $(n-\lambda-\mu-(2/p)K)K=0$ and K is constant. Furthermore we have

(1) $h^\alpha_{asi}=0$ for all a, s, i, and α,

(2) $\sum h^\alpha_{as}h^\beta_{as}=0$ for all $\alpha\neq\beta$,

(3) $\sum_{a,s}h^\alpha_{as}h^\alpha_{as}=\sum_{a,s}h^\beta_{as}h^\beta_{as}$ for all α and β,

(4) $\sum h^\alpha_{as}h^\alpha_{bc}=0$ for all a, b, c, and s,

(5) $\sum h^\alpha_{as}h^\alpha_{tu}=0$ for all a, s, t, and u.

If we have b_0 and c_0 such that $\sigma(e_{b_0}, e_{c_0})\neq 0$, we set $e_{n+1}=(\sigma(e_{b_0}, e_{c_0}))/||\sigma(e_{b_0}, e_{c_0})||$. we choose $e_{n+2}, \cdots, e_{n+p}$ so that $e_{n+1}, \cdots, e_{n+p}$ form a local field of normal frames. Then from (4) we have $h^{n+1}_{as}=0$ and hence from (3) we have $h^\alpha_{as}=0$ for all α. So by Lemma 2.2 the immersion is a product immersion. Hence we assume the immersion is not a product immersion, i.e., $K\neq 0$. Then from the above discussion we have $\sigma(e_b, e_c)=0$, i.e., $h^\alpha_{bc}=0$ for all b, c, and α. Similarly we have $h^\alpha_{st}=0$. This implies that M is $S^m(1)\times S^{n-m}(1)$. We shall prove that the immersion is rigid. By the definition,

$$\sum h^{\alpha}_{abi}\omega^{i}=dh^{\alpha}_{ab}-\sum h^{\alpha}_{ai}\omega^{i}_{b}-\sum h^{\alpha}_{bi}\omega^{i}_{a}+\sum h^{\beta}_{ab}\omega^{\alpha}_{\beta} .$$

From $h^{\alpha}_{ab}=0$ and (1.7), we obtain $h^{\alpha}_{abi}=0$. Similarly $h^{\alpha}_{sti}=0$. These, together with (1), imply that the second fundamental form σ is parallel. From the assumption that the immersion is full and a result in [3], the normal space is spanned by the first normal space. Equation (1.9) shows that $\sum h^{\alpha}_{at}h^{\alpha}_{sb}=\sum h^{\alpha}_{ab}h^{\alpha}_{st}+\delta^{a}_{b}\delta_{st}$, which, together with $h^{\alpha}_{ab}=0$, implies $\sum h^{\alpha}_{at}h^{\alpha}_{sb}=\delta^{a}_{b}\delta_{st}$. Thus if we set $e_{(a,s)}=\sigma(e_a, e_s)$, then $e_{(a,s)}$, $1\leqq a\leqq m<s\leqq n$, form a local field of normal frames, with respect to which $h^{(a,s)}_{bt}=\delta^{a}_{b}\delta^{s}_{t}$ holds. From (1) we have

$$0=\sum h^{(a,s)}_{bti}\omega^{i}=dh^{(a,s)}_{bt}-\sum h^{(a,s)}_{bi}\omega^{i}_{t}-\sum h^{(a,s)}_{ti}\omega^{i}_{b}+\sum h^{\beta}_{bt}\omega^{(a,s)}_{\beta} ,$$

which implies

$$\omega^{(a\ s)}_{(b\ t)}=\delta^{a}_{b}\omega^{s}_{t}+\delta^{s}_{t}\omega^{a}_{b} .$$

So we have,

$$\omega^{(a,s)}_{(b,t)}=0,\ \omega^{(a\ s)}_{(a,t)}=\omega^{s}_{t} \text{ and } \omega^{(a,s)}_{(b,s)}=\omega^{a}_{b} ,$$

for all $a\neq b$ and $s\neq t$. Therefore the immersion is rigid. Q.E.D.

For a result of [7], we have the following.

THEOREM 4.3. *Let M_1(resp. M_2) be an m(resp. $n-m$)-dimensional compact and orientable Riemannian manifold and M be a Riemannian product of M_1 and M_2 minimally immersed into $S^{n+p}(1)$. If the normal connection is trivial and either M_1 or M_2 has positive Ricci curvature, then the immersion is a product immersion.*

PROOF. We may assume that the Ricci curvature of M_2 is positive. Similar to the proof of Theorem 4.2, we have $\sum(\Delta h^{\alpha}_{as})h^{\alpha}_{as}\geqq\mu K+\sum h^{\alpha}_{as}h^{\beta}_{ai}R^{\beta}_{\alpha si}$. Since the normal connection is trivial, we obtain $R^{\beta}_{\alpha si}=0$. So we have $\sum(\Delta h^{\alpha}_{as})h^{\alpha}_{as}\geqq\mu K$. Since M is compact and orientable, we obtain $K=0$. From Lemma 2.2, the immersion is a product immersion. Q.E.D.

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