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Minimal Models in Proper Birational Geometry

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Introduction

In classical algebraic geometry, the following theorem due to Castelnuovo-Enriques-Zariski is fundamental, [9].

THEOREM A. *A non-singular projective surface S is minimal if S is relatively minimal and if S is not a ruled surface.*

In view of Enriques' criterion on ruled surfaces, the condition that S is not ruled may be replaced by $\kappa(S)\geqq 0$. Here, $\kappa(S)$ denotes the Kodaira dimension of S. Thus, we obtain

THEOREM B. *A non-singular projective surface S is minimal if S is relatively minimal surface with $\kappa(S)\geqq 0$.*

In this paper we shall consider analogues of the above facts in *proper birational geometry*. The category in which we shall work is that of schemes over the field of complex numbers $\boldsymbol{C}$.

In place of birational morphism and birational map in the classical theory, we shall use *proper birational morphism* and *strictly birational map* or *proper birational map*, respectively (see [2]). Thus for open surfaces, we shall define the concepts of *relatively minimal surface* and *minimal surface*. Using the notion of logarithmic Kodaira dimension we shall establish a theorem analogous to Theorem B (Theorem 1). Moreover, the notion of *∂-manifold* $(\bar{V}, D)$ will be introduced which consists of a non-singular complete algebraic variety $\bar{V}$ and a divisor with normal crossings D on $\bar{V}$. We shall study algebraic geometry for ∂-manifolds. The notions of relatively ∂-minimal model and properly ∂-minimal or ∂-minimal model will be introduced. For a ∂-surface $(\bar{S}, D)$ with $\bar{\kappa}(\bar{S}-D)=2$, an analogue of Theorem B will be established (Theorem 2).

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Finally, we shall discuss how to determine minimal completions of a given surface S with $\bar{\kappa}(S)\geqq 0$ and we shall give a precise definition of the logarithmic Chern number $\bar{c}_1^2(S)$ of a surface S.

The author would like to express his heartfelt thanks to Dr. T. Fujita. Without his advice, Lemma 1 would not be formulated.

§ 1. For simplicity, we use the following conventions.

Manifold means a non-singular algebraic variety and *surface* means a manifold of dimension 2. But curve is understood to be an algebraic variety of dimension 1. Namely, a curve may have singularities.

First, we introduce the concept of relatively *minimal manifold* which may not be complete. Fix a manifold V. V is called a relatively *minimal manifold* if and only if any proper birational morphism φ: $V\rightarrow V_1$, V_1 being a manifold, turns out to be isomorphic. Note that this definition of minimality coincides with that of Zariski's minimality when V is complete (see [9]). We prove the existence of relatively minimal model in our sense, that is, in proper birational geometry.

PROPOSITION 1. *For a given manifold V, there exist a relatively minimal manifold V_* and a proper birational morphism ψ: $V\rightarrow V_*$.*

In order to prove this, we aim to define a subspace $B(V)$ of $H^2(\bar{V})$ which stands for $H^2(\bar{V}, \boldsymbol{Q})$ as follows: Let $\bar{V}$ be a smooth completion of V with boundary D. $\sum \Gamma_j=D$ is a sum of irreducible components Γ_j of D. The number of Γ_j's is indicated by $r(D)$. We have an exact sequence of homology groups:

$$\begin{aligned}&\longrightarrow H_{2n-1}(D)\longrightarrow H_{2n-1}(\bar{V})\longrightarrow H_{2n-1}(\bar{V}, D)\\&\longrightarrow H_{2n-2}(D)\longrightarrow H_{2n-2}(\bar{V})\longrightarrow H_{2n-2}(\bar{V}, D)\end{aligned}$$

where $n=\dim V$. Then by means of Poincaré duality and Lefschetz duality, it yields the following exact sequence:

$$\begin{aligned}0&\longrightarrow H^1(\bar{V})\longrightarrow H^1(V)\longrightarrow H_{2n-2}(D)=\bigoplus \boldsymbol{Q}\Gamma_j\\&\longrightarrow H^2(\bar{V})\longrightarrow H^2(V)\longrightarrow H_{2n-3}(D)\,.\end{aligned}$$

We denote the image of $H^2(\bar{V})\rightarrow H^2(V)$ by $B(\bar{V}, D)$. Then we have the following exact sequences:

$$(*)\quad \begin{aligned}&0\longrightarrow H^1(\bar{V})\longrightarrow H^1(V)\longrightarrow H_{2n-2}(D)\longrightarrow H^2(\bar{V})\longrightarrow B(\bar{V}, D)\longrightarrow 0\,,\\&0\longrightarrow B(\bar{V}, D)\longrightarrow H^2(V)\longrightarrow H_{2n-3}(D)\,.\end{aligned}$$

It has been shown that $B(\bar{V}, D)$ depends only on V (for example, see

[1] p. 3). Here we shall give an elementary proof. Let $\bar{V}^1$ be another completion of V with smooth boundary D^1 such that the identity $V\to V$ defines a birational morphism $f\colon \bar{V}^1\to\bar{V}$. Then, letting $\sum \Delta_j$ be the irreducible decomposition of D^1, we have the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \oplus \boldsymbol{Q}\Delta_j & \longrightarrow & H^2(\bar{V}^1) & \longrightarrow & B(\bar{V}^1, D^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f^2_*} & & \downarrow{\scriptstyle \bar{f}^2_*} & & \\
\cdots & \longrightarrow & \oplus \boldsymbol{Q}\Gamma_i & \longrightarrow & H^2(\bar{V}) & \longrightarrow & B(\bar{V}, D) & \longrightarrow & 0
\end{array}$$

in which the horizontal sequences are exact. On the other hand, f^2_* is surjective, since f^2_* is the dual of $f^{*2}\colon H^2(\bar{V})\to H^2(\bar{V}^1)$ which is injective (see [8]). Hence, $\bar{f}^2_*\colon B(\bar{V}^1, D^1)\to B(\bar{V}, D)$ is also surjective. Moreover, from the exact commutative diagram:

$$\begin{array}{ccccc}
0 & \longrightarrow & B(\bar{V}^1, D^1) & \longrightarrow & H^2(V) \\
 & & \downarrow{\scriptstyle \bar{f}^2_*} & & \downarrow\wr \\
0 & \longrightarrow & B(\bar{V}, D) & \longrightarrow & H^2(V)
\end{array}$$

we infer that $\bar{f}^2_*$ is also injective. Hence, $B(\bar{V}^1, D^1)\simeq B(\bar{V}, D)$.

By $B(V)$ we denote $B(\bar{V}, D)$ and we write $\beta(V)=\dim B(V)$. From the exact sequence (*), and the formula [3, p. 529], we derive the following

Formula. $\beta(V)=b_2(\bar{V})-r(D)+\bar{q}(V)-q(\bar{V})$.

Now let $f\colon \bar{V}\to\bar{V}_1$ be a birational morphism between complete manifolds $\bar{V}$ and $\bar{V}_1$. Putting $F=f(\operatorname{Supp} R_f)$ where R_f is the ramification divisor of f, we get $\operatorname{codim} F\geqq 2$, and $\operatorname{Supp} R_f=f^{-1}(F)$ by Zariski's Main Theorem. Note that $f^{-1}(F)$ is the support of the ramification divisor R_f. By $\operatorname{codim} F\geqq 2$, we have $H^2(\bar{V}_1-F)=B(\bar{V}_1-F)=H^2(\bar{V}_1)$. Since $V_0=\bar{V}-f^{-1}(F)\simeq V_1-F$, we infer that

$$b_2(\bar{V}_1)=\beta(\bar{V}_1-F)=\beta(V_0)=b_2(\bar{V})-r(f^{-1}(F))+\bar{q}(V_0)-q(\bar{V}) .$$

On the other hand, $\bar{q}(V_0)=\bar{q}(\bar{V}_1-F)=\bar{q}(\bar{V}_1)=q(\bar{V}_0)$, since $\operatorname{codim} F\geqq 2$. Thus we obtain

$$(**)\qquad b_2(\bar{V})=b_2(\bar{V}_1)+r(f^{-1}(F))=b_2(\bar{V}_1)+r(R_f) .$$

Accordingly, we conclude that $b_2(\bar{V})\geqq b_2(\bar{V}_1)$ and that $b_2(\bar{V})=b_2(\bar{V}_1)$ if and only if f is isomorphic.

LEMMA 1. *Let $f\colon V\to V_1$ be a proper birational morphism, V and*

V_1 being manifolds. Then $\beta(V)\geqq\beta(V_1)$. Moreover, if $\beta(V)=\beta(V_1)$, then f is isomorphic.

PROOF. We choose suitable completions $\bar{V}$ and $\bar{V}_1$ of V and V_1 with smooth boundaries D and D_1, respectively, such that the rational map g: $\bar{V}\to\bar{V}_1$ defined by f is a morphism. Then by the formula, we have

$$\beta(V)-\beta(V_1)=b_2(\bar{V})-b_2(\bar{V}_1)-r(D)+r(D_1)\ .$$

Let $Z=(R_g)_{\mathrm{red}}$ and let $\sum Z_j$ be the irreducible decomposition of Z. Put $X=\{\sum Z_j;\ Z_j\subset D\}$ and $Y=\{\sum Z_i;\ Z_i\not\subset D\}$. Then $Z=X+Y$. Hence by the above formula (**), we have

$$b_2(\bar{V})-b_2(\bar{V}_1)=r(Z)=r(X)+r(Y)\ .$$

Furthermore, $r(X)=r(D)-r(D_1)$ and Y coincides with the closure of $\mathrm{Supp}\,R_f$. Thus we obtain

$$\beta(V)-\beta(V_1)=r(Y)=r(\text{the closure of } \mathrm{Supp}\,R_f)\ .$$

This completes the proof of Lemma 1.

Needless to say, Proposition 1 is derived easily from Lemma 1.

In general, for a given variety V, there exists a relatively minimal manifold V_* which is properly birationally equivalent to V. Such a V_* is called *a relatively minimal model* of V.

Next we shall give a definition of minimal model. Let V be a manifold. V is called *minimal* or *properly minimal manifold* if and only if any strictly birational or any proper birational map (see [2, 3] φ: $V^1\to V$, V^1 being a manifold, turns out to be a morphism, respectively.

It is clear that a minimal manifold is properly minimal and that a properly minimal manifold is relatively minimal. For a given variety V, a properly minimal or minimal manifold that is proper birationally equivalent to V is called a *properly minimal model* or *manimal model* of V, respectively. A properly minimal model is unique, if it exists. When a relatively minimal manifold has only one relatively minimal model, it is a properly minimal model. For a given properly minimal manifold V^* that is proper birationally equivalent to an algebraic variety V, we have

$$\mathrm{PBir}(V)=\mathrm{PBir}(V^*)=\mathrm{Aut}(V^*)\ .$$

§ **2.** The following theorem is a counterpart of Theorem B in proper birational geometry.

THEOREM 1. *Let S be a surface with $\bar{\kappa}(S)\geqq 0$. Then S is relatively minimal if and only if S is minimal.*

PROOF. Suppose that S is a relatively minimal but not minimal surface with $\bar{\kappa}(S)\geqq 0$. Then there exists a strictly birational map $\varphi: S^1 \to S$ such that S^1 is a surface and that φ is not defined at p of S^1. $\varphi(p)$ is a (reducible) curve by Z.M.T. By the definition of strictly rational map, we have a proper birational morphism $\mu: S_2 \to S^1$, S_2 being a surface, and a birational morphism $g: S_2 \to S$ such that $g=\varphi\circ\mu$. Since μ is proper, $\mu^{-1}(p)$ is a complete curve, which is an exceptional curve of the first kind. Denote by E_i the irreducible components of $\mu^{-1}(p)$, hence $\mu^{-1}(p)=\sum E_i$. We may assume that E_1 is an irreducible exceptional curve of the first kind. If $g(E_1)$ is a point, we contract E_1 to a non-singular point and thus we obtain a surface S_3 and a proper birational morphism $\lambda: S_2 \to S_3$, i.e., S_2 is a blowing up of S_3. Then $g'=g\circ\lambda^{-1}$ and $\mu'=\mu\circ\lambda^{-1}$ are both morphisms, since $g'(p)$ and $\mu'(p)$ are points (see Figure 1). Hence, we can replace S_2 by S_3. After a finite succession of

$$\begin{tikzcd}
S^1 \arrow[rr, "\varphi"] & & S \\
& S_2 \arrow[ul, "\mu"'] \arrow[ur, "g"] \arrow[d, "\lambda"] & \\
& S_3 \arrow[uul, bend left, "\mu'"] \arrow[uur, bend right, "g'"'] &
\end{tikzcd}$$

FIGURE 1

such replacements, we have an irreducible exceptional curve of the first kind E_1 such that $C_1=g(F_1)$ is a curve. Since E_1 is complete, so is C_1. Hence C_1 is a closed curve in $\bar{S}$ which is a completion of S with smooth boundary D, hence $C_1\cap D=\varnothing$. By $K=K(\bar{S})$ we denote a canonical divisor on $\bar{S}$. From $E_1^2=-1$, follows $C_1^2\geqq -1$. Precisely speaking, let $\nu_i(i=1, \cdots, s)$ denote the multiplicities of C_1 at (infinitely near) singular points of C_1. Then

$$(*)\qquad \begin{aligned} C_1^2 &= -1+\sum \nu_j^2+t\ , \\ (K, C_1) &= -1-\sum \nu_j - t\ , \\ 2\pi(C_1)-2 &= \sum \nu_j(\nu_j-1)\ . \end{aligned}$$

These follow from the fact that g is composed of blowing ups (see [9]). t equals the number of blowing ups whose centers are non-singular (infinitely near) points of C_1.

Thus we have two cases.

Case 1: $C_1^2=-1$ and $(K, C_1)=-1$. Then C_1 is an irreducible exceptional curve of the first kind, which is contractible. This contradicts the relative minimality.

Case 2: $C_1^2 \geqq 0$. Then $(K, C_1) \leqq -2$. Since $C_1 \cap D = \emptyset$, we have $(K+D, C_1)=(K, C_1) \leqq -2$. We use the following lemma.

LEMMA 2. *Let C be an irreducible curve on a complete surface $\bar{S}$ and D a divisor. Assume that $C^2 \geqq 0$.*

(i) *If $\kappa(D, \bar{S}) \geqq 0$, then $(D, C) \geqq 0$.*

(ii) *If $C^2>0$ and $\kappa(D, \bar{S})>0$, then $(D, C)>0$.*

Here, $\kappa(D, \bar{S})$ means the D-dimension of $\bar{S}$, (see [8]).

PROOF. After replacing D by some multiple of D, we may assume that D is effective. We write $D=\sum r_i C_i$ where the C_i are irreducible components of D. By assumption, $(C_i, C) \geqq 0$ for any i. As for (ii), we may assume that $\dim |D| \geqq 1$. Let p be a point of C. Then $|D|_p = \{D_1 \in |D|; p \in \operatorname{Supp} D_1\} \neq \emptyset$. Hence, a member Δ of $|D|_p$ is written as $sC+\sum r_i C_i$, where we use the following convention: If C is not a component of Δ, we put $s=0$ and choose C_1 such that $C \cap C_1 \neq \emptyset$. And if C is a component of Δ, the C_i are different from C. Thus

$$(D, C)=sC^2+\sum r_i(C_i, C)>0 . \qquad \text{Q.E.D.}$$

Now we proceed with the proof of Theorem 1.

By Lemma 2 (i), we get $\bar{\kappa}(S)=-\infty$.

This contradicts the hypothesis and we complete the proof.

REMARK. If $\bar{q}(S)>0$, we have the quasi-Albanese map $\alpha_S: S \rightarrow \mathscr{A}_S$, [3]. Let S be a surface that has no minimal model. Then by Theorem 1, it follows that $\bar{\kappa}(S)=-\infty$. Hence, $\alpha_S(S)$ is a curve Δ. Thus the curve C_1 constructed in the proof of Theorem 1 is contained in a fiber of α_S: $S \rightarrow \Delta$. Hence, $C_1^2 \leqq 0$, and so $C_1^2=0$ by Case 2.

A manifold V is called *strongly minimal* if and only if any strictly rational map φ: $W \rightarrow V$, W being a manifold, turns out to be a morphism. For example, a manifold that does not contain any complete rational curves is strongly minimal. In particular, *an affine manifold is strongly minimal.* Hence, an affine plane is a strongly minimal surface, whose logarithmic Kodaira dimension is $-\infty$.

§3. Let $\bar{V}$ be a complete manifold and D a divisor with normal crossings on $\bar{V}$. We say that $\bar{V}$ is a *completion of $V=\bar{V}-D$ with* *ordinary boundary* D. In the previous papers [2], [3], we assumed that each component of D is non-singular and called $\bar{V}$ a completion of V with smooth boundary D. In this paper, we employ the following terminology: *∂-manifold* means a couple $(\bar{V}, D)$ consisting of a complete manifold $\bar{V}$ and a divisor D with normal crossings on $\bar{V}$. Now we shall introduce the category of ∂-manifolds. A morphism $f: (\bar{V}, D) \to (\bar{V}_1, D_1)$ is understood as a morphism $f: \bar{V} \to \bar{V}_1$ satisfying that $f^{-1}(D_1) \subset D$. In other words, putting $V = \bar{V} - D$ and $V_1 = \bar{V}_1 - D_1$, $f \mid V$ is a morphism of V into V_1. Moreover, a rational map $\varphi: (\bar{V}, D) \to (\bar{V}_1, D_1)$ is understood as a rational map $\varphi: \bar{V} \to \bar{V}_1$ such that $\varphi | V$ is a strictly rational map from V to V_1. $f: (\bar{V}, D) \to (\bar{V}_1, D_1)$ is a proper morphism or map if $f | V$: $V \to V_1$ is proper.

In order to avoid confusion, morphism, rational map, $\cdots$ in this category are written as *∂-morphism, rational ∂-map*, $\cdots$.

Next we introduce the notion of minimality in the category of ∂-manifolds. We define $(\bar{V}, D)$ to be relatively ∂-minimal if any proper birational ∂-morphism $(\bar{V}, D) \to (\bar{V}_1, D_1)$ turns out to be isomorphic. Given $(\bar{V}, D)$, by Lemma 1, we have a relatively ∂-minimal $(\bar{V}_*, D_*)$ such that there exists a proper birational ∂-morphism $(\bar{V}, D) \to (\bar{V}_*, D_*)$. Such a $(\bar{V}_*, D_*)$ is called a *relatively ∂-minimal model* of $(\bar{V}, D)$.

Suppose that $(\bar{S}, D)$ is a relatively ∂-minimal surface. Then each component C of D does not satisfy the condition that $C^2 = -1$, $\pi(C) = 0$ and $(C, D') = 1$ or 2, where $C + D' = D$. If a divisor D with normal crossings has the same property as above, D is called a *minimal boundary* of $S = \bar{S} - D$. It is easy to verify that *a ∂-surface $(\bar{S}, D)$ is relatively ∂-minimal if and only if $S = \bar{S} - D$ is relatively minimal and D is a minimal boundary.*

In what follows, we use the following symbol $[C: D] = (C, D')$, when C is a component of a boundary D and $C + D' = D$.

PROPOSITION 2. *Let $(\bar{S}, D)$ be a relatively ∂-minimal surface and assume that $\kappa(\bar{S}) \geqq 0$ or $\bar{\kappa}(S) = 2$ where $S = \bar{S} - D$, as usual. Then any proper birational ∂-map $\varphi: (\bar{S}^1, D^1) \to (\bar{S}, D)$ turns out to be a morphism.*

PROOF. Suppose that φ is not defined at $p \in D^1$. As in the proof of Theorem 1, we have a ∂-surface $(\bar{S}_2, D_2)$ and proper birational ∂-morphisms $\mu: (\bar{S}_2, D_2) \to (\bar{S}^1, D^1)$ and $g: (\bar{S}_2, D_2) \to (\bar{S}, D)$ such that $g = \varphi \circ \mu$. We may assume that there exists an irreducible exceptional curve of the first kind E_1 on $\bar{S}_2$ such that $C_1 = g(E_1)$ is also a curve. Since S is minimal by Theorem 1 and since g is proper, C_1 is a component of D. $F := g^*(C_1) - E_1$ (subtraction as divisor) is effective and g-exceptional. Put

$C_1+D'=D$. Then,

$$[C_1: D]=(C_1, D')=(g^*(C_1), g^*(D'))=(E_1, g^*(D')) .$$

Noting that $(g^*(D'), E_1)=(g^{-1}(D'), E_1)$, we define B' and F' by $B'+E_1=D_2$ and $g^{-1}(D')+F'=B'$ where B' and F' are effective divisors. Thus,

$$(E_1, g^*(D'))\leqq(E_1, g^*(D'))+(E_1, F')=(E_1, B')=[E_1: D_2] .$$

Contracting E_1 to a non-singular point x, we have a ∂-surface $(\bar{S}_3, D_3)$ such that $\bar{S}_2$ is a blowing up of $\bar{S}_3$ at x, which defines a proper birational ∂-morphism λ: $(\bar{S}_2, D_2)\to(\bar{S}_3, D_2)$. Define $\rho=\mu\circ\lambda^{-1}$, which is a ∂-morphism.

$$\begin{tikzcd}
\bar{S}^1 \arrow[rr, "\varphi"] & & \bar{S}\supset C_1 \\
& \bar{S}_2\supset E_1 \arrow[ul, "\mu"'] \arrow[ur, "g"'] \arrow[dl, "\lambda"] & \\
\bar{S}_3 \arrow[uu, "\rho"] & &
\end{tikzcd}$$

FIGURE 2

On the other hand, D^1 has only normal crossings and hence, $D_3=\rho^{-1}(D_1)$ has only normal crossings, too. Therefore, $[E_1: D_2]=(E_1, B')=$ the multiplicity of D_3, which is smaller than 3. Accordingly, we obtain

$$[C_1: D]=(C_1, D')\leqq(E_1, B')\leqq 2 .$$

Case 1: C_1 is a non-singular curve. Then

$$(K+D, C_1)=2\pi(C_1)-2+(D', C_1)\leqq-2+2=0 .$$

Recalling that D is a minimal boundary, we see that $C_1^2\geqq0$ and $[C_1: D]\leqq2$. Hence, $(K, C_1)=2\pi(C_1)-2-C_1^2\leqq-2$. This implies that $\kappa(\bar{S})=\kappa(K, \bar{S})=-\infty$ by Lemma 2 (i). Therefore, by the classification theory due to Enriques, $\bar{S}$ is an irrational ruled surface or a rational surface. In the former case, we consider the Albanese fibered surface $\alpha=\alpha_S: \bar{S}\to Y$, where $Y=\alpha(\bar{S})$, $\pi(Y)$ equals $q(\bar{S})$. Since $C_1=\boldsymbol{P}^1$, $\alpha(C_1)$ is a point $a\in Y$. For a general point $y\in Y$, define $C_y=\alpha^{-1}(y)$, which satisfies that $\pi(C_y)=0$ and $(C_y, D')=(C_1, D')\leqq2$. Thus we see that

$$\bar{\kappa}(C_y-C_y\cap D')\leqq0 .$$

Hence, by Theorem 4 ([2] p. 184), $\bar{\kappa}(S)\leqq1$. This contradicts the hypothesis.

Next we assume that $\bar{S}$ is rational. Then by Riemann-Roch Theorem, we have

$$\dim |C_1| \geqq C_1(C_1 - K)/2 \geqq 1 .$$

A general member C_u of $|C_1|$ satisfies that $\pi(C_u)=0$ and $(C_u, D')=(C_1, D')\leqq 2$. Hence, from the same argument as in the former case, it follows that $\bar{\kappa}(S)\leqq 1$. This is a contradiction.

Case 2: C_1 is a singular curve. Then $g|E_1: E_1 \to C_1$ is a proper birational morphism, which is a reduction of singularities of C_1. Let p_1 be a singular point of C_1. p_1 is an ordinary double point, because D is a divisor with normal crossings. Hence, we have two points x and x' on $\bar{S}_2$ such that $g(x)=g(x')=p_1$. Put $E_2=g^{-1}(p_1)\subset \bar{S}_2$. Then $E_2 \subset B'$ and $(E_2, E_1)=2$. If C_1 had another singular point p_2, then B' would have another component $E_3=g^{-1}(p_2)$ satisfying that $(E_3, E_1)=2$. Hence,

$$2 \geqq (B', E_1) \geqq (E_2+E_3, E_1)=4 .$$

This would be a contradiction. Define B'' by $B'=B''+E_2$. Then by the same reasoning as before, we have $(B'', E_1)=0$. Hence, we conclude that C_1 is a connected component of D with only one double point. This implies that $\pi(C_1)=1$ and so

$$(***) \qquad (K+D, C_1)=2\pi(C_1)-2=0 .$$

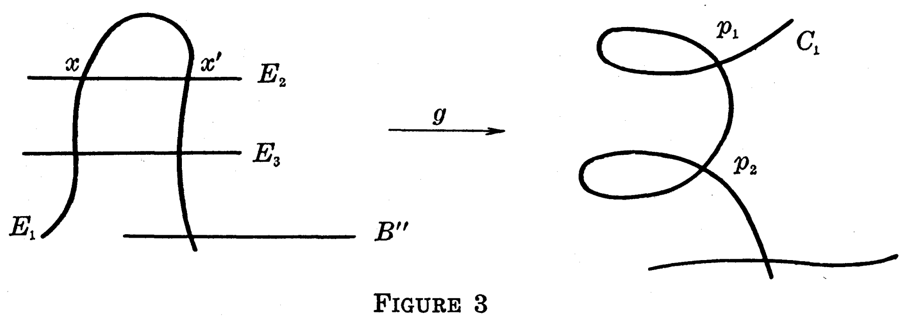


FIGURE 3

Since $C_1^2 \geqq -1+2^2=3$, we have $(K, C_1)\leqq -3$. Hence, $\kappa(\bar{S})=-\infty$. Thus in view of the hypothesis, we have $\bar{\kappa}(S)=2$. By Lemma 2 (ii), we obtain $(K+D, C_1)>0$. This contradicts $(***)$.

A ∂-manifold $(\bar{V}, D)$ is called *properly ∂-minimal* (resp. ∂-minimal) *manifold* if and only if any proper birational ∂-map (resp. any birational ∂-map): $(\bar{V}_1, D_1)\to(\bar{V}, D)$ turns out to be a morphism. Thus, Proposition 2 is restated as follows: *A relatively minimal ∂-surface $(\bar{S}, D)$ with $\bar{\kappa}(\bar{S}-D)=2$ or $\kappa(\bar{S})\geqq 0$ is properly ∂-minimal.*

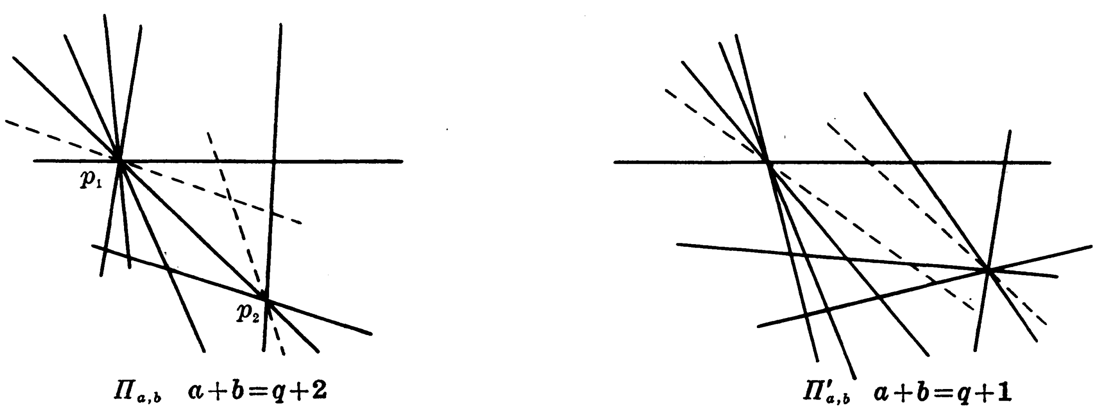
THEOREM 2. *Let* $(\bar{S}, D)$ *be a relatively* ∂*-minimal surface with* $\bar{\kappa}(\bar{S}-D)=2$ *or* $\kappa(\bar{S})\geqq 0$. *Suppose that any exceptional curve* C' *of the first kind which is not contained in* D *satisfies that* $(C', D)\geqq 3$. *Then* $(\bar{S}, \bar{D})$ *is* ∂*-minimal.*

PROOF. We use the same notation as in the proofs of Proposition 2 and Theorem 1. The case we have to consider here is the case in which $C_1 \not\subset D$. Then we have

$$(K+D, C_1)=-1-\sum^{s}\nu_j-t+(D, C_1)\ .$$

Since $(D, C_1)\leqq 2$, we get $(K+D, C_1)\leqq 1$. The equality holds if and only if $s=t=0$ and $(D, C_1)=2$. This case does not occur by the hypothesis. Hence, $(K+D, C_1)\leqq 0$. By hypothesis again, we have $C_1^2\geqq 0$. Using the same argument as before, we obtain $\bar{\kappa}(S)\leqq 1$ and $\kappa(\bar{S})=-\infty$. Q.E.D.

Example (cf. [6]). Let Δ be a union of lines $\Delta_0, \cdots, \Delta_q$ in $\boldsymbol{P}^2$. Define $S=\boldsymbol{P}^2-\cup\Delta_j$. Since S is affine, S is strongly minimal. Put $\Sigma_3=\{p\in\boldsymbol{P}^2;\ \mathrm{mult}_p(\Delta)\geqq 3\}$ and let Σ_3 consist of s points $p_1, \cdots, p_s$. By blowing $\boldsymbol{P}^2$ up at centers $p_1, \cdots, p_s$, we have the standard completion $\bar{S}$ of S with smooth boundary D. Assume $\bar{\kappa}(S)=2$. If Δ is of type $\prod_{a,b}$, $K+D$ is not ample. In fact, letting Δ_0 be a line connecting p_1 with p_2, the proper transform Δ_0' of Δ_0 is an exceptional curve of the first kind such that $[\Delta_0': D]=2$. Hence, D is not a minimal boundary. If Δ is not of type $\prod_{a,b}$, then $K+D$ is ample, by Theorem 3 [6]. We shall look for an irreducible exceptional curve C_1 of the first kind which is not contained in D such that $(C_1, D)=1$ or 2. Such a C_1 satisfies the condition that $\mu(C_1)\cap\Delta$ consists of two points. Then it is easy to see that $\mu(C_1)$ is also a line. Further, $(K+C_1+D, C_1)=0$, i.e., $K+C_1+D$ is not ample. Hence $\mu(C_1)+\Delta$ is of type $\prod_{a,b}$. In this case we say that Δ is of type $\prod'_{a-1,b-1}$. We conclude that if Δ is neither of type $\prod_{a,b}$ nor of type



$\Pi_{a,b}\quad a+b=q+2$ $\qquad$ $\Pi'_{a,b}\quad a+b=q+1$

FIGURE 4

$\Pi'_{a,b}$, then the ∂-surface $(\bar{S}, D)$ consisting of the standard completion $\bar{S}$ and its boundary D is ∂-minimal.

§4. We shall study relatively ∂-minimal surfaces $(\bar{S}, D)$ with $\bar{\kappa}(\bar{S}-D)\geqq 0$. First we recall the definition of a *canonical blowing up*. For a ∂-surface $(\bar{S}, D)$, letting $p \in D$, we define the blowing up $\lambda: \bar{S}^1 = Q_p(\bar{S}) \to \bar{S}$ and put $D^1 = \lambda^{-1}(D)$. If p is a double point of D, $\lambda: (\bar{S}^1, D^1) \to (\bar{S}, D)$ is called a canonical blowing up.

PROPOSITION 3. *Let $(\bar{S}, D)$ and $(\bar{S}_1, D_1)$ be relatively ∂-minimal surfaces such that $S=\bar{S}-D=\bar{S}_1-D_1$. We assume that $\bar{\kappa}(S)\geqq 0$. Let $\varphi: (\bar{S}, D) \to (\bar{S}_1, D_1)$ be a birational ∂-map. Then there exists a composition of canonical blowing ups $\mu: (\bar{S}_2, D_2) \to (\bar{S}, D)$ such that $g=\varphi\circ\mu$ is a proper birational ∂-morphism. g is also a composition of canonical blowing ups.*

PROOF. Let $\mu: (\bar{S}_2, D_2) \to (\bar{S}, D)$ be a proper birational ∂-morphism such that $g=\varphi\circ\mu$ is a proper birational ∂-morphism with $\mu|_S = \mathrm{id}$. Then μ is a composition of blowing ups. We shall prove that a non-canonical blowing up in the decomposition of μ is not necessary to eliminate the points of indeterminacy of φ. Let λ be the first non-canonical blowing up in μ. Namely, there exists a composition of canonical blowing ups $\mu_1: (\bar{S}_4, D_4) \to (\bar{S}, D)$ and a proper birational ∂-morphism $\mu_2: (\bar{S}_2, D_2) \to (\bar{S}_3, D_3)$ such that a non-canonical blowing up $\lambda: (\bar{S}_3, D_3) \to (\bar{S}_4, D_4)$ satisfies $\mu = \mu_1\circ\lambda\circ\mu_2$ (see Figure 5). The center of λ is denoted by $w \in D_3$. Let Γ

$$\begin{tikzcd}
 & \bar{S} \arrow[rr, "\varphi"] & & \bar{S}_1 & \\
\bar{S}_4 \arrow[ur, "\mu_1"] & & & \bar{S}_5 \arrow[u, "g_2"'] \arrow[r, "\sigma"] & \bar{S}_6 \\
\bar{S}_3 \arrow[u, "\lambda"] & & & & \\
 & \bar{S}_2 \arrow[ul, "\mu_2"] \arrow[uuu, "\mu"] \arrow[uuurr, "g"] \arrow[uurr, "g_1"'] & & &
\end{tikzcd}$$

FIGURE 5

be the irreducible component of D_3 which contains w. Putting $F = \mu_2^{-1}(\lambda^{-1}(w))$, we shall prove that $g(F)$ is a point, in other words, F is g-exceptional. We assume that $g(F)$ is a curve. Let Γ^* be the proper transform of Γ by $(\lambda\circ\mu_2)^{-1}$. Then $g(\Gamma^*)$ is a point. Actually, if $g(\Gamma^*)$ is a curve, then $g(F)$ remains to be an exceptional curve of the first

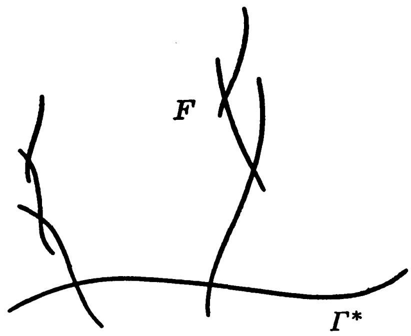


FIGURE 6

kind with $[g(F): D_1]=1$. This contradicts the hypothesis that D_1 is a minimal boundary. We write g as a composition of blowing ups $g=h_1\circ\cdots\circ h_m$. Then there exists an i such that $h_i\circ\cdots\circ h_m(\Gamma^*)$ is a curve $\tilde{\Gamma}$ and $h_{i-1}(\tilde{\Gamma})=$a point. Denoting $g_1=h_i\circ\cdots\circ h_m$ and $g_2=h_1\circ\cdots\circ h_{i-1}$, we see that $\tilde{F}=g_1(F)$ remains to be an exceptional curve of the first kind satisfying that $[\tilde{F}: g_1(D_2)]=1$. If $\tilde{F}$ is reducible, we can contract some curves in $\tilde{F}$ to obtain the irreducible $\tilde{F}$. Thus we may assume $\tilde{F}$ to be irreducible. Moreover, $[\tilde{\Gamma}: g_1(D_2)]\leqq 2$, since $g_2(\tilde{\Gamma})$ is a point and $g_2(g_1(D_2))$ is a divisor with normal crossings.

We can contract $\tilde{\Gamma}$ to a non-singular point. Then we obtain a blowing up $\sigma: \bar{S}_5=g_1(\bar{S}_2)\to\bar{S}_6$. Letting $Z=\sigma(F)$ and $D_6=\sigma(g_1(D_2))$, we have $Z^2=0$, $Z\cong\boldsymbol{P}^1$, and $(Z, D_6)=1$. Since $(\bar{S}, D)$ is not properly ∂-minimal, $\bar{S}$ is a ruled surface by Theorem 2. Thus, Z is a fiber of the fiber space

$$\pi: \bar{S}\longrightarrow Y \quad\text{and so}\quad \bar{\kappa}(S)\leqq\bar{\kappa}(Z-Z\cap D_6)+\dim Y=-\infty\,.$$

This contradicts the hypothesis.

Hence, we may delete non-canonical blowing ups λ from $\mu: \bar{S}_2\to\bar{S}$. Thus we take a composition of canonical blowing ups μ such that $g=\varphi\circ\mu$ is a proper birational morphism. We shall prove that g is also a composition of canonical blowing ups. Note the following

LEMMA 3. *Let $g: (\bar{S}, D)\to(\bar{S}_1, D_1)$ be a proper birational ∂-morphism. Suppose that g is a composition of α canonical blowing ups and β non-canonical blowing ups. Then*

$$(K(\bar{S})+D)^2=(K(\bar{S}_1)+D_1)^2-\beta\,.$$

Proof is easy and omitted.

We proceed with the proof of Proposition 3. By Lemma 3 applied to μ and g, we have $(K(\bar{S}_2)+D_2)^2=(K(\bar{S})+D)^2$ and $(K(\bar{S}_2)+D_2)^2\leqq(K(\bar{S}_1)+D_1)^2$. Thus we obtain

$$(K(\bar{S})+S)^2 \geqq (K(\bar{S}_1)+D_1)^2 .$$

Similarly, $(K(\bar{S}_1)+D_1)^2 \geqq (K(\bar{S})+D)^2$. Hence,

$$(K(\bar{S})+D)^2 = (K(\bar{S}_1)+D_1)^2 .$$

This implies that g is composed of canonical blowing ups. Q.E.D.

COROLLARY. *Let S be a surface with $\bar{\kappa}(S) \geqq 0$. Then $(K(\bar{S})+D)^2$ does not depend upon the choice of completions $\bar{S}$ of S with ordinary boundaries D, provided D's are minimal boundaries.*

For a surface S we define the *logarithmic Chern numbers* $\bar{c}_1^2(S)$ and $\bar{c}_2(S)$ as follows (see [6], [7]):

$$\bar{c}_1^2(S) = \sup\{c_1(\Theta(\log D))^2[\bar{S}]; (\bar{S}, D) \text{ being a } \partial\text{-surface such that } S = \bar{S} - D\}$$

$$\bar{c}_2(S) = \sup\{c_2(\Theta(\log D))[S]; \text{as above}\} .$$

Here $\Theta(\log D)$ is the dual sheaf of $\Omega^1(\log D)$. Note that $\bar{c}_2(S)$ is the Euler characteristic of S, [7]. Moreover, when $\bar{\kappa}(S) \geqq 0$, $\bar{c}_1^2(S) = (K(\bar{S})+D)^2$ where D is a minimal boundary.

By the proposition above, we shall determine all relatively ∂-minimal models $(\bar{S}, D)$ when $S = \bar{S} - D$ is given in the case of $\bar{\kappa}(S) = 1$.

First we define elementary transformations for a given ∂-surface $(\bar{S}, D)$. If there is an irreducible component C of D such that $C^2 = 0$, $\pi(C) = 0$, and $[C: D] = 1$ or 2, we consider a canonical blowing up $\lambda: (\bar{S}^1, D^1) \to (\bar{S}, D)$ whose center $p \in C \cap D'$, $D = C + D'$. The proper transform C' satisfies $C'^2 = -1$ and $\pi(C') = 0$. Hence, contracting C' to a non-singular point by a blowing up λ_1, we have a new ∂-surface $(\bar{S}_1, D_1)$. The birational map φ defined to be the composition of $\lambda^{-1}: (\bar{S}, D) \to (\bar{S}^1, D^1)$ and $\lambda_1: (\bar{S}^1, D^1) \to (\bar{S}_1, D_1)$ is called an elementary transformation of the first kind or the second kind, respectively, according to $[C: D] = 1$ or 2.

Irreducible components defined in the above figure have the following self-intersection numbers: $(\theta')^2 = \theta^2 - 1$, $(\theta_1')^2 = \theta_1^2 - 1$, $(\theta_2')^2 = \theta_2^2 + 1$, and $(\theta'')^2 = \theta^2$. To make things clear, we say that φ is an elementary transformation at p with axis C and φ is denoted by elm$[p, C]$. We can repeat elementary transformations at p (resp. p') with axis C (resp. E') and so on. A b-times composition of such transformations is written as elm$^b[p, C]$.

THEOREM 3. *Let $(\bar{S}, D)$ and $(\bar{S}_1, D_1)$ be relatively minimal ∂-surfaces where $S = \bar{S} - D = \bar{S}_1 - D_1$. Suppose that $\bar{\kappa}(S) = 1$. Then $(\bar{S}_1, D_1)$ is obtained from $(\bar{S}, D)$ by a finite succession of compositions of* elm$^m[p_i, C_i]$

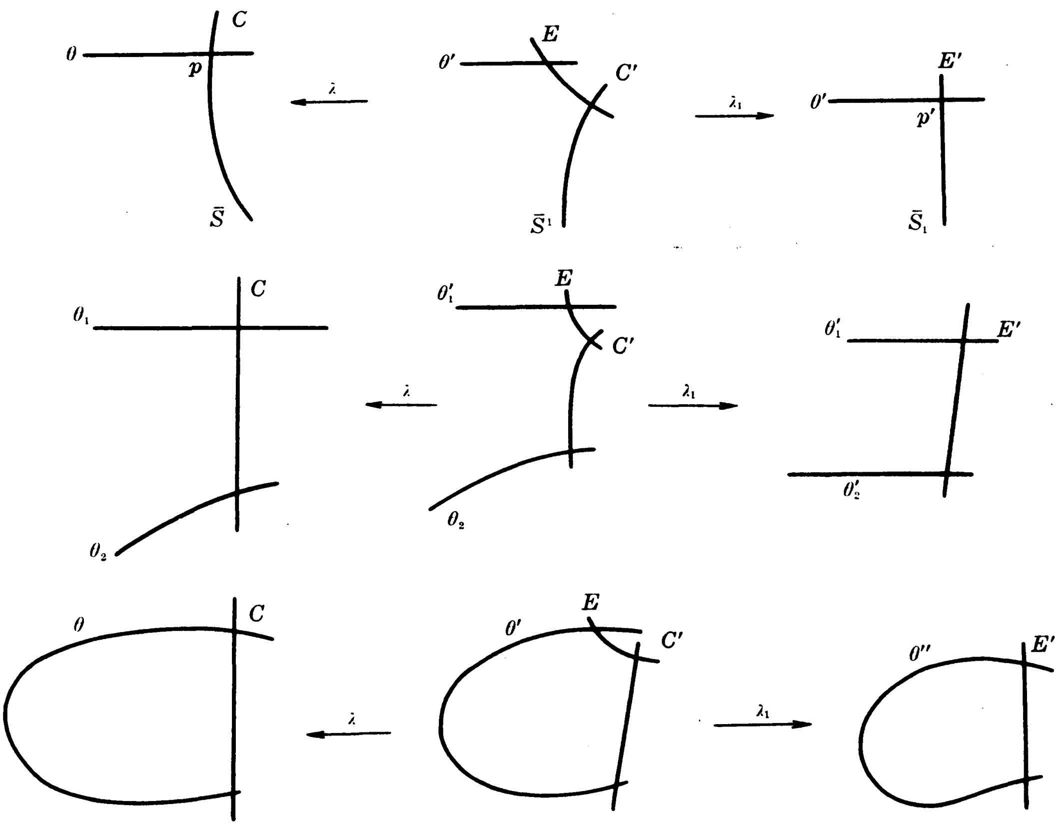


FIGURE 7

in which the C_i are parallel.

PROOF. By the fundamental theorem on logarithmic Kodaira dimension ([2] Theorem 5), we have the logarithmic canonical fibered surfaces $f: \bar{S} \to W$ and $f_1: \bar{S}_1 \to W_1$, W and W_1 being complete non-singular curves. Assume that the identity: $S \to S$ induces a proper birational map $\varphi: \bar{S} \to \bar{S}^1$ that is not a morphism. φ induces the linear isomorphism from $T_m(S) = H^0(m(K(S_1)+D_1))$ into $T_m(S) = H^0(m(K(S)+D))$, which determines an isomorphism $\varphi: W \to W_1$. Thus we have the following diagram.

$$\begin{array}{ccc} \bar{S} & \xrightarrow{\varphi} & \bar{S}_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ W & \xrightarrow{\psi} & W_1 \end{array}$$

FIGURE 8

Take a point $p_1 \in \bar{S}_1$ at which φ^{-1} is not defined. Then by the proof of Proposition 2, we have an irreducible exceptional curve of the second kind C on $\bar{S}$ such that $C^2 \geqq 0$ and $[C: D] \leqq 2$. Since ψ is an isomorphism,

C is a fiber of $f: \bar{S} \to W$. Hence, $C^2=0, \pi(C)=0$ and $[C: D]=2$. Set $w=f(C)$ and $u=\psi(w)$. We eliminate the points of indeterminacy of φ by a composition of canonical blowing ups $\mu: \bar{S}_2 \to \bar{S}$. $f\circ\mu: \bar{S}_2 \to W$ is the logarithmic canonical fibered surface of $\bar{S}_2$ and we write $g=\varphi\circ\mu$ as usual. The reduced divisor $\mu^{-1}(C)$ is written as $C^*+\sum E_j$, in which C^* is the proper transform of C. By hypothesis, $g(C^*)$ is a point q. Hence, if $C^{*2} \leqq -2$, then there exists an irreducible exceptional curve of the first kind in $g^{-1}(q)$, say E_1. After contracting such E_1, we conclude that $C^{*2}=-1$. This implies that the centers of μ belong to one of the two points $C\cap D'$ where $D=C+D'$. Thus we have the following figure:

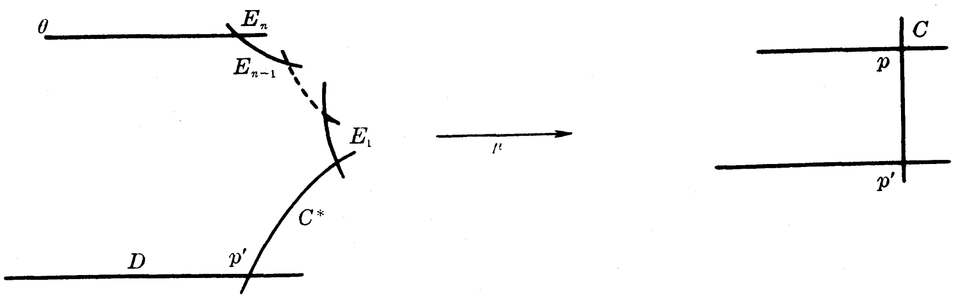


FIGURE 9

After changing the indices of E_j's, if necessary, we assume $C^*\cap E_1 \neq \emptyset$, $E_1\cap E_2 \neq \emptyset, \cdots, E_{n-1}\cap E_n \neq \emptyset$, $n+1=r(g^{-1}(C))$. If $E_1^2=-1$, then contract C^*. The proper transform E_1' has the vanishing self-intersection number. This yields that $n=1$ and $\varphi=\mathrm{elm}[p, C]$ locally around C. If $E_1^2=-2$ and $E_j^2=-2$ or -1 for any $j\in[2, n]$, then repeat contractions starting from C^*. Thus we finally have

$$E_1^2=E_2^2=\cdots=E_n^2=-2\ , \quad \text{and} \quad E_n^2=-1\ .$$

It is easy to see that φ is expressed near C as an n-times composition of elementary transformations of the second kind. In other words, $\varphi=\mathrm{elm}^n[p, C]$ locally around C. The final case is the case where there exists i such that $E_i^2 \leqq -3$. Let E_l correspond to a fiber C_1 by g. Then $E_n+E_{n-1}+\cdots+E_{l+1}$ is an exceptional curve of the first kind that is g-exceptional. Hence, there exists an irreducible exceptional curve of the first kind E_k, $n\geqq k\geqq l+1$. Contract such an E_k and repeat. At last we may assume that $l=n$. Thus, $C^*+E_1+\cdots+E_{n-1}$ is an exceptional curve of the first kind, which is a bamboo. If there exists $E_j(1\leqq j\leqq E_{n-1})$ which is an exceptional curve of the first kind, contract it. Even after such contractions, $C^*+E_1+\cdots+E_{n-1}$ has the same property, i.e., $C^{*2}=-1$ and it is exceptional. Thus, we may assume that $E_j^2\leqq -2$ for any

$j \in [1, n-1]$. We claim that $E_j^2 = -2$ for any $j \in [1, n-1]$. Actually, if there exists E_m such that $E_m^2 \leqq -3$, we let i be the minimal number among such m. Contract C^* and let E_1' be the proper transform of E_1, which is an exceptional curve of the first kind, if $1 \leqq i-1$. Then contract E_1', again. Continuing these contractions we arrive at the following bamboo

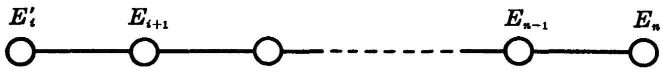


FIGURE 10

where $E_i'^2 \leqq -2$, $E_{i+1}^2 \leqq -2, \cdots, E_{n-1}^2 \leqq -2$. Thus $E_i' + E_{i+1} + \cdots + E_{n-1}$ could not be an exceptional curve of the first kind. This contradicts the fact that $C^* + E_1 + \cdots + E_{n-1}$ is exceptional. Performing the same processes, we conclude that

$$\varphi = \mathrm{elm}^n[p, C] \circ \mathrm{elm}^m[q, C'] \circ \cdots \circ \cdots .$$

COROLLARY. *Let S be a surface with $\bar{\kappa}(S) \geqq 1$. Take an arbitrary completion $\bar{S}$ of S with ordinary boundary D such that D is a minimal boundary. Then $(K(\bar{S}))^2$, $(K(\bar{S}), K(\bar{S})+D)$, $(K(\bar{S})+D)^2$ do not depend upon the choice of ∂-surface $(\bar{S}, D)$ such that $S = \bar{S} - D$.*

Thus we can define *logarithmic Chern numbers* of S with $\bar{\kappa}(S) \geqq 1$ as follows:

$$c_1^2(S) = (K(\bar{S}))^2 ,$$
$$\bar{c}_1 c_1(S) = (K(\bar{S}), K(\bar{S}) + D) ,$$
$$\bar{c}_1^2(S) = (K(\bar{S}) + D)^2 .$$

EXAMPLE. Let $S = A^2 - V(x^2 - y^3)$. Then $S = P^2 - C_1 \cup C_2$, C_1 being the infinite line and C_2 being the closure of $V(x^2 - y^3)$. By a 6-times composition of blowing ups, we have a completion $\bar{S}$ of S with smooth boundary D.

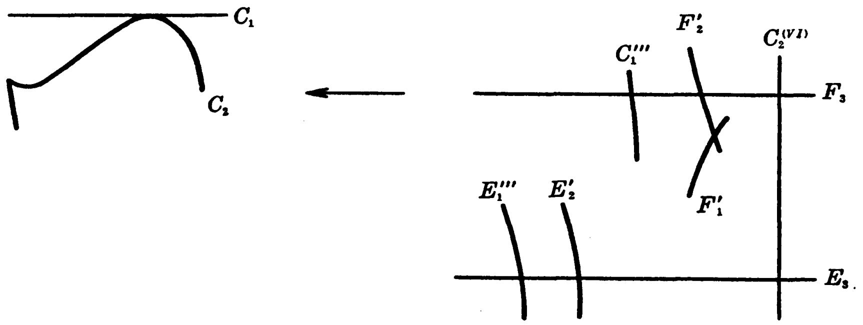


FIGURE 11

Here, the entity X^a is the proper transform of X^{a-1} by a blowing up. Let H represent a line on $\boldsymbol{P}^2$ and the total transform of Y is denoted by the same symbol Y. Then

$$D+K \sim H-F_2-E_3-F_3 ,$$
$$K \sim -3H+E_1+E_2+E_3+F_1+F_2+F_3 .$$

Hence,

$$c_1^2(S)=3, \ \bar{c}_1 c_1(S)=0, \ \bar{c}_1^2(S)=-2 .$$

The configurations of minimal boundaries of S are as follows:

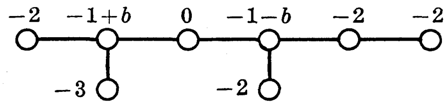


FIGURE 12

Each point indicates the irreducible component of D with its self-intersection number.

REMARK. The determination of all ∂-surfaces $(\bar{S}, D)$ with $S=\bar{S}-D$ for a given surface S is rather difficult when $\bar{\kappa}(S)=0$. But it can be done in a similar way to Theorem 3. For this, we refer the reader to [5].

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