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A Proof of the Classical Kronecker Limit Formula

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Introduction

Let z, w be complex numbers. We assume that imaginary part of z is positive. Set

$$\xi(s, w, z)=\sum_{m,n}{}' |m+nz+w|^{-2s} ,$$

where summation with respect to m, n ranges over all pairs of integers such that $m+nz+w\neq 0$.

Put

$$\eta(z)=e[z/24]\prod_{n=1}^{\infty}(1-e[nz]) ,$$

$$\vartheta_1(w, z)=2e[z/12](\sin \pi w)\eta(z)\prod_{n=1}^{\infty}(1-e[w+nz])(1-e[-w+nz]) ,$$

where we write $e[z]=\exp(2\pi iz)$. Furthermore, we set $\xi'=d\xi/ds$. A version of the classical Kronecker limit formula is given as follows (see e.g., [9]).

If $w\notin \boldsymbol{Z}+\boldsymbol{Z}z$,

$$\xi'(0, w, z)=-\log\left|\frac{\vartheta_1(w, z)}{\eta(z)}\exp\frac{\pi iw(w-\bar{w})}{z-\bar{z}}\right|^2 .$$

If $w\in \boldsymbol{Z}+\boldsymbol{Z}z$,

$$\xi'(0, w, z)=-\log\{4\pi^2|\eta(z)|^4\} .$$

For the proofs of the Kronecker limit formula, we refer to [4] and papers quoted there. In this note we present a proof of the formula which makes use of the theory of the *double gamma function*. The author takes this opportunity to make an addendum of the reference to

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his previously published paper [6]. After the main part of [6] was written down, the author received a preprint of [5]. There are some overlaps between results of [5] and [6].

NOTATION. As usual we denote by $\boldsymbol{Z}$ and $\boldsymbol{C}$ the ring of rational integers and the field of complex numbers. We denote by γ the Euler constant, by Γ the gamma function and by ψ the logarithmic derivative of the gamma function. We put

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{and} \quad B_n=B_n(0) .$$

The Riemann zeta function is denoted by ζ.

§1. We review the definition and basic properties of the double gamma function. For details, we refer to [1], [2] and [3] (see also [7]). Let z be a complex number and $\omega=(\omega_1, \omega_2)$ be a pair of complex numbers. For a while we assume that z, ω_1 and ω_2 are all *with positive real part.* For a complex number $w \in C-(-\infty, 0]$, we put $w^s=\exp(s \log w)$, where $\log w=\log|w|+i \arg w$ ($|\arg w|<\pi$). Set

$$\zeta_2(s, z, \omega)=\sum_{m\ n=0}^{\infty}(z+m\omega_1+n\omega_2)^{-s} \qquad (\operatorname{Re} s>2) .$$

It is known that ζ_2 is continued to a meromorphic function in the whole complex plane which is holomorphic except for simple poles at $s=2$ and $s=1$.

Put

$$\log \Gamma_2^*(z, \omega)=\zeta_2'(0, z, \omega), \quad \text{where} \quad \zeta_2'=d\zeta_2/ds .$$

It is shown that $z=0$ is a simple pole of $\Gamma_2^*(z, \omega)$. Put

$$1/\rho_2(\omega)=\text{residue at } z=0 \text{ of } \Gamma_2^*(z, \omega) \quad \text{and}$$

$$\frac{\Gamma_2(z, \omega)}{\rho_2(\omega)}=\Gamma_2^*(z, \omega)=\exp \zeta_2'(0, z, \omega) .$$

It is immediate to see that Γ_2 satisfies the difference equations:

$$(1.1) \quad \begin{aligned} &\Gamma_2(z+\omega_1, \omega)/\Gamma_2(z, \omega)=(2\pi)^{1/2}\Gamma(z/\omega_2)^{-1}\exp\{(1/2-z/\omega_2)\log\omega_2\} \\ &\Gamma_2(z+\omega_2, \omega)/\Gamma_2(z, \omega)=(2\pi)^{1/2}\Gamma(z/\omega_1)^{-1}\exp\{(1/2-z/\omega_1)\log\omega_1\} . \end{aligned}$$

It follows easily from the difference equations that

$$(1.2) \quad \begin{aligned} &\Gamma_2(\omega_1, \omega)=(2\pi/\omega_2)^{1/2} , \qquad \Gamma_2(\omega_2, \omega)=(2\pi/\omega_1)^{1/2} , \\ &\Gamma_2(\omega_1+\omega_2, \omega)=(2\pi)(\omega_1)^{-1/2}(\omega_2)^{-1/2} . \end{aligned}$$

§ **2.**

Proposition 1. *Assume* $z,\ w>0$.

(1) $\Gamma_2(w,\ (1,\ z))$

$$=(2\pi)^{w/2}\exp(\{(w-w^2)/2z-w/2\}\log z+(w^2-w)\gamma/2z)$$
$$\times\Gamma(w)\prod_{n=1}^{\infty}\frac{\Gamma(w+nz)}{\Gamma(1+nz)}\exp\{(w-w^2)/2nz+(1-w)\log nz\}$$

(2) $\rho_2((1,\ z))$

$$=(2\pi)^{3/4}\exp\{-\gamma/12z-z/12+z\zeta'(-1)+(z/12-1/4+1/12z)\log z\}$$
$$\times\prod_{n=1}^{\infty}(2\pi)^{1/2}\Gamma(1+nz)^{-1}\exp\{1/12nz+(1/2+nz)\log nz-nz\}\ .$$

Proof. To simplify the notation, set

$$\Gamma^*(w,\ z)=\Gamma_2(w,\ (1,\ z))/\rho_2((1,\ z))\ .$$

It follows from (1.1) that

$$\log\Gamma^*(w,\ z)-\log\Gamma^*(w+z,\ z)=\log\{\Gamma(w)/(2\pi)^{1/2}\}\ .$$

Hence

$$\log\Gamma^*(w,\ z)=\log\Gamma^*(w+nz,\ z)+\sum_{m=0}^{n-1}\log\{\Gamma(w+mz)/(2\pi)^{1/2}\}\ . \tag{2.1}$$

Recall the following asymptotic expansion:

$$\log\Gamma(z+a)\sim(z+a-1/2)\log z-z+\log(2\pi)^{1/2}$$
$$+\sum_{m=2}^{\infty}\frac{(-1)^m B_m(a)}{m(m-1)}z^{1-m}\qquad(z\mapsto+\infty)\ . \tag{2.2}$$

We transform (2.1) as follows:

$$\log\Gamma^*(w,\ z)=\log\{\Gamma(w)/(2\pi)^{1/2}\}+\log\Gamma^*(w+nz,\ z)$$
$$+\sum_{m=1}^{n-1}\{\log\Gamma(w+mz)-(mz+w-1/2)\log mz+mz-\log(2\pi)^{1/2}$$
$$-B_2(w)/2mz\}$$
$$+\sum_{m=1}^{n-1}\{(mz+w-1/2)\log mz-mz+B_2(w)/2mz\}\ .$$

Thus

(2.3) $\log \Gamma^*(w, z) - \log \{\Gamma(z)/(2\pi)^{1/2}\}$

$$- \sum_{m=1}^{n-1} \{\log \Gamma(w+mz)/(2\pi)^{1/2} - (mz+w-1/2)\log mz + mz - B_2(w)/2mz\}$$

$$= \log \Gamma^*(w+nz, z) + (w-1/2)\log \Gamma(n) + z\sum_{m=1}^{n-1}(m\log m - m)$$

$$+ \frac{n(n-1)}{2} z\log z + (w-1/2)(n-1)\log z + \frac{B_2(w)}{2z}\sum_{m=1}^{n-1} 1/m \ .$$

We denote by $LG(w)$ the function of w given by the following formula if w is positive (cf. (1.12) of [7]):

$$LG(w) = \frac{1}{2\pi i}\int_{I(\varepsilon,\infty)} \frac{\exp(-wt)}{1-\exp(-t)} \frac{\log t}{t^2} dt + \frac{\gamma - \pi i}{2} B_2(w) \qquad (0<\varepsilon<2\pi) \ ,$$

where $I(\varepsilon, \infty)$ is the integral path consisting of (∞, ε), counterclockwise circle of radius ε around the origin and $(\varepsilon, +\infty)$. Since $LG(w) - LG(w+1) = w\log w - w$ (see Lemma 2 of [7]), we have

$$\sum_{m=1}^{n-1} \{m\log m - m\} = LG(1) - LG(n) \ .$$

Under the assumption that both w and z are positive it follows easily from (1.16) of [7] that

$$\log \Gamma^*(w, z) = \frac{1}{z} LG(w) - B_1 \log\{\Gamma(w)/(2\pi)^{1/2}\} - \psi(w) B_2 z/2 + O(1/w)$$
$$\text{if} \quad w \longrightarrow +\infty \ .$$

It follows from (iv) of Lemma 2 of [7] that

$$LG(n) = -\frac{n^2}{2}\log n + \frac{3}{4}n^2 - B_1(n\log n - n) - \frac{B_2}{2}\log n + O\Big(\frac{1}{n}\Big) \ ,$$

$$LG(w+nz) = -\frac{1}{2}B_2(w+nz)\log(nz) + \frac{3}{4}n^2z^2 + nzB_1(w) + O\Big(\frac{1}{n}\Big) \ .$$

Furthermore, in view of (2.2),

$$\log \frac{\Gamma(w+nz)}{(2\pi)^{1/2}} = (nz+w-1/2)\log(nz) - nz + O\Big(\frac{1}{n}\Big) \ ,$$

$$\log \Gamma(n) = (n-1/2)\log n - n + \log(2\pi)^{1/2} + O\Big(\frac{1}{n}\Big) \ ,$$

$$\psi(w+nz) = \log(nz) + O\Big(\frac{1}{n}\Big) \ ,$$

$$\sum_{m=1}^{n-1} 1/m = \log n + \gamma + O\left(\frac{1}{n}\right).$$

Thus we obtain the asymptotic expansion of the right side of (2.3) when $n \to +\infty$. Since the left side of (2.3) is convergent when $n \to +\infty$, diverging terms in the right side of (2.3) must cancel each other. Hence we have

$$\begin{aligned}(2.4)\ \log \Gamma^*(w, z) - \log\{\Gamma(w)/(2\pi)^{1/2}\} \\ - \sum_{m=1}^{\infty}\{\log \Gamma(w+mz)/(2\pi)^{1/2} - (mz+w-1/2)\log mz + mz - B_2(w)/2mz\} \\ = (w-1/2)\log(2\pi)^{1/2} + LG(1)z - (w-1/2)\log z + B_2(w)\gamma/2z \\ - \frac{1}{2z}B_2(w)\log z + (w-1/2)/2\log z - z\log z/12 .\end{aligned}$$

Set $w=1$ in (2.4). Since $\Gamma^*(1, z) = \Gamma_2(1, (1, z))/\rho_2((1, z)) = (2\pi/z)^{1/2}/\rho_2((1, z))$ and $LG(1) = (1/12) - \zeta'(-1)$ (see (ii) of Lemma 2 of [7]), we obtain the second part of Proposition 1. The first part of Proposition 1 now follows easily from (2.4) and the equality $\Gamma(w, (1, z)) = \Gamma^*(w, z)\rho_2((1, z))$.

COROLLARY TO PROPOSITION 1.

(1) *Via infinite product expansion* (1) *of Proposition* 1, $\Gamma_2(w, (1, z))$ *is continued analytically to a holomorphic function in the domain* $\{(w, z); z \in C - (-\infty, 0], w \neq -(m+nz)(m, n=0, 1, 2, \cdots)\}$.

(2) *Via infinite product expansion* (2) *of Proposition* 1, $\rho_2((1, z))$ *is continued analytically to a holomorphic function in the domain* $\{z; z \in C - (-\infty, 0]\}$.

PROPOSITION 2. *Assume* $\operatorname{Im} z > 0$.

(1) $\rho_2((1, -z))\rho_2((1, z)) = (2\pi)^{3/2} z^{-1/2} \eta(z) \exp\{\pi i(1/4 + 1/12z)\}$.

(2) *Set*

$$\Gamma^*(w, z) = \frac{\Gamma_2(w, (1, z))}{\rho_2((1, z))}.$$

Then

$$\begin{aligned}\Gamma^*(w, z)\Gamma^*(1-w, -z)\Gamma^*(1+z-w, z)\Gamma^*(w-z, -z) \\ = \frac{\eta(z)}{\vartheta_1(w, z)} \exp \pi i\{-1/6z + (w-w^2)/z\} .\end{aligned}$$

PROOF. The formula (1) is an easy consequence of (2) of Proposition 1 in view of the equalities $\log(-z) = \log z - \pi i$ and $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$.

It follows from (1) of Proposition 1 that

$$\Gamma_2(w, (1, z))\Gamma_2(1-w, (1, -z))$$
$$=(2\pi/z)^{1/2}\exp \pi i\{(w-w^2)/2z+(1-w)/2\}\frac{\pi}{\sin \pi w}\prod_{n=1}^{\infty}\frac{1-e^{2n\pi i z}}{1-e^{2\pi i(w+nz)}}.$$

A straightforward computation now shows the validity of (2).

§3. Let $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a matrix of size 2 and let $x=(x_1, x_2)'$ be a column vector of size 2. Assume that all entries of A are with positive real part and that both x_1 and x_2 are non-negative and are not simultaneously equal to zero. Set

$$\zeta(s, A, x)=\sum_{m}\prod_{i=1}^{2}\left\{\sum_{j=1}^{2}a_{ij}(x_j+m_j)\right\}^{-s}, \tag{3.1}$$

where the summation with respect to m ranges over the set of all pairs $(m_1, m_2)'$ of non-negative integers. Put $\zeta'(s, A, x)=(d/ds)\zeta(s, A, x)$. Furthermore, we set $(w_1, w_2)'=Ax$ and $\boldsymbol{a}_i=(a_{i1}, a_{i2})(i=1, 2)$.

PROPOSITION 3. *The notation and assumption being as above,*

$$\begin{aligned}\zeta'(0, A, x)=&\log\left\{\frac{\Gamma_2(w_1, \boldsymbol{a}_1)}{\rho_2(\boldsymbol{a}_1)}\right\}+\log\left\{\frac{\Gamma_2(w_2, \boldsymbol{a}_2)}{\rho_2(\boldsymbol{a}_2)}\right\}\\ &+\det A\ B_2(x_1)\{\log a_{12}-\log a_{22}\}/4a_{12}a_{22}\\ &+\det A\ B_2(x_2)\{\log a_{21}-\log a_{11}\}/4a_{11}a_{21},\end{aligned} \tag{3.2}$$

where log *is understood to be a holomorphic function on* $C-(-\infty, 0)$ *which is real valued on the positive real axis.*

PROOF. If all entries of A are positive, the proposition is a special case of Proposition 1 of [8]. It is immediate to see that Proposition 1 of [8] remains to be valid under the weaker hypothesis that all entries of A are with positive real part.

Suppose A is of the form

$$A=\begin{pmatrix} 1 & z_1 \\ 1 & z_2 \end{pmatrix} \quad (z_1, z_2 \in C-(-\infty, 0]). \tag{3.3}$$

Then $(x_1+m_1)+z_i(x_2+m_2)\in C-(-\infty, 0]$ for any non-negative integers m_1, m_2. Hence $\zeta(s, A, x)$ is defined by (3.1) even if z_1 and z_2 are not with positive real part. It is shown that $\zeta'(0, A, x)$ is a holomorphic function of z_1 and z_2 in the domain $(C-(-\infty, 0])\times(C-(-\infty, 0])$.

COROLLARY TO PROPOSITION 3. *Assume A is of the form* (3.3). *Then* (3.2) *remains to be valid for all* $z_1, z_2 \in C-(-\infty, 0]$.

§ 4. Now we are ready to derive the Kronecker limit formula. We use the notation in the introduction without further comment. Since $\xi(s, w+m+nz, z)=\xi(s, w, z)(m, n \in Z)$, we may put $w=u+vz$ $(0 \leqq u, v<1)$. Assume u and v are not simultaneously equal to zero. Then

$$\begin{aligned}
\xi(s, w, z) &= \sum_{(m,n)\in Z^2} |m+nz+w|^{-2s} \\
&= \sum_{m,n\geqq 0} |m+nz+w|^{-2s} + \sum_{m,n\geqq 0} |-1-m+nz+w|^{-2s} \\
&\quad + \sum_{m,n\geqq 0} |m-(1+n)z+w|^{-2s} + \sum_{m,n\geqq 0} |-1-m-(1+n)z+w|^{-2s} \\
&= \zeta\left(s, \begin{pmatrix} 1 & z \\ 1 & \bar{z} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) + \zeta\left(s, \begin{pmatrix} 1 & -z \\ 1 & -\bar{z} \end{pmatrix}, \begin{pmatrix} 1-u \\ v \end{pmatrix}\right) \\
&\quad + \zeta\left(s, \begin{pmatrix} 1 & -z \\ 1 & -\bar{z} \end{pmatrix}, \begin{pmatrix} u \\ 1-v \end{pmatrix}\right) + \zeta\left(s, \begin{pmatrix} 1 & z \\ 1 & \bar{z} \end{pmatrix}, \begin{pmatrix} 1-u \\ 1-v \end{pmatrix}\right) \\
&\quad \text{(cf. (3.1))} .
\end{aligned}$$

Applying Corollary to Proposition 3, we have, if one puts $\Gamma^*(w, z)= \Gamma_2(w, (1, z))/\rho_2((1, z))$,

$$\begin{aligned}
&\xi'(0, w, z) \\
&\quad = \log |\Gamma^*(w, z)|^2 - \frac{z-\bar{z}}{4z\bar{z}} B_2(u)(\log z - \log \bar{z}) \\
&\quad + \log |\Gamma^*(1-w, -z)|^2 + \frac{z-\bar{z}}{4z\bar{z}} B_2(1-u)\{\log(-z)-\log(-\bar{z})\} \\
&\quad + \log |\Gamma^*(w-z, -z)|^2 + \frac{z-\bar{z}}{4z\bar{z}} B_2(u)\{\log(-z)-\log(-\bar{z})\} \\
&\quad + \log |\Gamma^*(1+z-w, z)|^2 - \frac{z-\bar{z}}{4z\bar{z}} B_2(1-u)(\log z - \log \bar{z}) .
\end{aligned}$$

Since $\operatorname{Im} z>0$, $\log(-z)=\log z-\pi i$, $\log(-\bar{z})=\pi i+\log \bar{z}$. In view of (2) of Proposition 2, we have

$$\begin{aligned}
&\xi'(0, w, z) \\
&\quad = \log \left| \frac{\eta(z)}{\vartheta_1(w, z)} \exp \pi i\left(-\frac{1}{6z}+\frac{w-w^2}{z}\right)\right|^2 + \frac{\bar{z}-z}{z\bar{z}} B_2(u)\pi i \\
&\quad = -\log \left| \frac{\vartheta_1(w, z)}{\eta(z)} \exp\left(\pi i w \frac{w-\bar{w}}{z-\bar{z}}\right)\right|^2 .
\end{aligned}$$

Next assume $w=u+vz=0$.

Then

$$\begin{aligned}\xi(s,0,z)&={\sum}'|m+nz|^{-2s}\\&=\sum_{m,n\geq 0}|1+m+nz|^{-2s}+\sum_{m,n\geq 0}|-1-m+nz|^{-2s}\\&\quad+\sum_{m,n\geq 0}|m-z-nz|^{-2s}+\sum_{m,n\geq 0}|-1-m-z-nz|^{-2s}+|z|^{-2s}\zeta(2s)\\&=\zeta\left(s,\begin{pmatrix}1&z\\1&\bar z\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)+\zeta\left(s,\begin{pmatrix}1&-z\\1&-\bar z\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)\\&\quad+\zeta\left(s,\begin{pmatrix}1&-z\\1&-\bar z\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right)+\zeta\left(s,\begin{pmatrix}1&z\\1&\bar z\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+|z|^{-2s}\zeta(2s)\,.\end{aligned}$$

Applying Corollary to Proposition 3 and taking (1.2) into account, we have

$$\begin{aligned}&\xi'(0,0,z)\\&=\log\{(2\pi)^5|z|^{-3}|\rho_2((1,z))\rho_2((1,-z))|^{-4}\}+\frac{\bar z-z}{z\bar z}B_2\pi i\\&\quad+\log|z|+2\log(2\pi)^{-1/2}\,.\end{aligned}$$

In view of (1) of Proposition 2, we have

$$\begin{aligned}&\xi'(0,0,z)\\&=\log\{(2\pi)^{-1}|z|^{-1}|\eta(z)|^{-4}|\exp\pi i/12z|^{-4}\}+\frac{\bar z-z}{6z\bar z}\pi i+\log|z|-\log(2\pi)\\&=\log|\eta(z)|^{-4}-\log(4\pi^2)\,.\end{aligned}$$

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