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Perturbation Theory for Cosine Families on Banach Spaces

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Introduction.

Let X be a Banach space and let $B(X)$ denote the set of all bounded linear operators from X into itself. A one-parameter family $\{C(t); t \in R = (-\infty, \infty)\}$ in $B(X)$ is called a *cosine family* on X if it satisfies the following conditions:

(0.1) $$C(t+s)+C(t-s)=2C(t)C(s) \quad \text{for all} \quad t, s \in R ;$$

(0.2) $$C(0)=I \quad \text{(the identity operator)} ;$$

(0.3) $$C(t)x: R \to X \quad \text{is continuous for every} \quad x \in X .$$

The associated *sine family* $\{S(t); t \in R\}$ is the one-parameter family in $B(X)$ defined by

(0.4) $$S(t)x=\int_0^t C(s)x ds \quad \text{for } x \in X \text{ and } \quad t \in R .$$

The *infinitesimal generator* A of $\{C(t); t \in R\}$ is defined by

(0.5) $$Ax=\lim_{h\to 0} 2h^{-2}(C(h)-I)x$$

whenever the limit exists. The set of elements x for which $\lim_{h\to 0} 2h^{-2}(C(h)-I)x$ exists is the domain of A, denoted by $D(A)$.

The purpose of this paper is to prove some perturbation theorems for cosine families. That is, we give several sufficient conditions such that if A is the infinitesimal generator of a cosine family on X and B is a linear operator in X, then $\overline{A+B}$ (the closure of $A+B$) is also the infinitesimal generator of a cosine family on X. The results are related to those of Takenaka-Okazawa [4] and Travis-Webb [6] which are included in ours.

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§1. The main results.

Let $\{C(t); t \in R\}$ be a cosine family on X with the infinitesimal generator A. We introduce a class of perturbing operators for A.

DEFINITION. A linear operator B in X is said to be of class $\mathfrak{P}(A)$ if

(i) there exists a dense linear subset D of X such that $D \subset D(A) \cap D(B)$, $C(t)D \subset D$ for $t \in R$ and $BC(t)x$: $R \to X$ is continuous for every $x \in D$,

(ii) for some $\mu > 0$

$$\sup\left\{\int_0^\infty e^{-\mu t}\left\|\int_0^t BC(s)xds\right\|dt;\ x \in D,\ \|x\| \leqq 1\right\} < \infty . \tag{1.1}$$

Let $B \in \mathfrak{P}(A)$ and set

$$\begin{cases} L_\lambda = \sup\left\{\int_0^\infty e^{-\lambda t}\left\|\int_0^t BC(s)xds\right\|dt;\ x \in D,\ \|x\| \leqq 1\right\} \quad \text{for} \quad \lambda \geqq \mu , \\ L_\infty = \lim_{\lambda \to \infty} L_\lambda . \end{cases} \tag{1.2}$$

By condition (ii), $0 \leqq L_\infty \leqq L_\mu < \infty$.

Our first result is the following

THEOREM 1.1. *Let A be the infinitesimal generator of a cosine family $\{C(t); t \in R\}$ on X. If $B \in \mathfrak{P}(A)$, then for each ε with $|\varepsilon| < L_\infty^{-1}$, where L_∞ is defined by* (1.2), *the closure $\overline{(A+\varepsilon B)|_D}$ of $(A+\varepsilon B)|_D$ is the infinitesimal generator of a cosine family on X and $D(\overline{(A+\varepsilon B)|_D}) = D(A)$.*

The following properties about $\{C(t); t \in R\}$ are well known (see [5], [1] and [3]):

(1.3) there exist constants $K \geqq 1$ and $\omega \geqq 0$ such that $\|C(t)\| \leqq Ke^{\omega|t|}$ for $t \in R$;

$$C(-t) = C(t) \quad \text{for} \quad t \in R ; \tag{1.4}$$

(1.5) the infinitesimal generator A is a densely defined closed linear operator;

(1.6) $C(t)D(A) \subset D(A)$ and $C(t)Ax = AC(t)x$ for $t \in R$ and $x \in D(A)$;

(1.7) $\int_0^t\left[\int_0^s C(r)xdr\right]ds \in D(A)$ and $A\int_0^t\left[\int_0^s C(r)xdr\right]ds = C(t)x - x$ for $t \in R$ and $x \in X$;

(1.8) if $\lambda > \omega$ then $\lambda^2 \in \rho(A)$ (the resolvent set of A) and

$$R(\lambda^2; A)x=\lambda^{-1}\int_0^\infty e^{-\lambda t}C(t)xdt=\int_0^\infty e^{-\lambda t}S(t)xdt$$

for $x\in X$, where $R(\lambda^2; A)=(\lambda^2 I-A)^{-1}$ (the resolvent of A).

Before giving the proof of Theorem 1.1 we prove the following

LEMMA 1.2. *Let A be the infinitesimal generator of a cosine family $\{C(t); t\in R\}$, and let D be a dense linear subset of X such that $D\subset D(A)$ and $C(t)D\subset D$ for $t\in R$. Then $\overline{A|_D}=A$.*

PROOF. Since A is a closed extension of $A|_D$, it is clear that $A|_D$ is closable and $\overline{A|_D}\subset A$. To show $D(A)\subset D(\overline{A|_D})$, let $x\in D(A)$ and choose $x_n\in D$ with $\lim_{n\to\infty}x_n=x$. Since $C(r)x_n\in D$ and $(\overline{A|_D})C(r)x_n=AC(r)x_n=C(r)Ax_n$ is continuous in $r\in R$, the closedness of $\overline{A|_D}$ implies that $\int_0^s C(r)x_n dr\in D(\overline{A|_D})$ and $(\overline{A|_D})\int_0^s C(r)x_n dr=\int_0^s C(r)Ax_n dr$. Using again the closedness of $\overline{A|_D}$, we have $\int_0^t\left[\int_0^s C(r)x_n dr\right]ds\in D(\overline{A|_D})$ and hence

$$(\overline{A|_D})\int_0^t\left[\int_0^s C(r)x_n dr\right]ds=A\int_0^t\left[\int_0^s C(r)x_n dr\right]ds=C(t)x_n-x_n$$

by (1.7). Letting $n\to\infty$, we have that $\int_0^t\left[\int_0^s C(r)xdr\right]ds\in D(\overline{A|_D})$ and $(\overline{A|_D})\int_0^t\left[\int_0^s C(r)xdr\right]ds=C(t)x-x$ for $t\in R$. Since

$$2t^{-2}\int_0^t\left[\int_0^s C(r)xdr\right]ds\to x \quad\text{and}\quad 2t^{-2}(C(t)x-x)\to Ax \quad\text{as}\quad t\to 0\,,$$

we obtain that $x\in D(\overline{A|_D})$ and $(\overline{A|_D})x=Ax$. Q.E.D.

PROOF OF THEOREM 1.1. For each non-negative integer n and $t\in R$ we define a bounded linear operator $C_n(t)$ on D as follows:

$$(1.9)\qquad C_0(t)x=C(t)x\,,\qquad C_n(t)x=\int_0^t \bar{C}_{n-1}(t-s)\left[\int_0^s BC(r)xdr\right]ds \quad\text{for}\quad x\in D\,,$$

where $\bar{C}_{n-1}(t)$ denotes the extension of $C_{n-1}(t)$ onto X. To observe that $C_n(t)$ are well defined and bounded on D, we show that for every n and $x\in D$, $C_n(t)x\colon R\to X$ is continuous and

$$(1.10)\qquad \|C_n(t)x\|\leqq K(L_\lambda)^n e^{\lambda|t|}\|x\| \quad\text{for}\quad \lambda\geqq\max\{\omega,\mu\} \quad\text{and}\quad t\in R\,,$$

where L_λ is the constant in (1.2). Let $x\in D$ and $\lambda\geqq\max\{\omega,\mu\}$. Since $BC(t)x\colon R\to X$ is continuous and $\bar{C}_0(t)=C(t)$ for $t\in R$,

$$C_1(t)x=\int_0^t C(t-s)\left[\int_0^s BC(r)xdr\right]ds$$

is well defined and continuous on R. Moreover, by (1.3) and the definition of L_λ,

$$\|C_1(t)x\|\leqq\left|\int_0^t Ke^{\lambda|t-s|}\left\|\int_0^s BC(r)xdr\right\|ds\right|$$
$$\leqq Ke^{\lambda|t|}\int_0^{|t|} e^{-\lambda s}\left\|\int_0^s BC(r)xdr\right\|ds\leqq KL_\lambda e^{\lambda|t|}\|x\| \quad \text{for} \quad t\in R\,.$$

Since D is dense in X, $C_1(t)$ can be extended onto X. Suppose that (1.10) holds for $n=k$ and $C_k(t)z\colon R\to X$ is continuous for each $z\in D$. Then $C_k(t)$ has the extension $\bar{C}_k(t)$ for each $t\in R$, $\bar{C}_k(t)z\colon R\to X$ is continuous for every $z\in X$ and $\|\bar{C}_k(t)\|\leqq K(L_\lambda)^k e^{\lambda|t|}$ for $t\in R$. Therefore $C_{k+1}(t)x=\int_0^t \bar{C}_k(t-s)\left[\int_0^s BC(r)xdr\right]ds$ is well defined and continuous on R, and

$$\|C_{k+1}(t)x\|\leqq K(L_\lambda)^k\left|\int_0^t e^{\lambda|t-s|}\left\|\int_0^s BC(r)xdr\right\|ds\right|\leqq K(L_\lambda)^{k+1}e^{\lambda|t|}\|x\|$$

for $t\in R$. This shows that for every non-negative integer n, $C_n(t)x\colon R\to X$ is well defined and continuous for each $x\in D$, and (1.10) holds. Consequently, for every $n\geqq 0$, $\bar{C}_n(t)x\colon R\to X$ is continuous for each $x\in X$ and

(1.11) $$\|\bar{C}_n(t)\|\leqq K(L_\lambda)^n e^{\lambda|t|} \quad \text{for} \quad \lambda\geqq\max\{\omega,\mu\} \quad \text{and} \quad t\in R\,.$$

Let $|\varepsilon|<L_\infty^{-1}$ and choose $\lambda_0\geqq\max\{\omega,\mu\}$ such that $|\varepsilon|L_{\lambda_0}<1$. Then $\|\varepsilon^n\bar{C}_n(t)\|\leqq K(|\varepsilon|L_{\lambda_0})^n e^{\lambda_0|t|}$ by (1.11) and hence $\sum_{n=0}^\infty \varepsilon^n\bar{C}_n(t)$ converges uniformly in t on every compact interval. Define $\hat{C}(t)\in B(X)$ by

(1.12) $$\hat{C}(t)=\sum_{n=0}^\infty \varepsilon^n\bar{C}_n(t) \quad \text{for} \quad t\in R\,.$$

We now prove that $\{\hat{C}(t); t\in R\}$ is a cosine family on X. Clearly, $\hat{C}(t)x\colon R\to X$ is continuous for every $x\in X$, $\hat{C}(0)=I$ and

(1.13) $$\|\hat{C}(t)\|\leqq K(1-|\varepsilon|L_{\lambda_0})^{-1}e^{\lambda_0|t|} \quad \text{for} \quad t\in R\,.$$

To see that

(1.14) $$\hat{C}(t+s)+\hat{C}(t-s)=2\hat{C}(t)\hat{C}(s) \quad \text{for all} \quad s,t\in R\,,$$

we first show

(1.15) $$\bar{C}_n(t+s)+\bar{C}_n(t-s)=2\sum_{k=0}^n \bar{C}_k(t)\bar{C}_{n-k}(s)$$

for $t, s \in R$ and $n=0, 1, 2, \cdots$. Since $\bar{C}_0(t)=C(t)$, (0.1) shows that (1.15) holds for $n=0$. Assume that (1.15) holds. Then for every $x \in D$

$$\begin{aligned}
2\sum_{k=0}^{n+1}\bar{C}_k(t)\bar{C}_{n+1-k}(s)x &= 2\sum_{k=0}^{n}\bar{C}_k(t)\bar{C}_{n+1-k}(s)x+2\bar{C}_{n+1}(t)C(s)x \\
&= 2\sum_{k=0}^{n}\bar{C}_k(t)\int_0^s \bar{C}_{n-k}(s-r)\left[\int_0^r BC(\tau)xd\tau\right]dr+2\int_0^t \bar{C}_n(t-r)\left[\int_0^r BC(\tau)C(s)xd\tau\right]dr \\
&= \int_0^s [\bar{C}_n(t+s-r)+\bar{C}_n(t-s+r)]\left(\int_0^r BC(\tau)xd\tau\right)dr \\
&\quad +\int_0^t \bar{C}_n(t-r)\left[\int_0^{r+s} BC(\tau)xd\tau+\int_0^{r-s} BC(\tau)xd\tau\right]dr \\
&= \int_0^s \bar{C}_n(t+s-r)\left[\int_0^r BC(\tau)xd\tau\right]dr+\int_0^{-s}\bar{C}_n(t-s-r)\left[\int_0^r BC(\tau)xd\tau\right]dr \\
&\quad +\int_s^{t+s}\bar{C}_n(t+s-r)\left[\int_0^r BC(\tau)xd\tau\right]dr+\int_{-s}^{t-s}\bar{C}_n(t-s-r)\left[\int_0^r BC(\tau)xd\tau\right]dr \\
&= \bar{C}_{n+1}(t+s)x+\bar{C}_{n+1}(t-s)x\,. \quad \text{(We used (1.4) here.)}
\end{aligned}$$

Therefore (1.15) holds true for every n. By virtue of (1.15)

$$\begin{aligned}
\hat{C}(t+s)+\hat{C}(t-s) &= \sum_{n=0}^{\infty}\varepsilon^n(\bar{C}_n(t+s)+\bar{C}_n(t-s)) \\
&= 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\varepsilon^k\bar{C}_k(t)\varepsilon^{n-k}\bar{C}_{n-k}(s)=2\left(\sum_{m=0}^{\infty}\varepsilon^m\bar{C}_m(t)\right)\left(\sum_{m=0}^{\infty}\varepsilon^m\bar{C}_m(s)\right) \\
&= 2\hat{C}(t)\hat{C}(s) \quad \text{for all} \quad t, s \in R\,.
\end{aligned}$$

Thus (1.14) holds true, and hence $\{\hat{C}(t); t \in R\}$ is a cosine family on X.

Let $\hat{A}$ be the infinitesimal generator of $\{\hat{C}(t); t \in R\}$. By the definition of $\hat{C}(t)$ we have

$$\hat{C}(t)x=C(t)x+\varepsilon\int_0^t \hat{C}(t-s)\left[\int_0^s BC(r)xdr\right]ds \quad \text{for} \quad x \in D \quad \text{and} \quad t \in R\,. \tag{1.16}$$

Hence

$$\begin{aligned}
\lim_{t\to 0} 2t^{-2}(\hat{C}(t)x-x) &= \lim_{t\to 0} 2t^{-2}(C(t)x-x) \\
&\quad +\lim_{t\to 0} 2t^{-2}\varepsilon\int_0^t \hat{C}(t-s)\left[\int_0^s BC(r)xdr\right]ds=Ax+\varepsilon Bx
\end{aligned}$$

for $x \in D$. This shows that $\hat{A}$ is a closed extension of $(A+\varepsilon B)|_D$ and $\overline{(A+\varepsilon B)|_D}\subset\hat{A}$. It follows from (1.13), (1.8) and (1.16) that if $\lambda>\lambda_0$, then $\lambda^2 \in \rho(\hat{A})\cap\rho(A)$ and

$$\lambda R(\lambda^2; \hat{A})x-\lambda R(\lambda^2; A)x=\lambda R(\lambda^2; \hat{A})\varepsilon B_\lambda x \quad \text{for} \quad x \in D\,, \tag{1.17}$$

where $B_\lambda x=\int_0^\infty e^{-\lambda s}\left[\int_0^s BC(r)xdr\right]ds$. By our assumption (1.1), B_λ can be extended onto X and the extension $\bar{B}_\lambda$ satisfies

$$||\varepsilon\bar{B}_\lambda||\leqq|\varepsilon|L_{\lambda_0}<1 \quad \text{for every} \quad \lambda>\lambda_0 .$$

Thus $(I-\varepsilon\bar{B}_\lambda)^{-1}\in B(X)$ and $R(\lambda^2; \hat{A})(I-\varepsilon\bar{B}_\lambda)=R(\lambda^2; A)$ by (1.17), where $\lambda>\lambda_0$. Consequently $D(\hat{A})=D(A)$. To show $\overline{(A+\varepsilon B)|_D}\supset\hat{A}$, note that

$$||\hat{A}x||\leqq\lambda^2||\varepsilon\bar{B}_\lambda||\,||x||+||I-\varepsilon\bar{B}_\lambda||\,||Ax|| \quad \text{for} \quad x\in D(A)=D(\hat{A}) , \tag{1.18}$$

where $\lambda>\lambda_0$. (This is easily obtained from $R(\lambda^2; \hat{A})(I-\varepsilon\bar{B}_\lambda)=R(\lambda^2; A)$.) Let $x\in D(\hat{A})$. Since $D(\hat{A})=D(A)$ and $\overline{A|_D}=A$ by Lemma 1.2, there exist $x_n\in D$ such that $x_n\to x$ and $Ax_n\to Ax$ as $n\to\infty$. Then $[\overline{(A+\varepsilon B)|_D}]x_n=\hat{A}x_n\to\hat{A}x$ as $n\to\infty$ by (1.18), and hence $x\in D(\overline{(A+\varepsilon B)|_D})$ and $\hat{A}x=[\overline{(A+\varepsilon B)|_D}]x$. This completes the proof.

The following theorem is useful for applications, and a corresponding result for (C_0)-semigroups has been obtained by Voigt [7] and Miyadera [2].

THEOREM 1.3. *Let A be the infinitesimal generator of a cosine family $\{C(t); t\in R\}$ on X, and let B be a linear operator in X. Assume the following conditions* (a_1) *and* (a_2):

(a_1) *there exists a dense linear subset D of X such that $D\subset D(A)\cap D(B)$, $C(t)D\subset D$ for $t\in R$ and $BC(t)x: R\to X$ is continuous for every $x\in D$;*

(a_2) *there exists an $a>0$ such that*

$$\sup\left\{\left\|\int_0^a BC(s)xds\right\|; x\in D,\ ||x||\leqq1\right\}<\infty , \quad \textit{and} \tag{1.19}$$

$$\sup\left\{\int_0^a\left\|\int_0^t BC(s)xds\right\|dt; x\in D,\ ||x||\leqq1\right\}<\infty . \tag{1.20}$$

Set $K_\lambda=\sup\left\{\int_0^a e^{-\lambda t}\left\|\int_0^t BC(s)xds\right\|dt; x\in D,\ ||x||\leqq1\right\}$ *for* $\lambda\geqq0$ *and* $K_\infty=\lim_{\lambda\to\infty}K_\lambda$. *(Note that $0\leqq K_\infty\leqq K_0<\infty$.)*

Then for each ε with $|\varepsilon|<K_\infty^{-1}$, the closure $\overline{(A+\varepsilon B)|_D}$ of $(A+\varepsilon B)|_D$ is the infinitesimal generator of a cosine family on X and $D(\overline{(A+\varepsilon B)|_D})=D(A)$.

REMARK. Since $\int_0^a\left[\int_0^t BC(s)xds\right]dt=a\int_0^a BC(s)xds-\int_0^a sBC(s)xds$, (1.19) is replaced by

$$\sup\left\{\left\|\int_0^a sBC(s)xds\right\|; x\in D,\ ||x||\leqq1\right\}<\infty . \tag{1.19$'$}$$

To prove the theorem we prepare the following

LEMMA 1.4. *Under the hypothesis of Theorem* 1.3, *we have*

(1.21) $$\int_0^\infty e^{-\lambda t}\left\|\int_0^t BC(s)xds\right\|dt \leqq L'_\lambda \|x\| \quad for\ x\in D \quad and \quad \lambda>\omega+(\log K)/a$$

where K and ω are constants in (1.3) *and*

(1.22) $$L'_\lambda = K_\lambda[1+Ke^{-a(\lambda-\omega)}/(1-e^{-a(\lambda-\omega)})] + LK^3 e^{-a(\lambda-\omega)}/(\lambda-\omega)(1-Ke^{-a(\lambda-\omega)})^2 .$$

PROOF. For each $x\in D$, $2BC(r)C(s)x=BC(r+s)x+BC(r-s)x$ and hence $2\int_0^t BC(r)C(s)xdr=\int_s^{t+s}BC(r)xdr+\int_{-s}^{t-s}BC(r)xdr=\int_0^{t+s}BC(r)xdr+\int_0^{t-s}BC(r)xdr$. Similarly $2\int_0^s BC(r)C(t)xdr=\int_0^{t+s}BC(r)xdr+\int_0^{s-t}BC(r)xdr=\int_0^{t+s}BC(r)xdr-\int_0^{t-s}BC(r)xdr$ for $x\in D$. Adding both equalities we obtain

(1.23) $$\int_0^{t+s} BC(r)xdr=\int_0^t BC(r)C(s)xdr+\int_0^s BC(r)C(t)xdr$$

for $t, s\in R$ and $x\in D$.

Put $L=\sup\left\{\left\|\int_0^a BC(s)xds\right\|;\ x\in D,\ \|x\|\leqq 1\right\}$. Then we have

(1.24) $$\left\|\int_0^{na} BC(s)xds\right\|\leqq nLK(Ke^{a\omega})^n\|x\| \quad \text{for} \quad x\in D \quad \text{and} \quad n=1, 2, \cdots .$$

In fact, (1.24) holds for $n=1$. Assume that (1.24) holds for $n=k$. Then for $x\in D$

$$\begin{aligned}\left\|\int_0^{(k+1)a} BC(s)xds\right\| &= \left\|\int_0^{ka} BC(r)C(a)xdr+\int_0^a BC(r)C(ka)xdr\right\| \quad (\text{by } (1.23))\\ &\leqq kLK(Ke^{a\omega})^k\|C(a)x\|+L\|C(ka)x\|\leqq kLK(Ke^{a\omega})^{k+1}\|x\|\\ &\quad +LK(e^{a\omega})^k\|x\|\leqq(k+1)LK(Ke^{a\omega})^{k+1}\|x\| .\end{aligned}$$

Therefore (1.24) holds true.

Let $x\in D$ and $\lambda>\omega+a^{-1}\log K$. Then

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\left\|\int_0^t BC(s)xds\right\|dt &= \sum_{k=0}^\infty \int_{ka}^{(k+1)a} e^{-\lambda t}\left\|\int_0^t BC(s)xds\right\|dt\\ &= \sum_{k=0}^\infty e^{-\lambda ak}\int_0^a e^{-\lambda t}\left\|\int_0^{ak+t} BC(s)xds\right\|dt\\ &\leqq \sum_{k=0}^\infty e^{-\lambda ak}\int_0^a e^{-\lambda t}\left[\left\|\int_0^{ka} BC(r)C(t)xdr\right\|\right.\end{aligned}$$

$$+\left\|\int_0^t BC(r)C(ka)xdr\right\|\Big]dt \quad \text{(by (1.23))}$$

$$\leqq \sum_{k=0}^{\infty} e^{-\lambda ak}\Big[kLK(Ke^{a\omega})^k \int_0^a e^{-\lambda t}\|C(t)x\|\,dt + K_\lambda\|C(ka)x\|\Big] \quad \text{(by (1.24))}$$

$$\leqq K_\lambda\Big[1+K\sum_{k=1}^{\infty}(e^{-(\lambda-\omega)a})^k\Big]\|x\| + (LK^2/(\lambda-\omega))\sum_{k=0}^{\infty} k(Ke^{-a(\lambda-\omega)})^k\|x\|$$

$$=L'_\lambda\|x\| \,.$$

Q.E.D.

PROOF OF THEOREM 1.3. By virtue of Lemma 1.4, $B\in\mathfrak{P}(A)$ and $L_\lambda\leqq L'_\lambda$ for $\lambda>\omega+a^{-1}\log K$, where L_λ is the constant in (1.2). Since $\lim_{\lambda\to\infty} L'_\lambda=\lim_{\lambda\to\infty} K_\lambda=K_\infty$, we have $L_\infty=\lim_{\lambda\to\infty} L_\lambda\leqq K_\infty$. The conclusion follows from Theorem 1.1. Q.E.D.

§2. Applications.

As a consequence of Theorem 1.1 we have the following which is the main result in [4]:

COROLLARY 2.1. *Let $\{C(t);\, t\in R\}$ be a cosine family on X, with the infinitesimal generator A and the associated sine family $\{S(t);\, t\in R\}$. Assume that B is a linear operator in X satisfying*

(b$_1$) *$D(B)\supset D(A)$ and $BR(\mu^2; A)\in B(X)$ for some $\mu>\omega$, where ω is the constant in* (1.3), *and*

(b$_2$) *for some $\mu_0>0$*

$$\sup\left\{\int_0^\infty e^{-\mu_0 t}\|BS(t)x\|\,dt;\ x\in D(A),\ \|x\|\leqq 1\right\}<\infty \,.$$

Set $L(\lambda)=\sup\left\{\int_0^\infty e^{-\lambda t}\|BS(t)x\|\,dt;\ x\in D(A),\ \|x\|\leqq 1\right\}$ for $\lambda\geqq\mu_0$ and $L(\infty)=\lim_{\lambda\to\infty} L(\lambda)$.

Then for each ε with $|\varepsilon|<L(\infty)^{-1}$, $A+\varepsilon B$ (defined on $D(A)$) is also the infinitesimal generator of a cosine family on X.

PROOF. We show that (i) and (ii) (in the definition of $\mathfrak{P}(A)$) with D replaced by $D(A)$ are satisfied and $L(\infty)=L_\infty$ (the constant in (1.2)). By $BR(\mu^2; A)\in B(X)$, $BC(t)R(\mu^2; A)z=BR(\mu^2; A)C(t)z\colon R\to X$ is continuous for every $z\in X$. Thus (i) with D replaced by $D(A)$ is satisfied. (Note (1.5) and (1.6).) Next, by $S(t)R(\mu^2; A)z=R(\mu^2; A)\int_0^t C(s)zds$ and (b$_1$), we have $BS(t)R(\mu^2; A)z=BR(\mu^2; A)\int_0^t C(s)zds=\int_0^t BR(\mu^2; A)C(s)zds=\int_0^t BC(s)R(\mu^2; A)zds$ for $z\in X$ and $t\in R$, i.e.,

$$BS(t)x=\int_0^t BC(s)xds \quad \text{for} \quad x\in D(A) \quad \text{and} \quad t\in R \,. \tag{2.1}$$

Therefore (b_2) implies (ii) with D replaced by $D(A)$, and $L(\lambda)=L_\lambda$ for $\lambda \geqq \mu_0$ and then $L(\infty)=L_\infty$. Q.E.D.

As corollaries of Theorem 1.3 we have the following:

COROLLARY 2.2. *Let $\{C(t); t \in R\}$ be a cosine family on X, with the infinitesimal generator A and the associated sine family $\{S(t); t \in R\}$. Assume that B is a linear operator in X satisfying* (b_1).

(I) *If there exists an $a>0$ such that*

$$\sup\{\|BS(a)x\|; x \in D(A), \|x\| \leqq 1\} < \infty , \quad \textit{and} \tag{2.2}$$

$$\sup\left\{\int_0^a \|BS(t)x\|dt; x \in D(A), \|x\| \leqq 1\right\} < \infty , \tag{2.3}$$

then for each ε with $|\varepsilon| < K(\infty)^{-1}$, $A+\varepsilon B$ (defined on $D(A)$) is also the infinitesimal generator of a cosine family on X, where $K(\lambda)=\sup\left\{\int_0^a e^{-\lambda t}\|BS(t)x\|dt; x \in D(A), \|x\| \leqq 1\right\}$ for $\lambda \geqq 0$ and $K(\infty)=\lim_{\lambda\to\infty} K(\lambda)$.

(II) *If there exists a non-negative function $\varphi \in L^1(0, c)$ for some $c>0$ such that*

$$\textit{for a.e. } t \in (0, c), \quad \|BS(t)x\| \leqq \varphi(t)\|x\| \quad \textit{for all} \quad x \in D(A), \tag{2.4}$$

then for every complex number ε, $A+\varepsilon B$ is the infinitesimal generator of a cosine family on X.

PROOF. Similarly as in the proof of Corollary 2.1, (a_1) (in Theorem 1.3) with D replaced by $D(A)$ is satisfied, and (2.1) holds true. Thus (I) is a direct consequence of Theorem 1.3. We now prove (II). By our assumption there exists an $a \in (0, c)$ such that $\|BS(a)x\| \leqq \varphi(a)\|x\|$ for all $x \in D(A)$ and $\varphi(a) < \infty$. Moreover,

$$\begin{aligned} K(\lambda) &= \sup\left\{\int_0^a e^{-\lambda t}\|BS(t)x\|dt; x \in D(A), \|x\| \leqq 1\right\} \\ &\leqq \int_0^a e^{-\lambda t}\varphi(t)dt \to 0 \quad \text{as} \quad \lambda \to \infty ; \end{aligned}$$

hence $K(\infty)=0$. Therefore (II) follows from (I). Q.E.D.

REMARK. Theorem 3.2 in [4] is a corollary of Corollary 2.2 (II).

COROLLARY 2.3. *Let $\{C(t); t \in R\}$ be a cosine family on X, with the infinitesimal generator A and the associated sine family $\{S(t); t \in R\}$. Assume that B is a closed linear operator in X satisfying $S(t)X \subset D(B)$ for all $t \in R$.*

(I$'$) *If* $BS(\cdot)x \in L^1((0, a); X)$ *for every* $x \in X$, *where* a *is some positive number, then* $K(\infty)$ *defined in Corollary* 2.2 (I) *is finite, and for each* ε *with* $|\varepsilon| < K(\infty)^{-1}$, $A + \varepsilon B$ *is the infinitesimal generator of a cosine family on* X.

(II$'$) *If* $BS(\cdot)x \in L^p((0, a); X)$ *with* $p > 1$ *for every* $x \in X$, *where* a *is some positive number, then for every complex number* ε, $A + \varepsilon B$ *is the infinitesimal generator of a cosine family on* X.

PROOF. (I$'$) We first show that (2.2) and (2.3) are satisfied. By the closed graph theorem, $BS(t) \in B(X)$ for every $t \in R$ and hence (2.2) is satisfied. To see that (2.3) is satisfied, define an operator T from X into the Banach space $L^1((0, a); X)$ with norm $|||u||| = \int_0^a ||u(t)||\,dt$ as follows:

$$Tx = BS(\cdot)x \quad \text{for} \quad x \in X\,. \tag{2.5}$$

Clearly T is linear. To see that T is closed, let $||x_n - x|| \to 0$ and $\int_0^a ||BS(t)x_n - u(t)||\,dt = |||Tx_n - u||| \to 0$ as $n \to \infty$. Then there exists a subsequence $\{n_k\}$ such that $\lim_{k\to\infty} ||BS(t)x_{n_k} - u(t)|| = 0$ for a.e. $t \in (0, a)$. Since $\lim_{k\to\infty} ||S(t)x_{n_k} - S(t)x|| = 0$ for all $t \in R$, it follows from the closedness of B that $u(t) = BS(t)x$ for a.e. $t \in (0, a)$, i.e., $u = Tx$. Thus T is closed, and then T is bounded by the closed graph theorem. Consequently $\int_0^a ||BS(t)x||\,dt = |||Tx||| \leqq ||T||\,||x||$ for all $x \in X$ and (2.3) is satisfied.

We next prove that (b_1) in Corollary 2.1 is satisfied. Since

$$S(t+s) = S(t)C(s) + S(s)C(t) \quad \text{for} \quad t, s \in R \tag{2.6}$$

and $BS(t) \in B(X)$ for $t \in R$, $BS(t)x = BS(t-na)C(na)x + BS(na)C(t-na)x$ is Bochner integrable on $(na, (n+1)a)$ for each $n = 0, 1, 2, \cdots$ and $x \in X$. This shows that for each $x \in X$, $BS(t)x$ is Bochner integrable on every finite interval of $[0, \infty)$. Using $BS(t+s) = BS(t)C(s) + BS(s)C(t)$ instead of (1.23), the same argument as in the proof of Lemma 1.4 implies that $||BS(na)|| \leqq n\,||BS(a)||\,K(Ke^{a\omega})^n$ for $n = 1, 2, \cdots$ and then $e^{-\lambda t}BS(t)x$ is Bochner integrable on $(0, \infty)$ for every $x \in X$ if $\lambda > \omega + a^{-1}\log K$. Moreover $e^{-\lambda t}S(t)x$ is Bochner integrable on $(0, \infty)$ if $\lambda > \omega$. Therefore, by the closedness of B, we have that if $\lambda > \omega + a^{-1}\log K$, then $\int_0^\infty e^{-\lambda t}S(t)x\,dt \in D(B)$ for all $x \in X$. Combining this with (1.8), we obtain $D(A) = R(\lambda^2; A)X \subset D(B)$ and then $BR(\lambda^2; A) \in B(X)$ for every $\lambda > \omega$ by the closed graph theorem. Thus (b_1) is satisfied and the conclusion follows from Corollary 2.2 (I).

(II$'$) Define an operator T from X into the Banach space $L^p((0, a); X)$

by (2.5). Then we see that T is closed and then T is bounded. Hence

$$\int_0^a e^{-\lambda t}\|BS(t)x\|\,dt \leqq (1/\lambda q)^{1/q}\left(\int_0^a \|BS(t)x\|^p\,dt\right)^{1/p} \leqq (1/\lambda q)^{1/q}\|T\|\,\|x\|$$

for all $x \in X$, and $K(\infty)=\lim_{\lambda\to\infty} K(\lambda) \leqq \lim_{\lambda\to\infty}(1/\lambda q)^{1/q}\|T\|=0$. The conclusion follows from (I'). Q.E.D.

REMARK. The main result in [6] is included in Corollary 2.3 (II').

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