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# On the Partition Problem in an Algebraic Number Field

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Let $p(n)$ be the number of the partitions of $n$, then the generating function of $p(n)$ is given by

$$f(x)=\prod_{n=1}^{\infty}(1-x^n)^{-1}=1+\sum_{n=1}^{\infty}p(n)x^n \qquad (|x|<1) .$$

In 1917, Hardy and Ramanujan [1] proved the asymptotic formula for $p(n)$:

$$p(n)\sim\frac{1}{4\sqrt{3}\,n}\exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad (n\to\infty) .$$

In 1934, Wright [11] studied the partition problem of $n$ into $k$-th powers of integers. In this case, the generating function is

$$f_k(x)=\prod_{n=1}^{\infty}(1-x^{n^k})^{-1}=1+\sum_{n=1}^{\infty}p_k(n)x^n \qquad (|x|<1)$$

and Wright obtained the asymptotic formula for $p_k(n)$:

$$p_k(n)\sim\frac{A_k k^{1/2}n^{-3/2+1/(k+1)}}{(2\pi)^{(k+1)/2}(k+1)^{3/2}}\exp\left(A_k n^{1/(1+k)}\right) ,$$

where

$$A_k=(k+1)\left\{\frac{1}{k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right\}^{k/(k+1)} .$$

In 1950, Rademacher [7] suggested the problem of generalizing the partition function to algebraic number field. Three years later, in 1953, Meinardus [4] succeeded in obtaining the asymptotic formula for the partition function in a real quadratic field: Let $K$ be a real quadratic field and define the infinite product

$$f(\tau,\tau')=\prod_{\nu}(1-e^{-\nu\tau-\nu'\tau'})^{-1} ,$$

where the product is taken over all totally positive integers $\nu$ of $K$, $\nu'$

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is the conjugate of $\nu$, $\tau$ and $\tau'$ are complex parameters with positive real parts. $f(\tau, \tau')$ is expanded in a series:

$$f(\tau, \tau')=1+\sum_{\mu} P(\mu)e^{-\mu\tau-\mu'\tau'},$$

where $P(\mu)$ is the partition function corresponding to $p(n)$ in rational case. Meinardus proved that

$$P(\mu)\sim \exp\left\{3\left(\frac{\zeta(3)}{\sqrt{D}}N(\mu)\right)^{1/3}+\alpha_1\log^2 N(\mu)+\alpha_2\log N(\mu)+\beta(\mu)\right\}$$

as $N(\mu)\to\infty$, where $\alpha_1$, $\alpha_2$ are real constants, $\beta(\mu)$ is a number connected with $\mu$ and zeta functions with Grössencharacters, and $D$ is the absolute value of the discriminant of $K$.

In this paper, we shall generalize the partition problem to an algebraic number field and prove the asymptotic formula for the partition function, which includes all above results as special cases.

Let $K$ be an algebraic number field of degree $n$. This and the following notations will be used throughout this paper. Let $K^{(q)}$ $(q=1, \cdots, r_1)$ be the real conjugates of $K$, $K^{(p)}$, $K^{(p+r_2)}=\overline{K^{(p)}}$ $(p=r_1+1, \cdots, r_1+r_2)$ the pairs of complex conjugates of $K$, so that $n=r_1+2r_2$. We denote by $\mathfrak{O}$ the ideal consisting of all integers of $K$, by $\mathfrak{d}$ the differente of $K$, and by $D=N(\mathfrak{d})$ (norm of $\mathfrak{d}$) the absolute value of the discriminant of $K$. If $\mu$ is a number of $K$, we have an $n$-dimensional complex vector $(\mu^{(1)}, \cdots, \mu^{(n)})$ with real $\mu^{(q)}$ $(q=1, \cdots, r_1)$ and complex $\mu^{(p)}$, $\mu^{(p+r_2)}=\overline{\mu^{(p)}}$ $(p=r_1+1, \cdots, r_1+r_2)$, where $\mu^{(i)}$ is the conjugate of $\mu$ in $K^{(i)}$ $(i=1, \cdots, n)$. We shall denote this vector also by $\mu$. We shall consider more generally any $n$-dimensional complex vector $\xi=(\xi_1, \cdots, \xi_n)$ with real $\xi_1, \cdots, \xi_{r_1}$ and complex $\xi_{p+r_2}=\bar{\xi}_p$ $(p=r_1+1, \cdots, r_1+r_2)$. For such $\xi$, we write

$$S(\xi)=\sum_{i=1}^{n}\xi_i\,, \qquad N(\xi)=\prod_{i=1}^{n}\xi_i$$

and denote by $x(\xi)$ the $n$-dimensional real vector

$$x(\xi)=(\xi_1, \cdots, \xi_{r_1}, \mathrm{Re}(\xi_{r_1+1}), \cdots, \mathrm{Re}(\xi_{r_1+r_2}), \mathrm{Im}(\xi_{r_1+1}), \cdots, \mathrm{Im}(\xi_{r_1+r_2}))\,.$$

If $|\xi_i|\leqq a$ (or $\geqq a$) $(i=1, \cdots, n)$ for $\xi=(\xi_1, \cdots, \xi_n)$, then we write $|\xi|\leqq a$ (or $|\xi|\geqq a$). We call $\gamma$ of $K$ totally positive number, if $\gamma^{(1)}, \cdots, \gamma^{(r_1)}$ are positive. When $r_1=0$, totally positive number means non-vanishing number. We write $\gamma\succ 0$, if $\gamma$ is totally positive.

Let $y_1, \cdots, y_n$ be positive numbers such that $y_p=y_{p+r_2}$ $(p=r_1+1, \cdots, r_1+r_2)$. Let $z_1, \cdots, z_{r_1}$ be real numbers and $z_p$, $z_{p+r_2}=\bar{z}_p$ $(p=r_1+1, \cdots,$

$r_1+r_2$) the pairs of complex numbers. Let $\mathfrak{m}$ be an ideal, $\mu$ an integer. We define the function

$$f(y; z)=\prod_{\substack{\nu>0 \\ \nu\equiv\mu(\mathfrak{m})}}\{1-\exp(-S(|\nu|^k y)+2\pi i S(\nu^k z))\}^{-1},$$

where the product is taken over all totally positive integers $\nu$ congruent to $\mu$ modulo $\mathfrak{m}$. Let $\delta_1, \cdots, \delta_n$ be a basis of $\mathfrak{d}^{-1}$ and put

$$z_j=\sum_{i=1}^{n} x_i\delta_i^{(j)} \qquad (j=1, \cdots, n)$$

for real numbers $x_1, \cdots, x_n$. Using these $z_1, \cdots, z_n$ as the parameters in $f(y; z)$, we consider the integral

$$A(\nu; y)=\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} f(y; z)e^{-2\pi i S(\nu z)}\,dx_1\cdots dx_n$$

for totally positive integer $\nu$. Then we shall have the expansion of $f(y; z)$ in the series

$$f(y; z)=1+\sum_{\nu>0} A(\nu; y)e^{2\pi i S(\nu z)}. \tag{1}$$

We shall call this $A(\nu; y)$ the partition function and our main result is stated as follows;

MAIN THEOREM. *Let $\nu$ be totally positive and $\nu\in\mathfrak{m}_0^k$, where $\mathfrak{m}_0=(\mathfrak{m}, \mu)$. Put*

$$M=\left\{\pi^{r_2}\Gamma\left(\frac{1}{k}\right)^{r_1}\Gamma\left(\frac{2}{k}\right)^{r_2}\zeta\left(1+\frac{n}{k}\right)\frac{2^{2r_2(1-1/k)}N(\nu)^{1/k}}{k^{r_1(1-1/k)+r_2}\sqrt{D}N(\mathfrak{m})}\right\}^{k/(r_1+r_2+k)}$$

*and*

$$y_q=\frac{M}{k\nu^{(q)}} \qquad (q=1, \cdots, r_1),$$

$$y_p=\frac{\sqrt{M}}{|\nu^{(p)}|} \qquad (p=r_1+1, \cdots, r_1+r_2).$$

*Then we have*

$$A(\nu; y)\sim\frac{D^{1/2}M^{r_1/2}k^{2r_2-r_1/2+1/2}N(\mathfrak{m}_0)^k}{2^{r_1/2}\pi^{n/2}(2+k)^{r_2}(k+r_1)^{1/2}N(\nu)}\exp\left(-\frac{2k^2r_2}{2+k}+M+R(y)\right),$$

*as $N(\nu)\to\infty$. $R(y)$ is the sum of the residues of certain special functions defined in § 6 below.*

In § 1, after seeing the convergence of $f(y; z)$, we shall prove, in

Theorem 1.2, the expansion of $f(y; z)$ in the series (see (1) above). Then we shall write

$$A(\nu; y)=f(y; 0)\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}H(z)e^{-2\pi iS(\nu z)}dx_1\cdots dx_n\,, \tag{2}$$

where $H(z)=f(y; z)/f(y; 0)$. Making use of the properties of a certain exponential sum defined in § 2, we shall divide the domain of integration of (2) into a finite number of sets, $E_1(\gamma_j)$, $E_1(\sigma_i)$, $E_2$ and $E_3$ (see definitions in § 3). On each set, we shall estimate $H(z)$, which will give our main result.

Roughly speaking, the estimation of $H(z)$ will be established by two different methods. One of them is function-theoretical method (§§ 4–9), which is partly due to Rademacher [6]. We shall apply Hecke-Rademacher's transformation formula to our function $f(y; z)$ (§ 4), and obtain a fundamental formula for $\log f(y; z)$:

$$\log f(y; z)=\sum_{\lambda}\frac{1}{2\pi i}\int_{(\sigma)}\Psi_{\lambda}(s; y, z)ds \tag{3}$$

(see § 5). In § 6, we shall prove some function-theoretical lemmas on the properties of (3). After these lemmas, we shall be able to estimate $H(z)$ on the $E_1(\gamma_j)$, $E_1(\sigma_i)$ and $E_2$ (§§ 7–9). In particular, we shall see that $H(z)$ on the $E_1(\gamma_j)$ will lead to the main term of our asymptotic formula for $A(\nu; y)$.

Our second method for estimating $H(z)$ is the application of trigonometrical sums, by which we shall be able to estimate $H(z)$ on the remaining set $E_3$ (§§ 10, 11). In § 10, we shall introduce the Farey division defined by Siegel [8], and prove two theorems on a trigonometrical sum $S(z; N)$. Theorem 10.1 is analogous to that given by Siegel [8] and Theorem 10.2 is a simple consequence of three Lemmas 10.1$\sim$10.3, which will be quoted from Siegel [8] and Mitsui [5]. Using these theorems, we shall prove Theorem 11.1 concerning the difference

$$\mathrm{Re}\,S(z; N)-S(0; N) \tag{4}$$

and, after reducing the estimation of $H(z)$ to that of (4), we shall easily obtain the desired result (Theorem 11.2).

Collecting all above results, we shall prove, in § 12, the asymptotic formula for partition function, and, after this, we shall give two corollaries for special cases. Finally, in § 13, we shall obtain an estimation of $R(y)$.

Here we add some definitions and notations used in this paper. Let

$\gamma$ be a number of $K$ and put $\gamma\mathfrak{d}=\mathfrak{b}/\mathfrak{a}$ with integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $(\mathfrak{a}, \mathfrak{b})=1$. We call $\mathfrak{a}$ the denominator of $\gamma$ and denote this relation by $\gamma\to\mathfrak{a}$. A small Roman letter $c$ means positive constant which depends only on $K$. It does not always mean the same constant at each time it appears. We also use $c_1, c_2, \cdots$ in the same meaning. If $X$ and $Y$ are two numbers such that $|X|\leqq cY$, then we write $X=O(Y)$ or $X\ll Y$.

## § 1. Partition function and generating function.

Let $y_1, \cdots, y_n$ be positive numbers such that $y_p=y_{p+r_2}$ $(p=r_1+1, \cdots, r_1+r_2)$. Let $z_1, \cdots, z_{r_1}$ be real numbers, $z_p, z_{p+r_2}=\bar{z}_p$ $(p=r_1+1, \cdots, r_1+r_2)$ pairs of complex numbers. Let $\mathfrak{m}$ be an ideal of $K$, $\mu$ an integer. $\mathfrak{m}$ and $\mu$ are assumed to be given once for all. For these $y=(y_1, \cdots, y_n)$, $z=(z_1, \cdots, z_n)$, $\mathfrak{m}$, $\mu$ and an integer $k$ we put

$$e(\nu; y, z)=\exp\{-S(|\nu|^k y)+2\pi i S(\nu^k z)\}$$

and define the following function:

$$f(y; z)=f_k(y; z; \mathfrak{m}, \mu)=\prod_{\substack{\nu>0\\ \nu\equiv\mu(\mathfrak{m})}}(1-e(\nu; y, z))^{-1},$$

where the product is taken over totally positive integers $\nu$ congruent to $\mu$ modulo $\mathfrak{m}$. We also define the series:

$$g(y; z)=g_k(y; z; \mathfrak{m}, \mu)=\sum_{\substack{\nu>0\\ \nu\equiv\mu(\mathfrak{m})}} e(\nu; y, z).$$

THEOREM 1.1. *The product $f(y; z)$ and the series $g(y; z)$ are convergent uniformly in any $z_1, \cdots, z_n$ and $y_1, \cdots, y_n\geqq y_0>0$. $f(y; z)$ and $g(y; z)$ are connected in a formula*:

$$\log f(y; z)=\sum_{m=1}^{\infty}\frac{1}{m}g(my; mz). \tag{1.1}$$

PROOF. Let $y_0=\min(y_1, \cdots, y_n)$. Since the number of the integers $\nu$ such that $|\nu|\leqq N$ is $O(N^n)$, we have

$$\sum_{\nu>0} e(\nu; y, 0)\ll\sum_{N=1}^{\infty} N^n\exp(-cy_0N^k).$$

From this follow the convergence of $g(y; z)$ and that of $f(y; z)$. Moreover

$$\begin{aligned}\log f(y; z)&=-\sum_{\substack{\nu>0\\ \nu\equiv\mu(\mathfrak{m})}}\log(1-e(\nu; y, z))\\ &=\sum_{\substack{\nu>0\\ \nu\equiv\mu(\mathfrak{m})}}\sum_{m=1}^{\infty}\frac{1}{m}e(\nu; my, mz).\end{aligned}$$

Since the last double series is absolutely convergent, we have

$$\log f(y; z)=\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\substack{\nu>0\\ \nu\equiv\mu(\mathfrak{m})}} e(\nu; my, mz)=\sum_{m=1}^{\infty}\frac{1}{m}g(my; mz)\ .$$

Let $\delta_1, \cdots, \delta_n$ be the basis of $\mathfrak{d}^{-1}$ and put

$$z_j=\sum_{i=1}^{n} x_i \delta_i^{(j)} \qquad (j=1, \cdots, n)$$

for real numbers $x_1, \cdots, x_n$. Using these $z_1, \cdots, z_n$ as the parameters in $f(y; z)$, we consider the integral

$$A(\nu; y)=\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} f(y; z)e^{-2\pi i S(\nu z)}dx_1\cdots dx_n\ ,$$

where $\nu$ is a totally positive integer.

THEOREM 1.2. *$A(\nu; y)$ is the series of the following form:*

$$A(\nu; y)=\sum_{\substack{\nu=\nu_1^k+\cdots+\nu_s^k\\ \nu_j>0,\nu_j\equiv\mu(\mathfrak{m})}} \exp\{-S((|\nu_1|^k+\cdots+|\nu_s|^k)y)\}\ , \tag{1.2}$$

*where the sum is extended over all partitions of $\nu$ into $k$-th powers of totally positive integers congruent to $\mu$ modulo $\mathfrak{m}$. Using the $A(\nu; y)$, we have the expansion of $f(y; z)$ in a series:*

$$f(y; z)=1+\sum_{\nu>0} A(\nu; y)e^{2\pi i S(\nu z)}\ , \tag{1.3}$$

*where the sum is taken over all totally positive integers.*

PROOF. Let $m$ be a positive number. The finite product

$$f_m(y; z)=\prod_{\substack{0<|\nu|<m\\ \nu>0,\nu\equiv\mu(\mathfrak{m})}}(1-e(\nu; y, z))^{-1}$$

can be expanded into the series:

$$f_m(y; z)=1+\sum_{\nu>0} A_m(\nu)e^{2\pi i S(\nu z)}\ ,$$

where

$$\begin{aligned}A_m(\nu)&=\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} f_m(y; z)e^{-2\pi i S(\nu z)}dx_1\cdots dx_n\\ &=\sum_{\substack{\nu=\nu_1^k+\cdots+\nu_s^k\\ \nu_j>0,\nu_j\equiv\mu(\mathfrak{m}),0<|\nu_j|<m}} \exp\{-S((|\nu_1|^k+\cdots+|\nu_s|^k)y)\}\ .\end{aligned}$$

On the other hand, we see that

$$\lim_{m\to\infty} f_m(y; z) = f(y; z) ,$$

which is uniformly convergent in $z_1, \cdots, z_n$. Hence

$$\lim_{m\to\infty} A_m(\nu) = \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} f(y; z) e^{-2\pi i S(\nu z)} dx_1 \cdots dx_n = A(\nu; y)$$

and (1.2) is proved. For any positive $\varepsilon$, we can take $m_0 = m_0(\varepsilon)$ for which the inequality

$$f_m(y; 0) \geqq f(y; 0) - \varepsilon \qquad (m \geqq m_0)$$

holds. Since $f_m(y; 0) \leqq f(y; 0)$, if follows that

$$f(y; 0) - \varepsilon \leqq 1 + \sum_{\nu>0} A_m(\nu) = f_m(y; 0) \leqq f(y; 0) ,$$

which implies the convergence of $\sum A(\nu; y)$ and

$$f(y; 0) = 1 + \sum_{\nu>0} A(\nu; y) .$$

Hence the series on the right of (1.3) is also convergent. For any $m$, we have

$$|f(y; z) - 1 - \sum_{\nu>0} A(\nu; y) e^{2\pi i S(\nu z)}|$$
$$\leqq |f(y; z) - f_m(y; z)| + |f_m(y; z) - 1 - \sum_{\nu>0} A(\nu; y) e^{2\pi i S(\nu z)}| .$$

The last term does not exceed

$$\sum_{\nu>0} |A(\nu; y) e^{2\pi i S(\nu z)} - A_m(\nu) e^{2\pi i S(\nu z)}|$$
$$= \sum_{\nu>0} A(\nu; y) - \sum_{\nu>0} A_m(\nu) = f(y; 0) - f_m(y; 0) .$$

Hence (1.3) is proved by letting $m$ to infinity.

In general, $A(\nu; y)$ is of the form of infinite series. If $K$ is totally real, then $A(\nu; y)$ is a finite sum and we see that

$$A(\nu; y) = e^{-S(\nu y)} P(\nu) ,$$

where $P(\nu)$ is the number of the partitions of $\nu$ into the $k$-th powers of the integers which are totally positive and congruent to $\mu$ modulo $\mathfrak{m}$. Hence we can regard $A(\nu; y)$ as the partition function generalized to the case of algebraic number field. We call $f(y; z)$ the generating function of the $A(\nu; y)$.

Now we define the set of $z = (z_1, \cdots, z_n)$;

$$E=\left\{z \mid z_i=\sum_{j=1}^{n} x_j \delta_j^{(i)},\ -\frac{1}{2}\leqq x_j<\frac{1}{2}\quad (j=1, \cdots, n)\right\} .$$

Let $\phi$ be the mapping on $E$ defined by $\phi(z)=(x_1, \cdots, x_n)$. Using this $\phi$, we write $A(\nu; y)$ in the following form;

$$A(\nu; y)=f(y; 0)\int_{\phi(E)} H(z)e^{-2\pi i S(\nu z)}dx_1 \cdots dx_n , \tag{1.4}$$

where

$$H(z)=f(y; z)/f(y; 0) .$$

## § 2. An exponential sum.

We define the sum

$$G(\gamma)=G(\gamma; \mathfrak{m}, \mu)=\frac{1}{N(\mathfrak{a}_1)}\sum_{\sigma} e^{2\pi i S(\sigma^k \gamma)} , \tag{2.1}$$

where $\gamma\to\mathfrak{a}$, $\mathfrak{a}_1=\mathfrak{a}/(\mathfrak{m}, \mathfrak{a})$ and $\sigma$ runs through the residues mod $\mathfrak{a}$ such that $\sigma\equiv\mu \pmod{(\mathfrak{m}, \mathfrak{a})}$. The number of the terms is equal to $N(\mathfrak{a}_1)$, so $|G(\gamma)|\leqq 1$. More precisely, we have

THEOREM 2.1.

$$G(\gamma)\ll N(\mathfrak{a}_1)^{-1/2k} .$$

PROOF. Let $\kappa$ be an integer such that $\kappa\in\mathfrak{m}$, $(\kappa, \mathfrak{m}\mathfrak{a}_1)=\mathfrak{m}$. Let $\xi$ run through the complete system of the residues mod $\mathfrak{a}_1$. Then we can replace $\sigma$ in the sum (2.1) by $\kappa\xi+\mu$:

$$G(\gamma)=\frac{1}{N(\mathfrak{a}_1)}\sum_{\xi \bmod \mathfrak{a}_1}\exp\{2\pi i S((\kappa\xi+\mu)^k\gamma)\} . \tag{2.2}$$

We denote by $T$ the sum in this right hand side. Further we write

$$\begin{aligned}\varphi(\xi)&=(\kappa\xi+\mu)^k-\mu^k=\kappa^k\xi^k+k\kappa^{k-1}\xi^{k-1}\mu+\cdots+k\mu^{k-1}\kappa\xi\\&=\alpha_1\xi^k+\alpha_2\xi^{k-1}+\cdots+\alpha_k\xi .\end{aligned}$$

In $\mathfrak{n}=(\alpha_1, \cdots, \alpha_k)$, then $\mathfrak{n}\subset\kappa\subset\mathfrak{m}$. Let $\tau$ be an integer such that $\tau\in\mathfrak{n}$, $(\tau, \mathfrak{a}\mathfrak{n})=\mathfrak{n}$. Let $\beta_1, \cdots, \beta_k$ be integers defined by congruences $\alpha_i\equiv\tau\beta_i \pmod{\mathfrak{a}\mathfrak{n}}$ $(i=1, \cdots, n)$. Then

$$|T|=|\sum_{\xi}\exp\{2\pi i S(\tau\gamma\varphi_0(\xi))\}| ,$$

where $\varphi_0(\xi)=\beta_1\xi^k+\cdots+\beta_k\xi$. If $\tau\gamma\to\mathfrak{a}_2$, then $\mathfrak{a}_2=\mathfrak{a}/(\mathfrak{a}, \mathfrak{n})$ since $(\tau/\mathfrak{n}, \mathfrak{a})=1$. Hence $\mathfrak{a}_1\subset\mathfrak{a}_2$ and

$$|T|=N(\mathfrak{a}_3)|\sum_{\lambda \bmod \mathfrak{a}_2} \exp\{2\pi i S(\tau\gamma\varphi_0(\lambda))\}| ,$$

where $\mathfrak{a}_3=\mathfrak{a}_1/\mathfrak{a}_2$. Since $(\beta_1, \cdots, \beta_k, \mathfrak{a})=1$, we have

$$\sum_{\lambda \bmod \mathfrak{a}_2} \exp\{2\pi i S(\tau\gamma\varphi_0(\lambda))\} \ll N(\mathfrak{a}_2)^{1-1/2k}$$

(Hua [3], Theorem 1) and

$$T \ll N(\mathfrak{a}_3)N(\mathfrak{a}_2)^{1-1/2k} .$$

Since $(\mathfrak{a}, \kappa)=(\mathfrak{a}, \mathfrak{m})$,

$$N(\mathfrak{a}_3) \ll N((\mathfrak{a}, \mathfrak{n})) \leqq N((\mathfrak{a}, k\mu^{k-1}\kappa)) \ll 1$$

and

$$T \ll N(\mathfrak{a}_1)^{1-1/2k} .$$

Thus the proof is completed.

Now we define two sets of the numbers of $K$ as follows:

$$A_0=\{\gamma | \gamma \in E,\ G(\gamma)=1\} ,$$
$$A_1=\{\gamma | \gamma \in E,\ |G(\gamma)|=1,\ \gamma \notin A_0\} .$$

In view of Theorem 2.1, these are finite sets. The number of the elements of $A_0 \cup A_1$ depend on $K$ alone. We note that

$$T=N(\mathfrak{a}_1)\exp(2\pi i S(\mu^k\gamma)) \qquad (\gamma \in A_0 \cup A_1)$$

and, if $\nu \equiv \mu \pmod{\mathfrak{m}}$, then $S(\nu^k\gamma) \equiv S(\mu^k\gamma) \pmod 1$ for $\gamma \in A_0 \cup A_1$. If we put

$$\delta_A=\max_{\gamma \in A_1}\{\cos 2\pi S(\mu^k\gamma)\} , \tag{2.3}$$

then $\delta_A<1$ and $\delta_A$ depends on $K$ alone.

THEOREM 2.2. *If* $\gamma \in A_0$, $\gamma \to \mathfrak{a}$, *then* $\mathfrak{a}|\mathfrak{m}_0^k$, *where* $\mathfrak{m}_0=(\mathfrak{m}, \mu)$. *Conversely, if* $\gamma \to \mathfrak{a}$, $\mathfrak{a}|\mathfrak{m}_0^k$, *then* $\gamma \in A_0$.

PROOF. We consider $T$ in (2.2);

$$T=\sum_{\xi \bmod \mathfrak{a}_1} \exp\{2\pi i S((\kappa\xi+\mu)^k\gamma)\} .$$

Put $\mathfrak{c}=(\kappa, \mu)$ and let $\lambda$ be an integer such that $\lambda \in \mathfrak{c}$, $(\lambda, \mathfrak{c}\mathfrak{a})=\mathfrak{c}$. Then there exist $\rho_1$ and $\rho_2$ such that $\kappa \equiv \lambda\rho_1 \pmod{\mathfrak{c}\mathfrak{a}}$, $\mu \equiv \lambda\rho_2 \pmod{\mathfrak{c}\mathfrak{a}}$. Hence

$$T=\sum_{\xi} \exp\{2\pi i S(\lambda^k\gamma(\rho_1\xi+\rho_2)^k)\} .$$

If $\lambda^k \in \mathfrak{a}$, then $T=N(\mathfrak{a}_1)$. If $\lambda^k \notin \mathfrak{a}$, then $\lambda^k\gamma \to \mathfrak{b} \neq 1$ and $(\rho_1, \rho_2, \mathfrak{b})=1$. If there exists a prime ideal $\mathfrak{p}$ dividing $\mathfrak{b}$ but not $\rho_1$, then we can take $\xi_0$ such that $\rho_1\xi_0+\rho_2\not\equiv 0 \pmod{\mathfrak{p}}$. Hence $S(\lambda^k\gamma(\rho_1\xi_0+\rho_2)^k)$ is not a rational integer and $T\neq N(\mathfrak{a}_1)$. If there exists a prime ideal $\mathfrak{p}$ dividing $\mathfrak{b}$ and $\rho_1$, then $\mathfrak{p}\nmid\rho_2$ and the term of $T$ with $\xi=0$, $\exp\{2\pi i S(\lambda^k\gamma\rho_2^k)\}$, is not equal to 1. Hence $T\neq N(\mathfrak{a}_1)$. Thus we have proved that $\gamma\in A_0$ if and only if $\mathfrak{a}|\lambda^k$. Since $(\lambda, \mathfrak{c}\mathfrak{a})=\mathfrak{c}=(\kappa, \mu)$ and $\mathfrak{m}|\kappa$, if $\mathfrak{a}|(\mathfrak{m}, \mu)^k$, then $\mathfrak{a}|(\kappa, \mu)^k$, which implies $\mathfrak{a}|\lambda^k$. Conversely, if $\mathfrak{a}|\lambda^k$, then $\mathfrak{a}|(\kappa, \mu)^k$ and so $\mathfrak{a}|(\mathfrak{m}, \mu)^k$, since $(\kappa, \mathfrak{a})=(\mathfrak{m}, \mathfrak{a})$. Thus the proof is completed.

REMARK. The number of the elements of $A_0$ is equal to $N(\mathfrak{m}_0)^k$.

## § 3. Division of $E$.

Let $E_0$ be the group of units of $K$ and $E(\mathfrak{a})$ be the group of units mod $\tilde{\mathfrak{a}}$, that is, the group of units congruent to 1 modulo $\tilde{\mathfrak{a}}$, where $\tilde{\mathfrak{a}}$ is the product of an ideal $\mathfrak{a}$ and all infinite primes. We denote by $w$ the number of the roots of unity in $K$, $w(\mathfrak{a})$ the number of the roots of unity in $E(\mathfrak{a})$. Let $\varepsilon_1, \cdots, \varepsilon_r$ $(r=r_1+r_2-1)$ be the fundamental units of $E_0$. The absolute value of the determinant

$$\begin{vmatrix} e_1\log|\varepsilon_1^{(1)}|, \cdots, e_r\log|\varepsilon_1^{(r)}| \\ \cdots\cdots\cdots\cdots \\ e_1\log|\varepsilon_r^{(1)}|, \cdots, e_r\log|\varepsilon_r^{(r)}| \end{vmatrix},$$

where $e_i=1$ $(i=1, \cdots, r_1)$ and $e_j=2$ $(j=r_1+1, \cdots, r)$, is called the regulator of $K$ and denoted by $R$. Similarly we denote by $R(\mathfrak{a})$ the absolute value of the determinant

$$\begin{vmatrix} e_1\log|\eta_1^{(1)}|, \cdots, e_r\log|\eta_1^{(r)}| \\ \cdots\cdots\cdots\cdots \\ e_1\log|\eta_r^{(1)}|, \cdots, e_r\log|\eta_r^{(r)}| \end{vmatrix},$$

where $\eta_1, \cdots, \eta_r$ is the fundamental units of $E(\mathfrak{a})$. It is seen that the order of the factor group $E_0/E(\mathfrak{a})$ is equal to

$$(E_0: E(\mathfrak{a}))=\frac{w}{w(\mathfrak{a})}\frac{R(\mathfrak{a})}{R}.$$

LEMMA 3.1. *Let $\alpha_1, \cdots, \alpha_{r+1}$ be positive numbers such that*

$$\alpha_1e_1+\cdots+\alpha_{r+1}e_{r+1}=1 \qquad (e_{r+1}=n-(e_1+\cdots+e_r)).$$

*Let $\nu$ be a number of $K$. Then by a suitable choice of $\eta\in E(\mathfrak{m})$, we have*

$$c_1|N(\nu)|^{\alpha_q}\leq|\nu^{(q)}\eta^{(q)k}|\leq c_2|N(\nu)|^{\alpha_q} \qquad (q=1, \cdots, r+1).$$

PROOF. Let $\eta_1, \cdots, \eta_r$ be the fundamental units of $E(\mathfrak{m})$ and consider the system of $r+1$ linear equations with $r$ unknowns $X_1, \cdots, X_r$;

$$k\sum_{j=1}^{r} X_j \log|\eta_j^{(q)}| = \alpha_q \log|N(\nu)| - \log|\nu^{(q)}| \qquad (q=1, \cdots, r+1) .$$

We easily see that there exists a solution $t_1, \cdots, t_r$ and

$$k\sum_{j=1}^{r}[t_j]\log|\eta_j^{(q)}| + \log|\nu^{(q)}| = \alpha_q \log|N(\nu)| + O(1) \qquad (q=1, \cdots, r+1) ,$$

so

$$|\nu^{(q)}\prod_{j=1}^{r}(\eta_j^{(q)})^{[t_j]k}| = e^{O(1)}|N(\nu)|^{\alpha_q} \qquad (q=1, \cdots, r+1) .$$

Hence the lemma is proved if we take the unit $\eta = \prod_{j=1}^{r}\eta_j^{[t_j]}$.

Now we return to our $A(\nu; y)$. We assume that $N(\nu)$ $(\nu \succ 0)$ is sufficiently large.

Put

$$\alpha = \frac{1}{r_1+r_2+k}\left(1+\frac{k}{n}\right), \qquad \beta = \frac{1}{2(r_1+r_2+k)}\left(1+\frac{2k}{n}\right), \tag{3.1}$$

then $r_1\alpha + 2r_2\beta = 1$. Hence by Lemma 3.1,

$$c_1N(\nu)^{\alpha} < \nu^{(q)}\varepsilon^{(q)k} < c_2N(\nu)^{\alpha} \qquad (q=1, \cdots, r_1) ,$$
$$c_3N(\nu)^{\beta} < |\nu^{(p)}\varepsilon^{(p)k}| < c_4N(\nu)^{\beta} \qquad (p=r_1+1, \cdots, r_1+r_2)$$

with a suitable unit $\varepsilon$ in $E(\mathfrak{m})$. On the other hand, by the definition of $A(\nu; y)$,

$$A(\nu; y) = A(\nu\varepsilon^k; y)$$

for any unit $\varepsilon$ in $E(\mathfrak{m})$. Therefore, taking $\nu\varepsilon^k$ instead of $\nu$ if necessary, we may assume by Lemma 3.1 that our $\nu$ satisfies the inequalities

$$\begin{aligned} c_1N(\nu)^{\alpha} &\leqq \nu^{(q)} \leqq c_2N(\nu)^{\alpha} \qquad (q=1, \cdots, r_1) , \\ c_3N(\nu)^{\beta} &\leqq |\nu^{(p)}| \leqq c_4N(\nu)^{\beta} \qquad (p=r_1+1, \cdots, r_1+r_2) . \end{aligned} \tag{3.2}$$

We put

$$M = \left\{\pi^{r_2}\Gamma\left(\frac{1}{k}\right)^{r_1}\Gamma\left(\frac{2}{k}\right)^{r_2}\zeta\left(1+\frac{n}{k}\right)\frac{2^{2r_2(1-1/k)}N(\nu)^{1/k}}{k^{r_1(1-1/k)+r_2}\sqrt{D}N(\mathfrak{m})}\right\}^{k/(r_1+r_2+k)} \tag{3.3}$$

and determine our parameters $y_1, \cdots, y_n$ by

$$y_q = \frac{M}{k\nu^{(q)}} \qquad (q=1, \cdots, r_1) ,$$

$$y_p = \frac{\sqrt{M}}{|\nu^{(p)}|} \qquad (p = r_1+1, \cdots, r_1+r_2)\,.$$

By (3.1), (3.2) and (3.3),

$$c_1 M^{-k/n} \leqq y_j \leqq c_2 M^{-k/n} \qquad (j=1, \cdots, n)$$

and

$$M = \frac{2^{2r_2}\pi^{r_2}\Gamma(1/k)^{r_1}\Gamma(2/k)^{r_2}}{k^{r_1+r_2}2^{2r_2/k}\sqrt{D}\,N(\mathfrak{m})}\zeta\Big(1+\frac{n}{k}\Big)(y_1 \cdots y_n)^{-1/k}\,. \tag{3.4}$$

Let $\gamma_1, \cdots, \gamma_f$ $(f = N(\mathfrak{m}_0)^k)$ and $\sigma_1, \cdots, \sigma_g$ be the elements of $A_0$ and $A_1$, respectively. Let $\Delta$ be a number such that $1/3 < \Delta < 1/2$. For each $\gamma_j$ in $A_0$, we define the subsets of $E$ as follows;

$$E_1(\gamma_j) = \Big\{ z \;\Big|\; \frac{2\pi}{y_p}|z_p - \gamma^{(p)}| \leqq M^{-\Delta} \;\; (p=1, \cdots, n) \;\; \text{for any } \gamma \equiv \gamma_j \pmod{\mathfrak{d}^{-1}} \Big\}\,,$$

$$E_2(\gamma_j) = \{ z \mid |z_i - \gamma^{(i)}| y_i^{-1} \leqq C_0 \quad (i=1, \cdots, n) \;\; \text{for any } \gamma \equiv \gamma_j \pmod{\mathfrak{d}^{-1}} \} - E_1(\gamma_j)\,,$$

where $C_0$ is a constant such that

$$y_j \leqq C_0 M^{-k/n} \qquad (j=1, \cdots, n)\,. \tag{3.5}$$

Similarly we define $E_1(\sigma_i)$ and $E_2(\sigma_i)$ for $\sigma_i \in A_1$ $(i=1, \cdots, g)$. Finally we put

$$E_2 = \bigcup_{j=1}^{f} E_2(\gamma_j) \bigcup_{i=1}^{g} E_2(\sigma_i)\,,$$

$$E_3 = E - \bigcup_{j=1}^{f}(E_1(\gamma_j)+E_2(\gamma_j)) - \bigcup_{i=1}^{g}(E_1(\sigma_i)+E_2(\sigma_i))\,.$$

According to this division of $E$, we divide the integral in (1.4) as follows:

$$\int_{\phi(E)} H(z)e^{-2\pi i S(\nu z)}dx_1 \cdots dx_n = \sum_{j=1}^{f}\int_{\phi(E_1(\gamma_j))} + \sum_{i=1}^{g}\int_{\phi(E_1(\sigma_i))} + \int_{\phi(E_2)} + \int_{\phi(E_3)}\,. \tag{3.6}$$

REMARK. It is easily seen that

$$g(y; \gamma_j + \tau) = g(y; \tau) \qquad (j=1, \cdots, f)\,,$$

$$f(y; \gamma_j + \tau) = f(y; \tau) \qquad (j=1, \cdots, f)\,,$$

$$g(y; \sigma_i + \tau) = \exp\{2\pi i S(\mu^k \sigma_i)\} g(y; \tau) \qquad (i=1, \cdots, g)\,.$$

## § 4. Hecke-Rademacher's transformation formula.

We put

$$\tau_q = y_q - 2\pi i z_q \qquad (q=1, \cdots, r_1),$$
$$\tau_p = 2y_p \qquad (p=r_1+1, \cdots, r_1+r_2).$$

Denoting by $\hat{\nu}$ the ideal numbers introduced by Hecke [2], we define the series:

$$\Phi(y, z; \psi) = \sum_{\substack{\hat{\nu}\neq 0 \\ \hat{\nu}\in \mathfrak{m}_0}} \psi(\hat{\nu}/\hat{\alpha}) \exp\{-\sum_{q=1}^{r_1+r_2} |\hat{\nu}^{(q)}|^k \tau_q + 2\pi i \sum_{p=r_1+1}^{n} (\hat{\nu}^{(p)})^k z_p\},$$

where $\mathfrak{m}_0 = (\mathfrak{m}, \mu)$, $\mathfrak{m} = \mathfrak{m}_0 \mathfrak{n}$, $\mathfrak{m}_0 = (\hat{\alpha})$, the sum is over all non-zero ideal numbers $\hat{\nu}$ in $\mathfrak{m}_0$, and $\psi$ is a character mod $\tilde{\mathfrak{n}}$ for ideal numbers. The character $\psi$ has the properties;

(i) If $(\mathfrak{n}, \hat{\mu}) \neq 1$, then $\psi(\hat{\mu}) = 0$.

(ii) If $\hat{\mu}/\hat{\nu} \in K$ and $\hat{\mu} \equiv \hat{\nu}$ (mod $\tilde{\mathfrak{n}}$), then $\psi(\hat{\nu}) = \psi(\hat{\mu})$.

The number of such characters is equal to $2^{r_1} h\varphi(\mathfrak{n})$, where $h$ is the ideal class number of $K$, $\varphi(\mathfrak{n})$ is Euler's function for ideals.

It is obvious that $\Phi(y, z; \psi)$ is absolutely convergent.

Let $\kappa$ be a number of $\mathfrak{m}_0$ such that $(\kappa/\mathfrak{m}_0, \mathfrak{n}) = 1$ and define $\mu_0$ by the congruence $\kappa\mu_0 \equiv \mu$ (mod $\mathfrak{m}$). We see $(\mu_0, \mathfrak{n}) = 1$. Since $(\kappa\mu_0/\hat{\alpha}, \mathfrak{n}) = 1$, we have

$$\sum_{\psi} \bar{\psi}(\kappa\mu_0/\hat{\alpha}) \psi(\hat{\nu}/\hat{\alpha}) = \begin{cases} 2^{r_1} h\varphi(\mathfrak{n}) & \text{if } \nu \equiv \mu \pmod{\tilde{\mathfrak{m}}}, \\ 0 & \text{if not,} \end{cases}$$

where $\psi$ runs through all characters mod $\tilde{\mathfrak{n}}$ for ideal numbers. Since we may assume $\kappa\mu_0 \succ 0$,

$$\frac{1}{2^{r_1} h\varphi(\mathfrak{n})} \sum_{\psi} \bar{\psi}(\kappa\mu_0/\hat{\alpha}) \Phi(y, z; \psi)$$
$$= \sum_{\substack{\nu \succ 0 \\ \nu \equiv \mu (\mathfrak{m})}} \exp\{-\sum_{q=1}^{r_1+r_2} |\nu^{(q)}|^k \tau_q + 2\pi i \sum_{p=r_1+1}^{n} (\nu^{(p)})^k z_p\}.$$

If $\nu \succ 0$, then

$$-\sum_{q=1}^{r_1+r_2} |\nu^{(q)}|^k \tau_q + 2\pi i \sum_{p=r_1+1}^{n} (\nu^{(p)})^k z_p = -S(|\nu|^k y) + 2\pi i S(\nu^k z).$$

Hence we have

$$g(y; z) = \frac{1}{2^{r_1} h\varphi(\mathfrak{n})} \sum_{\psi} \bar{\psi}(\kappa\mu_0/\hat{\alpha}) \Phi(y, z; \psi). \tag{4.1}$$

Let $\hat{\nu}$ and $\hat{\lambda}$ be non-zero ideal numbers. We say $\hat{\nu}$ and $\hat{\lambda}$ are associated with respect to $E(\mathfrak{n})$, if $\hat{\mathfrak{v}}/\hat{\lambda}$ is a unit in $E(\mathfrak{n})$. Let $H_1, \cdots, H_r$ be the fundamental units of $E(\mathfrak{n})$, $\Xi = e^{2\pi i/w(\mathfrak{n})}$. Since $\psi(E(\mathfrak{n}))=1$, we can write $\Phi(y, z; \psi)$ as follows;

$$(4.2) \qquad \Phi(y, z; \psi) = \sum_{(\hat{\mathfrak{v}})\mathfrak{n}} \psi(\hat{\mathfrak{v}}) \sum_{b=1}^{w(\mathfrak{n})} \sum_{b_1, \cdots, b_r = -\infty}^{\infty} \times \exp\{-\sum_{q=1}^{r_1+r_2} |H_1^{(q)b_1} \cdots H_r^{(q)b_r} \hat{\mathfrak{v}}^{(q)} \hat{\alpha}^{(q)}|^k \tau_q + 2\pi i \sum_{p=r_1+1}^{n} (H_1^{(p)b_1} \cdots H_r^{(p)b_r} \Xi^{(p)b} \hat{\mathfrak{v}}^{(p)} \hat{\alpha}^{(p)})^k z_p\},$$

where the summation $\sum_{(\hat{\mathfrak{v}})\mathfrak{n}}$ means that $\hat{\mathfrak{v}}$ runs through all integral ideal numbers not associated with each other with respect to $E(\mathfrak{n})$.

Let

$$\begin{pmatrix} e_1 & e_2 & \cdots & e_{r+1} \\ E_1^{(1)} & E_2^{(1)} & \cdots & E_{r+1}^{(1)} \\ & \cdots\cdots & & \\ E_1^{(r)} & E_2^{(r)} & \cdots & E_{r+1}^{(r)} \end{pmatrix}$$

be the inverse of the matrix

$$\begin{pmatrix} \frac{1}{n}, & \log|H_1^{(1)}|, & \cdots, & \log|H_r^{(1)}| \\ & \cdots\cdots\cdots\cdots & & \\ \frac{1}{n}, & \log|H_1^{(r+1)}|, & \cdots, & \log|H_r^{(r+1)}| \end{pmatrix}$$

and put

$$v_q = v_q(m_1, \cdots, m_r; a_{r_1+1}, \cdots, a_n) = \sum_{j=1}^{r} E_q^{(j)} (2\pi m_j + k \sum_{p=r_1+1}^{n} a_p \Theta_j^{(p)}) \quad (q=1, \cdots, r+1),$$

where

$$\Theta_j^{(p)} = \arg H_j^{(p)} \qquad (j=1, \cdots, r;\ p=r_1+1, \cdots, n).$$

Now we quote an essential lemma from Rademacher [6];

LEMMA 4.1. (*Hecke-Rademacher's transformation formula*). *Let $W_1, \cdots, W_{r_1}$ be complex numbers with positive real parts, and $W_{r_1+1}, \cdots, W_{r_1+r_2}$ positive numbers. Let $U_{r_1+1}, \cdots, U_n$ be complex numbers such that $U_{p+r_2} = \bar{U}_p$ $(p=r_1+1, \cdots, r_1+r_2)$. Then we have*

$$\sum_{b_1, \cdots, b_r = -\infty}^{\infty} \exp\{-\sum_{q=1}^{r+1} |H_1^{(q)b_1} \cdots H_r^{(q)b_r}|^k W_q + 2\pi i \sum_{p=r_1+1}^{n} (H_1^{(p)b_1} \cdots H_r^{(p)b_r})^k U_p\}$$

$$= \sum_{m_1,\cdots,m_r=-\infty}^{\infty} \sum_{\substack{a_{r_1+1},\cdots,a_n\geqq 0\\ a_p\cdot a_{p+r_2}=0}}^{\infty} \frac{2^{r_2}}{2\pi i k^r R(\mathfrak{n})} \prod_{p=r_1+1}^{n}\left(\frac{iU_p}{|U_p|}\right)^{a_p}$$

$$\times \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \prod_{q=1}^{r_1} \frac{\Gamma(s+iV_q)}{W_q^{s+iV_q}} \prod_{p=r_1+1}^{r_1+r_2} \frac{(2\pi|U_p|)^{l_p}}{(W_p^2+16\pi^2|U_p|^2)^{l_p/2+s+iV_p}}$$

$$\times \frac{\Gamma(l_p+2s+2iV_p)}{\Gamma(l_p+1)}$$

$$\times F\left(\frac{l_p}{2}+s+iV_p, \frac{l_p+1}{2}-s-iV_p, l_p+1; \frac{16\pi^2|U_p|^2}{W_p^2+16\pi^2|U_p|^2}\right)ds\ ,$$

*where* $\sigma_0>0$, $l_p=a_p+a_{p+r_2}$ $(p=r_1+1, \cdots, r_1+r_2)$, $V_q=v_q/e_q k$ $(q=1, \cdots, r+1)$ *and* $F(\alpha, \beta, \gamma; x)$ *is the hypergeometric function. The summation variables* $m_1, \cdots, m_r$ *run through all rational integers and* $a_{r_1+1}, \cdots, a_n$ *run through non-negative integers with the conditions* $a_p\cdot a_{p+r_2}=0$ $(p=r_1+1, \cdots, r_1+r_2)$.

PROOF. ([6], Hilfssatz 14).

Applying this lemma with $W_q=|\mathfrak{v}^{(q)}\hat{\alpha}^{(q)}|^k\tau_q$ $(q=1, \cdots, r+1)$ and $U_p=(\Xi^{(p)b}\hat{\mathfrak{v}}^{(p)}\hat{\alpha}^{(p)})^k z_p$ $(p=r_1+1, \cdots, n)$ to the inner sum of (4.2), and putting

$$\lambda(\hat{\mu})=\prod_{q=1}^{r+1}|\hat{\mu}^{(q)}|^{-iv_q}\prod_{p=r_1+1}^{n}\left(\frac{\hat{\mu}^{(p)}}{|\hat{\mu}^{(p)}|}\right)^{ka_p},$$

we have

$$\Phi(y, z; \psi)=\sum_{(\mathfrak{v})\mathfrak{n}}\sum_{b=1}^{w(\mathfrak{n})}\sum_{m_1,\cdots,m_r=-\infty}^{\infty}\sum_{\substack{a_{r_1+1},\cdots,a_n\geqq 0\\ a_p\cdot a_{p+r_2}=0}}^{\infty}\prod_{p=r_1+1}^{n}(2\pi i z_p)^{a_p}\prod_{p=r_1+1}^{n}\Xi^{(p)bka_p}$$

$$\times\frac{2^{r_2}}{k^r R(\mathfrak{n})}\frac{1}{2\pi i}\int_{(\sigma_0)}\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q)}{\tau_q^{s+iV_q}}\prod_{p=r_1+1}^{r_1+r_2}(\tau_p^2+16\pi^2|z_p|^2)^{-l_p/2-s-iV_p}$$

$$\times\frac{\Gamma(l_p+2s+2iV_p)}{\Gamma(l_p+1)}F\left(\frac{l_p}{2}+s+iV_p, \frac{l_p+1}{2}-s-iV_p, l_p+1; \frac{16\pi^2|z_p|^2}{\tau_p^2+16\pi^2|z_p|^2}\right)$$

$$\times\frac{\psi(\hat{\mathfrak{v}})\lambda(\hat{\mathfrak{v}})\lambda(\hat{\alpha})}{|N(\hat{\mathfrak{v}})|^{ks}N(\mathfrak{m}_0)^{ks}}ds\ ,$$

where we denote the integral $\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}$ by $\int_{(\sigma_0)}$.

Since

$$\sum_{b=1}^{w(\mathfrak{n})}\prod_{p=r_1+1}^{n}\Xi^{(p)bka_p}=\begin{cases} w(\mathfrak{n}) & \text{if } \prod_{p=r_1+1}^{n}\Xi^{(p)ka_p}=1\ , \\ 0 & \text{if not,} \end{cases}$$

we have finally

$$(4.3)\quad \Phi(y, z; \psi)=\sum_{(\mathfrak{v})\mathfrak{n}}\sum_{\{m\}}\sum_{\{a\}}{}' \prod_{p=r_1+1}^{n}(2\pi i z_p)^{a_p}\frac{2^{r_2}w(\mathfrak{n})}{k^r R(\mathfrak{n})}$$
$$\times\frac{1}{2\pi i}\int_{(\sigma_0)}\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q)}{\tau_q^{s+iV_q}}\prod_{p=r_1+1}^{r_1+r_2}(\tau_p^2+16\pi^2|z_p|^2)^{-l_p/2-s-iV_p}$$
$$\times\frac{\Gamma(l_p+2s+2iV_p)}{\Gamma(l_p+1)}F\left(\frac{l_p}{2}+s+iV_p, \frac{l_p+1}{2}-s-iV_p, l_p+1; \frac{16\pi^2|z_p|^2}{\tau_p^2+16\pi^2|z_p|^2}\right)$$
$$\times\frac{\psi\lambda(\hat{\mathfrak{v}})}{|N(\hat{\mathfrak{v}})|^{ks}}\frac{\lambda(\hat{\alpha})}{N(\mathfrak{m}_0)^{ks}}ds ,$$

where the sum $\sum'_{\{a\}}$ means that the summation variables $a_{r_1+1}, \cdots, a_n$ satisfy the following conditions

$$a_{r_1+1}, \cdots, a_n\geqq 0 , \quad a_p\cdot a_{p+r_2}=0 \qquad (p=r_1+1, \cdots, r_1+r_2) ,$$
$$\prod_{p=r_1+1}^{n}\Xi^{(p)ka_p}=1 .$$

This last condition means that $\lambda$ becomes Grössencharacter mod $\tilde{\mathfrak{n}}$ for ideal numbers. Therefore, the summation $\sum_{\{m\}}\sum'_{\{a\}}$ over $m_1, \cdots, m_r$ and $a_{r_1+1}, \cdots, a_n$ is regarded as the sum $\sum_\lambda$ taken over all Grössencharacters $\lambda$ mod $\tilde{\mathfrak{n}}$ for ideal numbers.

## § 5. Fundamental formulas for $g(y; z)$ and $\log f(y; z)$.

Following Rademacher [6], we put

$$G(s, l, x)=\frac{\Gamma(s+l)}{2^{s+l}\Gamma(l+1)}F\left(\frac{l+s}{2}, \frac{l+1-s}{2}, l+1; x\right) ,$$

where $s$ is complex variable, $l$ is non-negative rational integer and $0<x<1$ ([6], p. 368, (4.311)). As usual, we write $s=\sigma+it$. Then (4.3) is rewritten as follows:

$$(5.1)\quad \Phi(y, z; \psi)=\sum_{(\mathfrak{v})\mathfrak{n}}\frac{2^{r_2}w(\mathfrak{n})}{k^r R(\mathfrak{n})}\sum_\lambda\frac{1}{2\pi i}$$
$$\times\int_{(\sigma_0)}\prod_{p=r_1+1}^{n}(2\pi i z_p)^{a_p}\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q)}{(y_q-2\pi i z_q)^{s+iV_q}}\prod_{p=r_1+1}^{r_1+r_2}(y_p^2+4\pi^2|z_p|^2)^{-l_p/2-s-iV_p}$$
$$\times G\left(2s+2iV_p, l_p, \frac{4\pi^2|z_p|^2}{y_p^2+4\pi^2|z_p|^2}\right)\frac{\psi\lambda(\hat{\mathfrak{v}})\lambda(\hat{\alpha})}{N(\hat{\mathfrak{v}})^{ks}N(\mathfrak{m}_0)^{ks}}ds .$$

LEMMA 5.1. *Let $\sigma_0=5/4$. Then the series in the right hand side of (5.1) is absolutely convergent.*

PROOF. Using the well-known estimation for gamma function, we have

$$\frac{\Gamma(s+iV_q)}{(y_q-2\pi i z_q)^{s+iV_q}} \ll \frac{(1+|t+V_q|)^{\sigma-1/2}}{|y_q-2\pi i z_q|^{\sigma}} \times \exp\left\{-\left(\frac{\pi}{2}-|\arg(y_q-2\pi i z_q)|\right)|t+V_q|\right\} \qquad (q=1,\cdots,r_1) \tag{5.2}$$

([6], p. 381, (5.42)). If we put

$$G(s,l,x)=\frac{1}{2\sqrt{\pi}}\frac{\Gamma((s+l)/2)}{\Gamma((l-s+1)/2)}G_1(s,l,x)\,, \tag{5.3}$$

then $G_1(s,l,x)$ is regular in the strip $-1<\sigma<3$ and we have the estimation:

$$G_1(s,l,x)\ll e^{-(|t|/4)\sqrt{1-x}}\frac{1+l+|t|}{1+l}\cdot\frac{(1-x)^{-1/4}}{(1+(1/2)\sqrt{1-x})^{l/2}} \tag{5.4}$$

([6], p. 371, Hilfssatz 19) for $-1+\varepsilon\leqq\sigma\leqq3-\varepsilon$. (In [6], the estimation of $G_1$ of the form (5.4) was obtained for $0<\varepsilon\leqq\sigma\leqq3-\varepsilon$. But examining the proof of Hilfssatz 19 precisely, we can see that (5.4) holds for $-1+\varepsilon\leqq\sigma\leqq3-\varepsilon$.) Hence, by (5.3) and (5.4),

$$G(2s+2iV_p,l_p,x_p) \ll \exp(-\theta_p|t+V_p|)\frac{(1+l_p+|t+V_p|)^{2\sigma-1/2}}{l_p+1}\cdot\frac{(1-x_p)^{-1/4}}{(1+(1/2)\sqrt{1-x_p})^{l_p/2}} \tag{5.5}$$

for $-(1+\varepsilon)/2\leqq\sigma\leqq(3-\varepsilon)/2$, where we put

$$\theta_p=\frac{1}{2}\sqrt{1-x_p}\,,\quad x_p=\frac{4\pi^2|z_p|^2}{y_p^2+4\pi^2|z_p|^2}\qquad (p=r_1+1,\cdots,r_1+r_2)\,.$$

Hence, putting

$$\theta=\min_{p,q}\left(\theta_p,\frac{\pi}{2}-|\arg(y_q-2\pi i z_q)|\right),$$

we have, by (5.2) and (5.5), the estimation of the integrand in (5.1):

$$\begin{aligned}&\prod_{p=r_1+1}^{n}(2\pi i z_p)^{a_p}\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q)}{(y_q-2\pi i z_q)^{s+iV_q}}\\ &\times\prod_{p=r_1+1}^{r_1+r_2}(y_p^2+4\pi^2|z_p|^2)^{-l_p/2-s-iV_p}G(2s+2iV_p,l_p+1,x_p)\frac{\psi\lambda(\hat{\nu})\lambda(\hat{\alpha})}{|N(\hat{\nu})|^{ks}}\\ &\ll|N(\hat{\nu})|^{-k\sigma}\exp\left(-\theta\sum_{q=1}^{r_1+r_2}|t+V_q|\right)\prod_{q=1}^{r_1}\frac{(1+|t+V_q|)^{\sigma-1/2}}{|y_q-2\pi i z_q|^{\sigma}}\\ &\times\prod_{p=r_1+1}^{r_1+r_2}\frac{(1+l_p+|t+V_p|)^{2\sigma-1/2}}{l_p+1}\cdot\frac{(1-x_p)^{-1/4}}{(1+(1/2)\sqrt{1-x_p})^{l_p/2}}\frac{x_p^{l_p/2}}{(y_p^2+4\pi^2|z_p|^2)^{\sigma}}\,.\end{aligned}$$

In particular, if $\sigma=5/4$, then the last expression is replaced by

$$|N(\hat{\nu})|^{-k\sigma}\exp\Big(-\frac{\theta}{2}\sum_{q=1}^{r+1}|t+V_q|\Big)\prod_{p=r_1+1}^{r_1+r_2}x_p^{l_p/2}(1+l_p)\ .$$

Hence

$$\Phi(y,\ z;\ \psi)\ll\sum_{\{m\}}\sum_{\{l\}}\int_{-\infty}^{\infty}\exp\,(-c\sum_{q=1}^{+r1}|t+V_q|)dt\prod_{p=r_1+1}^{r_1+r_2}x_p^{l_p/2}(1+l_p)\ , \tag{5.6}$$

where $l_{r_1+1},\ \cdots,\ l_{r_1+r_2}$ run through all non-negative rational integers.

Here we see

$$V_q=V_q(m_1,\ \cdots,\ m_r)=\frac{1}{ke_q}\sum_{j=1}^{r}E_q^{(j)}(2\pi m_j+k\sum_{p=r_1+1}^{n}a_p\Theta_j^{(p)})\ .$$

Let $u_1,\ \cdots,\ u_r$ be real numbers and define

$$V_q^*=V_q^*(u_1,\ \cdots,\ u_r)=\frac{1}{ke_q}\sum_{j=1}^{r}E_q^{(j)}(2\pi u_j+k\sum_{p=r_1+1}^{n}a_p\Theta_j^{(p)})\ .$$

If $m_j\leqq u_j\leqq m_j+1$ $(j=1,\ \cdots,\ r)$, then

$$|V_q-V_q^*|\leqq\frac{2\pi}{k}\sum_{j=1}^{r}|E_q^{(j)}|\qquad(q=1,\ \cdots,\ r+1)\ ,$$

so we have

$$\exp\,(-c\sum_{q=1}^{r+1}|t+V_q|)\ll\exp\,(-c\sum_{q=1}^{r+1}|t+V_q^*|)\ .$$

Hence

$$\begin{aligned}\sum_{\{m\}}\int_{-\infty}^{\infty}&\exp\,(-c\sum_{q=1}^{r+1}|t+V_q|)\,dt\\ &\ll\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\,(-c\sum_{q=1}^{r+1}|t+V_q^*|)dt\,du_1\cdots du_r\ .\end{aligned} \tag{5.7}$$

If we change the variables of integration by putting

$$\xi_q=t+V_q^*\qquad(q=1,\ \cdots,\ r+1)\ ,$$

then its Jacobian

$$\frac{\partial(t,\ u_1,\ \cdots,\ u_r)}{\partial(\xi_1,\ \xi_2,\ \cdots,\ \xi_{r+1})}$$

is a constant. Hence from (5.6) and (5.7) it follows that

$$\Phi(y,\ z;\ \psi)\ll\sum_{\{l\}}\prod_{p=r_1+1}^{r_1+r_2}x_p^{l_p/2}(1+l_p)=\prod_{p=r_1+1}^{r_1+r_2}(1-x_p^{1/2})^{-2}\ll 1\ .$$

Thus the proof is completed.

From this lemma it follows that we can change the order of the summations in (5.1) and, moreover, we can invert the order of the summation $\sum_{(\mathfrak{v})\mathfrak{n}}$ and integration. Thus we have

$$\begin{aligned}(5.8)\qquad \Phi(y, z; \psi) = {} & \frac{2^{r_2}w(\mathfrak{n})}{k^r R(\mathfrak{n})} \cdot \frac{1}{2\pi i}\int_{(5/4)} \prod_{p=r_1+1}^{n} (2\pi i z_p)^{a_p} \\ & \times \prod_{q=1}^{r_1} \frac{\Gamma(s+iV_q)}{(y_q - 2\pi i z_q)^{s+iV_q}} \prod_{p=r_1+1}^{r_1+r_2} (y_p^2 + 4\pi^2|z_p|^2)^{-l_p/2-s-iV_p} \\ & \times G\left(2s+2iV_p, l_p, \frac{4\pi^2|z_p|^2}{y_p^2+4\pi^2|z_p|^2}\right) \frac{\lambda(\hat{\alpha})}{N(\mathfrak{m}_0)^{ks}} \sum_{\substack{(\mathfrak{v})\mathfrak{n} \\ \mathfrak{v}\neq 0}} \frac{\psi\lambda(\hat{\mathfrak{v}})}{|N(\hat{\mathfrak{v}})|^{ks}} ds .\end{aligned}$$

Now we consider the series

$$\sum_{(\mathfrak{v})\mathfrak{n}} \frac{\psi\lambda(\hat{\mathfrak{v}})}{|N(\hat{\mathfrak{v}})|^s} \qquad (\sigma > 1)$$

in the integrand in (5.8). Let $\varepsilon_1, \cdots, \varepsilon_m$ ($m = (E_0 : E(\mathfrak{n}))$) be the representatives of the factor group $E_0/E(\mathfrak{n})$. Then

$$\sum_{(\mathfrak{v})\mathfrak{n}} \frac{\psi\lambda(\hat{\mathfrak{v}})}{|N(\hat{\mathfrak{v}})|^s} = \sum_{(\mathfrak{v})} \frac{\psi\lambda(\hat{\mathfrak{v}})}{|N(\hat{\mathfrak{v}})|^s} \sum_{i=1}^{m} \psi\lambda(\varepsilon_i) ,$$

where the sum $\sum_{(\mathfrak{v})}$ is taken over all non-zero ideals $(\hat{\mathfrak{v}})$. Second sum in the right hand side is equal to $m$ if $\psi\lambda(\varepsilon_i) = 1$ ($i = 1, \cdots, m$) so that $\psi\lambda(\varepsilon) = 1$ for all units $\varepsilon$. The character $\psi\lambda$ with this property is called the Grössencharacter for ideals. In this case, the series

$$\zeta(s, \psi\lambda) = \sum_{(\mathfrak{v})} \frac{\psi\lambda(\hat{\mathfrak{v}})}{|N(\hat{\mathfrak{v}})|^s}$$

is called Hecke's zeta function, which was first defined by Hecke [2]. If $\psi\lambda$ is not a Grössencharacter for ideals, then it is easily seen that $\sum \psi\lambda(\varepsilon_i) = 0$. Hence if we put

$$\zeta^*(s, \psi\lambda) = \begin{cases} \zeta(s, \psi\lambda) & \text{if } \psi\lambda \text{ is a Grössencharacter for ideals ,} \\ 0 & \text{if not ,} \end{cases}$$

then

$$(5.9)\qquad \sum_{(\mathfrak{v})\mathfrak{n}} \frac{\psi\lambda(\hat{\mathfrak{v}})}{|N(\hat{\mathfrak{v}})|^s} = \frac{wR(\mathfrak{n})}{w(\mathfrak{n})R} \zeta^*(s, \psi\lambda) .$$

Combining (4.1), (5.8) and (5.9), we have

THEOREM 5.1. *If we put*

$$Z_\lambda(s;\mathfrak{m},\mu)=\frac{2^{r_2}w}{k^rR}\frac{\lambda(\hat{\alpha})}{N(\mathfrak{m}_0)^s}\frac{1}{2^{r_1}h\varphi(\mathfrak{n})}\sum_\psi\bar{\psi}(\kappa\mu_0/\hat{\alpha})\zeta^*(s,\psi\lambda)\ ,$$

$$(5.10)\quad P_\lambda(s;y,z)=\prod_{p=r_1+1}^{n}(2\pi iz_p)^{a_p}\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q)}{(y_q-2\pi iz_q)^{s+iV_q}}$$
$$\times\prod_{p=r_1+1}^{r_1+r_2}(y_p^2+4\pi^2|z_p|^2)^{-l_p/2-s-iV_p}G\left(2s+2iV_p,\ l_p,\ \frac{4\pi^2|z_p|^2}{y_p^2+4\pi^2|z_p|^2}\right),$$

*then we have*

$$g(y;z)=\sum_\lambda\frac{1}{2\pi i}\int_{(5/4)}P_\lambda(s;y,z)Z_\lambda(ks;\mathfrak{m},\mu)ds$$

*and*

$$\log f(y;z)=\sum_\lambda\frac{1}{2\pi i}\int_{(5/4)}P_\lambda(s;y,z)\zeta(1+ns)Z_\lambda(ks;\mathfrak{m},\mu)ds\ ,$$

*where $\zeta(s)$ is the Riemann zeta function.*

From now on, we put

$$\Psi_\lambda(s;y,z)=P_\lambda(s;y,z)\zeta(1+ns)Z_\lambda(ks;\mathfrak{m},\mu)\ .$$

## §6. Some function-theoretical lemmas.

It is well-known that, if $\psi\lambda$ is non-principal Grössencharacter mod $\tilde{\mathfrak{n}}$ for ideals, then $\zeta(s,\psi\lambda)$ is an entire function. If $\psi\lambda$ is principal, then $\zeta(s,\psi\lambda)$ has only one pole of order 1 at $s=1$. The residue of $\zeta(s,\psi\lambda)$ at $s=1$, which is denoted by $\operatorname{Res}_1\zeta(s,\psi\lambda)$, is

$$\operatorname{Res}_1\zeta(s,\psi\lambda)=\begin{cases}h\dfrac{2^{r+1}\pi^{r_2}}{w\sqrt{D}}\dfrac{\varphi(\mathfrak{n})}{N(\mathfrak{n})}R & \text{if}\quad \psi\lambda=1\ ,\\ 0 & \text{if}\quad \psi\lambda\neq 1\ .\end{cases}$$

By the definition (5.10), $Z_\lambda(ks;\mathfrak{m},\mu)$ has a pole of order 1 at $s=1/k$ if $\lambda=1$, and we easily see that

$$\operatorname{Res}_{1/k}Z_1(ks;\mathfrak{m},\mu)=\frac{2^{2r_2}\pi^{r_2}}{k^{r+1}\sqrt{D}N(\mathfrak{m})}\ .$$

$P_\lambda$ is regular in the strip $-1/4k\leqq\sigma\leqq 5/4$, except for a finite number of poles on the line $\sigma=0$. Therefore, if $\lambda\neq 1$, then $P_\lambda\cdot Z_\lambda$ is regular at $s=1/k$. $P_1\cdot Z_1$ has a pole of order 1 at $s=1/k$ and

$$\text{(6.1)}\quad \operatorname*{Res}_{1/k}\{P_1(s;\, y,\, z)Z_1(ks;\, \mathfrak{m},\, \mu)\}$$
$$= \frac{2^{2r_2}\pi^{r_2}\Gamma(1/k)^{r_1}}{k^{r_1+r_2}\sqrt{D}\,N(\mathfrak{m})}\prod_{q=1}^{r_1}(y_q-2\pi i z_q)^{-1/k}$$
$$\times\prod_{p=r_1+1}^{r_1+r_2}(y_p^2+4\pi^2|z_p|^2)^{-1/k}G\left(\frac{2}{k},\, 0,\, \frac{4\pi^2|z_p|^2}{y_p^2+4\pi^2|z_p|^2}\right).$$

In particular, if $z_1=\cdots=z_n=0$, then, noting that $G(s,\,0,\,0)=\Gamma(s)2^{-s}$, we have by (3.4)

$$\operatorname*{Res}_{1/k}\{P_1(s;\, y,\, 0)Z_1(ks;\, \mathfrak{m},\, \mu)\}$$
$$=\frac{2^{2r_2}\pi^{r_2}\Gamma(1/k)^{r_1}\Gamma(2/k)^{r_2}}{k^{r_1+r_2}2^{2r_2/k}\sqrt{D}\,N(\mathfrak{m})}(y_1\cdots y_n)^{-1/k}=\frac{M}{\zeta(1+n/k)}.$$

LEMMA 6.1.

$$\frac{1}{2\pi i}\int_{(5/4)}P_\lambda(s;\, y,\, z)Z_\lambda(ks;\, \mathfrak{m},\, \mu)ds$$
$$=\frac{1}{2\pi i}\int_{(-1/4k)}P_\lambda(s;\, y,\, z)Z_\lambda(ks;\, \mathfrak{m},\, \mu)ds$$
$$+\sum_{-1/4k\leqq\sigma\leqq5/4}\operatorname{Res}\{P_\lambda(s;\, y,\, z)Z_\lambda(ks;\, \mathfrak{m},\, \mu)\},$$

*where the last sum is the sum of the residues of* $P_\lambda\cdot Z_\lambda$ *at the poles lying in the strip* $-1/4k\leqq\sigma\leqq5/4$.

PROOF. To prove this lemma, it is sufficient to show that

$$\text{(6.2)}\qquad \int_{-1/4k+iT}^{5/4+iT}P_\lambda\cdot Z_\lambda ds \longrightarrow 0$$

as $|T|\to\infty$, where the integral is along the horizontal line from $-1/4k+iT$ to $5/4+iT$. If $|T|$ is large enough, and $s=\sigma+iT$ $(-1/4k\leqq\sigma\leqq5/4)$, then by (5.2) and (5.5)

$$P_\lambda(s;\, y,\, z)\ll\exp\Big(-c\sum_{q=1}^{r+1}|T+V_q|\Big).$$

As is well-known,

$$\zeta(s;\,\psi\lambda)\ll\prod_{q=1}^{r_1}(1+|T+v_q|)\prod_{p=r_1+1}^{r_1+r_2}(1+l_p+|2T+v_p|)^2\qquad(-1/2\leqq\sigma\leqq3/2).$$

Hence

$$(6.3)\qquad Z_\lambda(ks;\mathfrak{m},\mu)\ll\prod_{q=1}^{r_1}(1+|kT+v_q|)\prod_{p=r_1+1}^{r_1+r_2}(1+l_p+|2kT+v_p|)^2$$
$$\ll\prod_{q=1}^{r_1}(1+|T+V_q|)\prod_{p=r_1+1}^{r_1+r_2}(1+l_p+|T+V_p|)^2$$
$$(-1/4k\leqq\sigma\leqq 5/4)\,.$$

Therefore

$$\int_{-1/4k+iT}^{5/4+iT}P_\lambda\cdot Z_\lambda\,ds\ll e^{-c|T|}$$

and our assertion (6.2) follows at once.

If $0<\varepsilon<1/k$, then, by the similar argument,

$$\frac{1}{2\pi i}\int_{(5/4)}P_\lambda\cdot Z_\lambda\,ds=\frac{1}{2\pi i}\int_{(\varepsilon)}P_\lambda\cdot Z_\lambda\,ds+\begin{cases}0 & \text{if}\quad \lambda\neq 1\,,\\ \operatorname*{Res}_{1/k}(P_1\cdot Z_1) & \text{if}\quad \lambda=1\,.\end{cases}$$

Hence

$$(6.4)\qquad g(y;z)=\sum_\lambda\frac{1}{2\pi i}\int_{(\varepsilon)}P_\lambda\cdot Z_\lambda\,ds+\operatorname*{Res}_{1/k}(P_1\cdot Z_1)\,.$$

Now we consider $\Psi_\lambda(s;y,z)$. Since $\Psi_\lambda=\zeta(1+ns)P_\lambda\cdot Z_\lambda$, we obtain

$$(6.5)\qquad \operatorname*{Res}_{1/k}\Psi_1(s;y,z)=\frac{2^{2r_2}\pi^{r_2}\Gamma(1/k)^{r_1}}{k^{r_1+r_2}\sqrt{D}N(\mathfrak{m})}\zeta\left(1+\frac{n}{k}\right)$$
$$\times\prod_{q=1}^{r_1}(y_q-2\pi iz_p)^{-1/k}\prod_{p=r_1+1}^{r_1+r_2}(y_p^2+4\pi^2|z_p|^2)^{-1/k}G\left(\frac{2}{k},0,\frac{4\pi^2|z_p|^2}{y_p^2+4\pi^2|z_p|^2}\right)\,.$$

In particular,

$$\operatorname*{Res}_{1/k}\Psi_1(s;y,0)=\frac{2^{2r_2}\pi^{r_2}\Gamma(1/k)^{r_1}\Gamma(2/k)^{r_2}}{k^{r_1+r_2}2^{2r_2/k}\sqrt{D}N(\mathfrak{m})}\zeta\left(1+\frac{n}{k}\right)(y_1\cdots y_n)^{-1/k}=M\,.$$

LEMMA 6.2. *We have*

$$(6.6)\qquad \frac{1}{2\pi i}\int_{(5/4)}\Psi_\lambda(s;y,z)ds=\frac{1}{2\pi i}\int_{(-1/4k)}\Psi_\lambda(s;y,z)ds+\sum_{-1/4k\leqq\sigma\leqq 5/4}\operatorname{Res}\Psi_\lambda(s;y,z)\,,$$

$$(6.7)\qquad \frac{1}{2\pi i}\int_{(5/4)}\Psi_\lambda(s;y,z)ds=\frac{1}{2\pi i}\int_{(\varepsilon)}\Psi_\lambda(s;y,z)ds+\begin{cases}0 & \textit{if}\quad\lambda\neq 1\,,\\ \operatorname*{Res}_{1/k}\Psi_1 & \textit{if}\quad\lambda=1\,,\end{cases}\qquad(0<\varepsilon<1/k)$$

*and*

$$\log f(y; z)=\sum_{\lambda}\frac{1}{2\pi i}\int_{(\varepsilon)}\Psi_{\lambda}(s; y, z)ds+\operatorname*{Res}_{1/k}\Psi_{1}(s; y, z) .$$

PROOF. It is well-known that

(6.8) $$\zeta(1+ns)\ll(1+|t|)^{c}$$

for $-1/2\leqq\sigma\leqq 3$. Hence, using the estimation in the proof of Lemma 6.1, we easily see that

$$\int_{-1/4k+iT}^{5/i+4T}P_{\lambda}\cdot Z_{\lambda}\cdot\zeta(1+ns)ds\longrightarrow 0$$

as $|T|\to\infty$. From this follows (6.6). Similarly (6.7) is obtained.

LEMMA 6.3. *The series*

$$\sum_{\lambda}\frac{1}{2\pi i}\int_{(-1/4k)}\Psi_{\lambda}(s; y, z)ds$$

*is convergent.*

PROOF. Using the same notations as in § 5, we have by (5.2), (5.5), (6.3) and (6.8)

$$P_{\lambda}(s; y, z)\ll\exp\Big(-\theta\sum_{q=1}^{r+1}|t+V_{q}|\Big)\prod_{p=r_1+1}^{r_1+r_2}x_{p}^{l_p/2}(1+l_{p}) ,$$

$$Z_{\lambda}(ks; \mathrm{m}, \mu)\ll\prod_{q=1}^{r+1}(1+|t+V_{q}|)^{c}\prod_{p=r_1+1}^{r_1+r_2}(1+l_{p})^{c} ,$$

$$\zeta(1+ns)\ll(1+|t|)^{c} .$$

Hence

$$\sum_{\lambda}\frac{1}{2\pi i}\int_{(-1/4k)}\Psi_{\lambda}ds\ll\sum_{\{m\}}\sum_{\{l\}}\int_{-\infty}^{\infty}\exp\Big(-c\sum_{q=1}^{r+1}|t+V_{q}|\Big)dt\prod_{p=r_1+1}^{r_1+r_2}x_{p}^{l_p/2}(1+l_{p})^{c} ,$$

which is clearly convergent.

This lemma gives the convergence of the series

(6.9) $$R(y; z)=\sum_{\lambda}\sum_{\sigma=0}\operatorname{Res}\Psi_{\lambda}(s; y, z) ,$$

where $\sum_{\sigma=0}$ means the sum of the residues of $\Psi_{\lambda}$ at the poles on the line $\sigma=0$. We write

(6.10) $$\log f(y; z)=\sum_{\lambda}\frac{1}{2\pi i}\int_{(-1/4k)}\Psi_{\lambda}ds+R(y; z)+\operatorname*{Res}_{1/k}\Psi_{1} .$$

LEMMA 6.4. *Assume that the inequalities $|z_j|/y_j \leqq b$ $(j=1, \cdots, n)$ hold by a constant $b$, then, uniformly in $z_1, \cdots, z_n$, we have*

$$\sum_\lambda \frac{1}{2\pi i}\int_{(\varepsilon)} P_\lambda \cdot Z_\lambda ds = O(M^{\varepsilon k}) \qquad (0<\varepsilon<1/k), \tag{6.11}$$

$$\sum_\lambda \frac{1}{2\pi i}\int_{(-1/4k)} P_\lambda \cdot Z_\lambda ds = O(M^{-1/4}) .$$

*Similarly*

$$\sum_\lambda \frac{1}{2\pi i}\int_{(\varepsilon)} \Psi_\lambda ds = O(M^{\varepsilon k}) \qquad (0<\varepsilon<1/k), \tag{6.12}$$

$$\sum_\lambda \frac{1}{2\pi i}\int_{(-1/4k)} \Psi_\lambda ds = O(M^{-1/4}) .$$

PROOF. Since the proofs of these four formulas are similar, it will be sufficient to show (6.12). By (5.2), (5.5) and our assumption on $y$ and $z$, we have

$$P_\lambda(s;\, y,\, z) \ll (y_1 \cdots y_n)^{-\sigma} \exp\left(-c\sum_{q=1}^{r+1}|t+V_q|\right)$$
$$\times \prod_{p=r_1+1}^{r_1+r_2} x_p^{l_p/2}(1+l_p) .$$

From this and the estimations in the proof of Lemma 6.3, (6.12) can easily be obtained.

## § 7. Estimation of $H(z)$ on the $E_1(\gamma_j)$.

LEMMA 7.1. *Assume that $z$ in $E$ satisfies the inequalities*

$$2\pi|z_q|/y_q \leqq M^{-\Delta} \qquad (q=1, \cdots, r+1) .$$

*Then we have*

$$R(y;\, z) - R(y;\, 0) = O(M^{-\Delta/2}) .$$

PROOF. We have by (6.6) and (6.12)

$$\begin{aligned} R(y;\, z) - R(y;\, 0) \qquad\qquad\qquad\qquad & \qquad (7.1)\\ = \sum_\lambda \frac{1}{2\pi i}\left\{\int_{(\varepsilon)} - \int_{(-1/4k)}\right\}\{P_\lambda(s;\, y,\, z) - P_\lambda(s;\, y,\, 0)\} &\\ \times \zeta(1+ns) Z_\lambda(ks;\, \mathfrak{m},\, \mu) ds &\\ = \sum_\lambda \frac{1}{2\pi i}\int_{(\varepsilon)}\{P_\lambda(s;\, y,\, z) - P(s;\, y,\, 0)\}\zeta(1+ns)Z_\lambda(ks;\, \mathfrak{m},\, \mu) ds &\\ + O(M^{-1/4}) , & \end{aligned}$$

where $\varepsilon$ is a small positive number such that $0<\varepsilon<\Delta/2k$. By the definition of $P_\lambda(s; y, z)$,

$$P_\lambda(s; y, 0)=\begin{cases}\prod_{q=1}^{r_1}\dfrac{\Gamma(s+iV_q)}{y_q^{s+iV_q}}\prod_{p=r_1+1}^{r_1+r_2}\dfrac{G(2s+2iV_p, 0, 0)}{y_p^{2s+2iV_p}} & \text{if all } a_p \text{ are } 0\,, \\ 0 & \text{if not.}\end{cases}$$

Hence the last sum over the $\lambda$ in (7.1) can be written as follows:

$$\begin{aligned}(7.2)\qquad &\sum_{\{m\}}\sum_{\{a\}\neq\{0\}}\frac{1}{2\pi i}\int_{(\varepsilon)}P_\lambda(s; y, z)\zeta(1+ns)Z_\lambda(ks; \mathfrak{m}, \mu)ds \\ &+\sum_{\substack{\{m\}\\ \{a\}=\{0\}}}\frac{1}{2\pi i}\int_{(\varepsilon)}\{P_\lambda(s; y, z)-P_\lambda(s; y, 0)\}\zeta(1+ns)Z_\lambda(ks; \mathfrak{m}, \mu)ds \\ &=\Sigma_1+\Sigma_2\,,\end{aligned}$$

where the sum $\Sigma_1$ is taken over the Grössencharacters $\lambda$ for ideal numbers mod $\mathfrak{n}$ with $\{a_p\}\neq\{0\}$ and the sum $\Sigma_2$ is taken over the $\lambda$ with $\{a_p\}=\{0\}$. We put

$$x_p=\frac{4\pi^2|z_p|^2}{y_p^2+4\pi^2|z_p|^2}\qquad (p=r_1+1, \cdots, r_1+r_2)\,.$$

Then, by our assumption for $z$,

$$x_p\ll M^{-2\Delta}\qquad (p=r_1+1, \cdots, r_1+r_2)\,.$$

Moreover,

$$|\arg(y_q-2\pi iz_q)|\ll M^{-\Delta}\qquad (q=1, \cdots, r_1)\,.$$

Reviewing the proof of Lemma 6.4, we have, for $\sigma=\varepsilon$, the estimation of the integrand in the series $\Sigma_1$:

$$\begin{aligned}&P_\lambda(s; y, z)\zeta(1+ns)Z_\lambda(ks; \mathfrak{m}, \mu) \\ &\quad\ll M^{\varepsilon k}(1+|t|)^c\exp\Big(-\frac{1}{4}\sum_{q=1}^{r+1}|t+V_q|\Big)\prod_{p=r_1+1}^{r_1+r_2}x_p^{l_p/2}(1+l_p)\,.\end{aligned}$$

Hence

$$\begin{aligned}(7.3)\qquad \Sigma_1&\ll M^{\varepsilon k}\sum_{\{l\}\neq\{0\}}\prod_{p=r_1+1}^{r_1+r_2}x_p^{l_p/2}(1+l_p) \\ &\ll M^{\varepsilon k}\Big(\prod_{p=r_1+1}^{r_1+r_2}\sum_{l_p=0}^{\infty}x_p^{l_p/2}(1+l_p)-1\Big) \\ &=M^{\varepsilon k}\Big(\prod_{p=r_1+1}^{r_1+r_2}(1-x_p^{1/2})^{-2}-1\Big)\ll M^{\varepsilon k-\Delta}\,.\end{aligned}$$

As for $\Sigma_2$, we write

$$\Sigma_2=\sum_{\substack{\{m\}\\ \{l\}=0}}\frac{1}{2\pi i}\int_{(\varepsilon)}\prod_{q=1}^{r_1}\Gamma(s+iV_q)\zeta(1+ns)Z_\lambda(ks;\mathfrak{m},\mu)$$
$$\times\left\{\prod_{q=1}^{r_1}(y_q-2\pi i z_q)^{-s-iV_q}\prod_{p=r_1+1}^{r_1+r_2}\frac{G(2s+2iV_p,0,x_p)}{(y_p^2+4\pi^2|z_p|^2)^{s+iV_p}}\right.$$
$$\left.-\prod_{q=1}^{r_1}y_q^{-s-iV_q}\prod_{p=r_1+1}^{r_1+r_2}\frac{G(2s+2iV_p,0,0)}{y_p^{2s+2iV_p}}\right\}ds$$

and we must estimate the expressions in the braces { }. Consider

$$\prod_{q=1}^{r_1}(y_q-2\pi i z_q)^{-s-iV_q}=\prod_{q=1}^{r_1}y_q^{-s-iV_q}+\prod_{q=1}^{r_1}y_q^{-s-iV_q}\{\prod_{q=1}^{r_1}(1-iw_q)^{-s-iV_q}-1\}\,, \tag{7.4}$$

where $w_q=2\pi z_q/y_q$ $(q=1,\cdots,r_1)$. If we put, for a complex variable $z$,

$$\varphi(z)=\prod_{q=1}^{r_1}(1-izw_q)^{-s-iV_q}\,,$$

then $\varphi(z)$ is regular for $0\leqq z\leqq 1$ and

$$|\prod_{q=1}^{r_1}(1-iw_q)^{-s-iV_q}-1|=|\varphi(1)-\varphi(0)|$$
$$=\left|\int_0^1\varphi'(z)dz\right|\leqq\max_{0\leqq z\leqq 1}|\varphi'(z)|\,.$$

Since

$$\varphi'(z)=\varphi(z)\sum_{q=1}^{r_1}(s+iV_q)\frac{iw_q}{1-izw_q}\,,$$

we have

$$\max_{0\leqq z\leqq 1}|\varphi'(z)|\leqq\sum_{q=1}^{r_1}(1+|t+V_q|)|w_q|\max_{0\leqq z\leqq 1}|\varphi(z)|$$
$$\ll M^{-\Delta}\sum_{q=1}^{r_1}(1+|t+V_q|)\max_{0\leqq z\leqq 1}|\varphi(z)|\,.$$

By the definition of $\varphi(z)$,

$$|\varphi(z)|=\prod_{q=1}^{r_1}|1-izw_q|^{-\sigma}\exp(-|t+V_q|\tan^{-1}(zw_q))$$
$$\ll\exp\left(\frac{\pi}{8}\sum_{q=1}^{r_1}|t+V_q|\right).$$

Therefore

$$\max|\varphi'(z)|\ll M^{-\Delta}\sum_{q=1}^{r_1}(1+|t+V_q|)\exp\left(\frac{\pi}{8}\sum_{q=1}^{r_1}|t+V_q|\right).$$

From this and (7.4) we have

$$(7.5)\qquad \prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q)}{(y_q-2\pi iz_q)^{s+iV_q}}=\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q)}{y_q^{s++iV_q}} +O\Big(M^{-\Delta+k\varepsilon r_1/n}\prod_{q=1}^{r_1}(1+|t+V_q|)\exp\Big(-\frac{\pi}{8}\sum_{q=1}^{r_1}|t+V_q|\Big)\Big)\,.$$

Similarly we have

$$(7.6)\qquad (y_p^2+4\pi^2|z_p|^2)^{-s-iV_p}=y_p^{-2s-2iV_p}+O(M^{-2\Delta+2\varepsilon k/n}(1+|t+V_p|))\,.$$

Finally we consider

$$G(s,\,0,\,x)-G(s,\,0,\,0)\,.$$

Since

$$(7.7)\qquad G(s,\,0,\,x)=\frac{\Gamma(s/2)}{2\sqrt{\pi}\,\Gamma((1-s)/2)}\int_0^1 u^{-(s+1)/2}(1-u)^{(s-1)/2}(1-ux)^{-s/2}du$$

([6], p. 369, (4.315)), we must estimate

$$\int_0^1 u^{-(s+1)/2}(1-u)^{(s-1)/2}\{(1-ux)^{-s/2}-1\}du\,.$$

We consider this as a complex integral in $u$-plane and change the path of integration as in [6], p. 372. New path $C$ is an arc of a circle passing $u=0$ and $u=1$. On this $C$, the inequality

$$t\arg\frac{u(1-ux)}{1-u}\leqq-\frac{|t|}{4}\sqrt{1-x}$$

holds for $0<x<1$, and, moreover, $c_1\leqq|u/\mathrm{Re}\,u|\leqq c_2$. If we put

$$\psi(z)=(1-zux)^{-s/2}\,,$$

then

$$|(1-ux)^{-s/2}-1|\leqq\max_{0\leqq z\leqq 1}|\psi'(z)|\leqq\max_{0\leqq z\leqq 1}\left|\psi(z)\frac{s}{2}\cdot\frac{ux}{1-zux}\right| \ll x|su|\max\left\{|1-zux|^{-(\sigma/2)-1}\exp\Big(\frac{t}{2}\arg(1-zux)\Big)\right\}\,.$$

By the property of the path stated above, we have

$$(7.8)\qquad \int_0^1 u^{-(s+1)/2}(1-u)^{(s-1)/2}\{(1-ux)^{-s/2}-1\}du \ll x|s|\int_C|u|^{(1-\sigma)/2}|1-u|^{(\sigma-1)/2}\exp\Big(\frac{t}{2}\arg\frac{u(1-ux)}{1-u}\Big)|du| \ll x|s|e^{-(|t|/4)\sqrt{1-x}}\int_C|u|^{(1-\sigma)/2}|1-u|^{(\sigma-1)/2}|du|\,.$$

Since $c_1 \leqq |u/\mathrm{Re}\, u| \leqq c_2$ on $C$,

$$\int_C |u|^{(1-\sigma)/2}|1-u|^{(\sigma-1)/2}|du| \ll \int_0^1 x^{(1-\sigma)/2}(1-x)^{(\sigma-1)/2}dx \ll 1 . \tag{7.9}$$

By (7.7), (7.8) and (7.9) and the estimation for gamma function, we have

$$G(s, 0, x)-G(s, 0, 0) \ll x(1+|t|)^{\sigma+1/2}e^{-(|t|/4)\sqrt{1-x}}$$

or

$$G(2s+2iV_p, 0, x_p)=G(2s+2iV_p, 0, 0)+O(M^{-2\Delta}e^{-c|t+V_p|}) . \tag{7.10}$$

Similarly we have

$$G(s, 0, x) \ll e^{-(|t|/4)\sqrt{1-x}} . \tag{7.11}$$

By (7.5), (7.6), (7.10) and (7.11),

$$\begin{aligned}
&\prod_{q=1}^{r_1} \frac{\Gamma(s+iV_q)}{(y_q-2\pi i z_q)^{s+iV_q}} \prod_{p=r_1+1}^{r_1+r_2} (y_p^2+4\pi^2|z_p|^2)^{-s-iV_p} G(2s+2iV_p, 0, x_p) \\
&= \prod_{q=1}^{r_1} \frac{\Gamma(s+iV_q)}{y_q^{s+iV_q}} \prod_{p=r_1+1}^{r_1+r_2} y_p^{-2s-2iV_p} G(2s+2iV_p, 0, 0) \\
&\quad +O(M^{k\varepsilon-\Delta} \prod_{q=1}^{r_1+r_2} (1+|t+V_q|) \exp(-c\sum_{q=1}^{r+1}|t+V_q|)) .
\end{aligned}$$

Hence

$$\begin{aligned}
\Sigma_2 &\ll M^{k\varepsilon-\Delta} \sum_{\{m\}} \int_{-\infty}^{\infty} \prod_{q=1}^{r+1} (1+|t+V_q|)^c (1+|t|)^c \exp(-c\sum_{q=1}^{r+1}|t+V_q|)dt \\
&\ll M^{k\varepsilon-\Delta} .
\end{aligned} \tag{7.12}$$

Collecting (7.1), (7.2), (7.3) and (7.12), we have

$$R(y; z)-R(y; 0) \ll M^{-1/4}+M^{k\varepsilon-\Delta} \ll M^{-\Delta/2}$$

and the proof is completed.

THEOREM 7.2. *Assume that* $z \in E_1(\gamma_j)$. *Let* $\gamma$ *be a number such that* $\gamma \equiv \gamma_j \pmod{\mathfrak{d}^{-1}}$ *and* $\tau = z-\gamma$ *satisfies the inequalities*

$$2\pi|\tau_q|/y_q \leqq M^{-\Delta} \qquad (q=1, \cdots, r+1) .$$

*Put*

$$\begin{aligned}
w_q &= 2\pi\tau_q/y_q && (q=1, \cdots, r_1) , \\
w_p &= 2\pi|\tau_p|/y_p && (p=r_1+1, \cdots, r_1+r_2) .
\end{aligned}$$

*Then we have*

$$H(z)=\exp\left\{M\left(i\frac{1}{k}\sum_{q=1}^{r_1} w_q - Q - \frac{2+k}{2k^2}\sum_{p=r_1+1}^{r_1+r_2} w_p^2\right)\right\}(1+O(M^{-c_1})) ,$$

*where $Q$ is a positive definite quadratic form of $w_1, \cdots, w_{r_1}$ with determinant*

$$|Q| = \frac{k+r_1}{2^{r_1} k^{r_1+1}} .$$

**Proof.** By (6.5), (6.10) and (6.12) we have

$$\begin{aligned}
\log f(y; z) &= \log f(y; \tau) \\
&= \frac{2^{2r_2/k}}{\Gamma(2/k)^{r_2}} M \prod_{q=1}^{r_1} (1 - i w_q)^{-1/k} \prod_{p=r_1+1}^{r_1+r_2} (1+w_p^2)^{-1/k} G\left(\frac{2}{k}, 0, \frac{w_p^2}{1+w_p^2}\right) \\
&\quad + R(y; \tau) + O(M^{-1/4}) \\
&= M \prod_{q=1}^{r_1} (1 - i w_q)^{-1/k} \prod_{p=r_1+1}^{r_1+r_2} (1+w_p^2)^{-1/k} \left(1 + \frac{k-2}{2k^2} w_p^2\right) \\
&\quad + R(y; \tau) + O(M^{-1/4}) + O(M^{-c_2}) \\
&= M\left(1 + i \frac{1}{k} \sum_{q=1}^{r_1} w_q - Q - \frac{2+k}{2k^2} \sum_{p=r_1+1}^{r_1+r_2} w_p^2\right) + R(y; \tau) + O(M^{-c_3}) ,
\end{aligned}$$

where

$$Q = \frac{1+k}{2k^2} \sum_{q=1}^{r_1} w_q^2 + \frac{1}{k^2} \sum_{1 \le q < q_1 \le r_1} w_q w_{q_1} .$$

We easily see that $Q$ is positive definite and the determinant $|Q|$ of $Q$ is

$$|Q| = \frac{1}{(2k^2)^{r_1}} \begin{vmatrix} 1+k & 1 & \cdots & 1 \\ 1 & 1+k & \cdots & 1 \\ & \cdots\cdots & & \\ 1 & 1 & \cdots & 1+k \end{vmatrix} = \frac{k+r_1}{2^{r_1} k^{r_1+1}} .$$

Since

$$\log f(y; 0) = M + R(y; 0) + O(M^{-c_4}) ,$$

we have by Lemma 7.1

$$\log H(z) = M\left(i \frac{1}{k} \sum_{q=1}^{r_1} w_q - Q - \frac{2+k}{2k^2} \sum_{p=r_1+1}^{r_1+r_2} w_p^2\right) + O(M^{-c_5})$$

and the proof is completed.

## § 8. Estimation of $H(z)$ on the $E_1(\sigma_i)$.

In general, we have

(8.1) $$\begin{aligned} \log |H(z)| &= \operatorname{Re} \log f(y; z) - \log f(y; 0) \\ &\le \operatorname{Re} g(y; z) - g(y; 0) . \end{aligned}$$

THEOREM 8.1. *If* $z \in E_1(\sigma_i)$, *then we have*

$$|H(z)| \leqq \exp(-cM) .$$

PROOF. Assume that $z \in E_1(\sigma_i)$. We can take $\sigma$ such that $\sigma \equiv \sigma_i (\bmod \mathfrak{d}^{-1})$ and $\tau = z - \sigma$ satisfies the inequalities

$$2\pi |\tau_q| / y_q \leqq M^{-\Delta} \qquad (q = 1, \cdots, n) .$$

Since $\sigma_i \in A_1$,

$$g(y; z) = \exp\{2\pi i S(\mu^k \sigma_i)\} g(y; \tau) . \tag{8.2}$$

Defining $w_q$ $(q = 1, \cdots, r_1)$ and $w_p$ $(p = r_1 + 1, \cdots, r_1 + r_2)$ as in Theorem 7.2, we have by (6.1) and (6.4)

$$\begin{aligned} g(y; \tau) &= \frac{2^{2r_2/k}}{\Gamma(2/k)^{r_2}} \frac{M}{\zeta(1+n/k)} \prod_{q=1}^{r_1} (1 - i w_q)^{-1/k} \\ &\quad \times \prod_{p=r_1+1}^{r_1+r_2} (1 + w_p^2)^{-1/k} G\left(\frac{2}{k}, 0, \frac{w_p^2}{1+w_p^2}\right) + O(M^{\varepsilon k}) \\ &= \frac{M}{\zeta(1+n/k)} (1 + O(M^{-\Delta})) + O(M^{\varepsilon k}) . \end{aligned} \tag{8.3}$$

Combining (8.1), (8.2) and (8.3), we have

$$\begin{aligned} \log |H(z)| &\leqq (\cos 2\pi S(\mu^k \sigma_i) - 1) \frac{M}{\zeta(1+n/k)} + O(M^{1-\Delta}) \\ &\leqq (\delta_A - 1) \frac{M}{\zeta(1+n/k)} + O(M^{1-\Delta}) , \end{aligned}$$

where $\delta_A$ is the number defined in (2.3). Thus we have

$$\log |H(z)| \leqq -cM$$

for sufficiently large $M$.

## § 9. Estimation of $H(z)$ on $E_2$.

THEOREM 9.1. *If* $z \in E_2$, *then*

$$|H(z)| \leqq \exp(-cM^{1-2\Delta}) .$$

PROOF. Assume that $z \in E_2(\gamma_j)$. We can take $\gamma$ such that $\gamma \equiv \gamma_j$ $(\bmod \mathfrak{d}^{-1})$ and $\tau = z - \gamma$ satisfies the inequalities

$$cM^{-2\Delta} \leqq 4\pi^2 \sum_{i=1}^{n} |\tau_i|^2 / y_i^2 \leqq 4\pi^2 n C_0^2 .$$

Defining $w_q$, $w_p$ as in Theorem 8.1, we have

$$(9.1)\qquad g(y;\tau)=\frac{2^{2r_2/k}}{\Gamma(2/k)^{r_2}}\frac{M}{\zeta(1+n/k)}\prod_{q=1}^{r_1}(1-iw_q)^{-1/k} \times\prod_{p=r_1+1}^{r_1+r_2}(1+w_p^2)^{-1/k}G\Big(\frac{2}{k},\,0,\,\frac{w_p^2}{1+w_p^2}\Big)+O(M^{\varepsilon k})\,.$$

In the present case, we can not use the expansion by $\{w_q\}$ as is utilized in Theorem 8.1. We put

$$\sigma=2/k\,,\qquad x=x_p=w_p^2/(1+w_p^2)$$

and consider $G(\sigma, 0, x)$. It is known that

$$G(\sigma,0,x)=\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\sigma/2)}{\Gamma((3-\sigma)/2)}\int_0^1\frac{u^{(1-\sigma)/2}(1-u)^{(\sigma-1)/2}(1-ux-(1-u)x\sigma/2)}{(1-ux)^{1+\sigma/2}}du$$

([6], p. 370, (4.317)). Using the inequalities

$$(1-ux)^{-\sigma/2}\leqq(1-x)^{-\sigma/2}\,,$$

$$\frac{1-ux-(1-u)x\sigma/2}{1-ux}=1-\frac{\sigma}{2}x\frac{1-u}{1-ux}\leqq1-\frac{\sigma}{2}x(1-u)\,,$$

we have

$$\begin{aligned}(1-x)^{\sigma/2}G(\sigma,0,x)&\leqq\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\sigma/2)}{\Gamma((3-\sigma)/2)}\int_0^1u^{(1-\sigma)/2}(1-u)^{(\sigma-1)/2}\Big\{1-\frac{\sigma}{2}x(1-u)\Big\}du\\&=\frac{1}{2\sqrt{\pi}}\frac{\Gamma(\sigma/2)}{\Gamma((3-\sigma)/2)}\Big\{\Big(1-\frac{\sigma}{2}x\Big)\Gamma\Big(\frac{3-\sigma}{2}\Big)\Gamma\Big(\frac{\sigma+1}{2}\Big)\\&\quad+\frac{\sigma}{2}x\frac{1}{2}\Gamma\Big(\frac{5-\sigma}{2}\Big)\Gamma\Big(\frac{\sigma+1}{2}\Big)\Big\}\\&=\frac{1}{2\sqrt{\pi}}\Gamma\Big(\frac{\sigma}{2}\Big)\Gamma\Big(\frac{\sigma+1}{2}\Big)\Big\{1-\frac{\sigma}{2}x+\frac{\sigma}{4}\frac{3-\sigma}{2}x\Big\}\\&=\frac{\Gamma(\sigma)}{2^\sigma}\Big(1-\frac{\sigma}{8}(1+\sigma)x\Big)\,.\end{aligned}$$

Since

$$(1-x)^{\sigma/2}=(1+w_p^2)^{-1/k}\,,$$

we have

$$(1+w_p^2)^{-1/k}G\Big(\frac{2}{k},\,0,\,\frac{w_p^2}{1+w_p^2}\Big)\leqq\frac{\Gamma(2/k)}{2^{2/k}}\Big(1-\frac{w_p^2}{4k(1+b_0)}\Big)\,,$$

where $b_0=4\pi^2nC_0^2$. Hence, from (9.1) it follows that

$$|g(y;\tau)|\leqq\frac{M}{\zeta(1+n/k)}\prod_{q=1}^{r_1}(1+w_p^2)^{1/2k}\prod_{p=r_1+1}^{r_1+r_2}\left(1-\frac{w_p^2}{4k(1+b_0)}\right)+O(M^{\epsilon k})$$
$$\leqq\frac{M}{\zeta(1+n/k)}(1-c\sum_{q=1}^{n}w_q^2)+O(M^{\epsilon k})$$
$$\leqq\frac{M}{\zeta(1+n/k)}(1-cM^{-2\Delta})+O(M^{\epsilon k}) .$$

Combining this inequality and (8.1), we have for $z\in E_2(\gamma_j)$

$$\log|H(z)|\leqq|g(y;\tau)|-g(y;0)$$
$$\leqq -cM^{1-2\Delta}+O(M^{\epsilon k})$$
$$\leqq -cM^{1-2\Delta} .$$

The same result holds for $z\in E_2(\sigma_i)$ and the proof is completed.

## §10. Some results on trigonometrical sums.

Let $N$ be sufficiently large number. Let $V=V(N)$ be the set of $(u_1, \cdots, u_n)$ in $n$-dimensional space defined as follows:

$$0<u_q\leqq N \qquad (q=1,\cdots,r_1) ,$$
$$u_p^2+u_{p+r_2}^2\leqq N^2 \qquad (p=r_1+1,\cdots,r_1+r_2) .$$

We consider the trigonometrical sum

$$S(z;N)=\sum_{\substack{x(\nu)\in V\\ \nu\equiv\mu(\mathfrak{m})}}\exp(2\pi i S(\nu^k z)) , \tag{10.1}$$

where the sum is taken over the $\nu$ such that $x(\nu)\in V$ and $\nu\equiv\mu$ (mod $\mathfrak{m}$).

To estimate $S(z;N)$, we here introduce the Farey division of $E$. We put

$$H=N^{k-1+a} ,\quad T=N^{1-a}\quad (1/2<a<1)$$

and define the set

$$\Gamma=\{\gamma\mid\gamma\in E ,\quad \gamma\longrightarrow\mathfrak{a} ,\quad N(\mathfrak{a})\leqq T^n\} .$$

For each element $\gamma$ of $\Gamma$ with $\gamma\to\mathfrak{a}$ we define a subset $B_\gamma$ of $E$ as follows;

$$B_\gamma=\{z\mid z\in E ,\ N(\max(H|z-\gamma_1|, T^{-1}))\leqq N(\mathfrak{a})^{-1}\quad \text{for any } \gamma_1\equiv\gamma(\text{mod } \mathfrak{d}^{-1})\}$$

and put

$$B^0=E-\bigcup_{\gamma\in\Gamma}B_\gamma .$$

This division of $E$ into $B^0$ and $B_\gamma(\gamma \in \Gamma)$ is called the Farey division of $E$ with respect to $(H, T)$ ([5], § 2).

**THEOREM 10.1.** *Assume that $z \in B_\gamma$ with $\gamma \to \mathfrak{a}$. By the definition of $B_\gamma$, there exists $\gamma_1$ such that $\gamma_1 \equiv \gamma (\mathrm{mod}\ \mathfrak{d}^{-1})$ and $\tau = z - \gamma_1$ satisfies the inequality*

$$N(\max(T^{-1}, H|\tau|)) \leqq N(\mathfrak{a})^{-1} . \tag{10.2}$$

*Then we have*

$$S(z; N) = \frac{2^{r_2}}{\sqrt{D}\, N(\mathfrak{m})} G(\gamma) \int \cdots_{V} \int e^{2\pi i S(\eta^k \tau)} du_1 \cdots du_n + O(N^{n-a}) , \tag{10.3}$$

*where $G(\gamma)$ is the sum defined in* § 2, *and, in the integral in* (10.3), *$\eta = (\eta_1, \cdots, \eta_n)$ is defined by*

$$\eta_q = u_q \quad (q = 1, \cdots, r_1) ,$$
$$\eta_p = u_p + i u_{p+r_2} = \bar{\eta}_{p+r_2} \qquad (p = r_1 + 1, \cdots, r_1 + r_2) .$$

**PROOF.** The proof is analogous to that of [8], Lemma 7. First we see

$$S(z; N) = \sum_{\sigma} e^{2\pi i S(\sigma^k \tau)} \sum_{\substack{x(\lambda+\sigma) \in V \\ \lambda \in \mathfrak{a}, \lambda+\sigma \equiv \mu(\mathfrak{m})}} \exp\{2\pi i\, S((\lambda + \sigma)^k \tau)\} , \tag{10.4}$$

where $\sigma$ runs through the residues mod $\mathfrak{a}$ such that $\sigma \equiv \mu$ (mod$(\mathfrak{m}, \mathfrak{a})$) and the inner sum is over the $\lambda$ in $\mathfrak{a}$ such that $\lambda + \sigma \equiv \mu$ (mod $\mathfrak{m}$) and $x(\lambda + \sigma) \in V$. We take a number $\lambda_0 \in \mathfrak{a}$ such that $\lambda_0 + \sigma \equiv \mu$ (mod $\mathfrak{m}$) and write the inner sum of (10.4) as

$$\sum_{\substack{x(\lambda+\kappa) \in V \\ \lambda \in \mathfrak{a}_1 \mathfrak{m}}} \exp\{2\pi i\, S((\lambda + \kappa)^k \tau)\} ,$$

where $\kappa = \sigma + \lambda_0$ and $\mathfrak{a}_1 = \mathfrak{a}/(\mathfrak{a}, \mathfrak{m})$. In view of the inequality (10.2), we can take $\theta = (\theta_1, \cdots, \theta_n)$ satisfying the conditions

$$\theta_1, \cdots, \theta_{r_1} > 0 , \quad \theta_p = \theta_{p+r_2} > 0 \quad (p = r_1 + 1, \cdots, r_1 + r_2) ,$$
$$\theta \max(H|\tau|, T^{-1}) \leqq D^{-1/2n} ,$$
$$N(\theta) = \sqrt{D}\, N(\mathfrak{a}) .$$

By Minkowski's theorem, there exists a number $\alpha_0$ such that

$$\alpha_0 \in \mathfrak{a} , \quad 0 < |\alpha_0| \leqq \theta .$$

Put $\alpha_0 \mathfrak{a}^{-1} = \mathfrak{b}$, then $N(\mathfrak{b}) = N(\alpha_0) N(\mathfrak{a})^{-1} \leqq N(\theta) N(\mathfrak{a})^{-1} = \sqrt{D}$. Hence the number of the $\mathfrak{b}$ depends only on $K$. Let $\beta_1, \cdots, \beta_n$ be a basis of $\mathfrak{b}^{-1}$. Then $\alpha_i=\alpha_0\beta_i$ $(i=1, \cdots, n)$ is a basis of $\mathfrak{a}$. Since $\mathfrak{a}_1\mathfrak{m}\subset\mathfrak{a}$, we can take a basis $\rho_1\cdots, \rho_n$ of $\mathfrak{a}_1\mathfrak{m}$ such that

$$\rho_j=\sum_{i=j}^{n} a_{ji}\alpha_i \qquad (j=1, \cdots, n)\ ,$$

where the $a_{ji}$ are rational integers. We may further assume that $0\leqq a_{jl}\leqq a_{il}\leqq a_{ll}$ $(j\leqq i\leqq l\leqq n)$. Since $a_{11}\cdots a_{nn}=N(\mathfrak{a}_1\mathfrak{m})/N(\mathfrak{a})\leqq N(\mathfrak{m})$, we have $\rho_i=O(\theta)$ $(i=1, \cdots, n)$. Take $\lambda$ in the sum and write $\lambda=\sum m_i\rho_i$ with rational integers $m_1, \cdots, m_n$. We define $\eta=\sum \xi_i\rho_i$, where $\xi_1, \cdots, \xi_n$ are real numbers. If $m_i\leqq\xi_i\leqq m_i+1$ $(i=1, \cdots, n)$, then $\eta-\lambda\ll\theta$ and

$$\begin{aligned}(\eta+\kappa)^k\tau-(\lambda+\kappa)^k\tau &\ll|\tau|\theta(|\eta+\kappa|^{k-1}+|\lambda+\kappa|^{k-1})\\ &\ll H^{-1}N^{k-1}=N^{-a}\ .\end{aligned}$$

Hence we have

$$\exp\{2\pi i\ S((\lambda+\kappa)^k\tau)\}=\int_{m_i}^{m_i+1}\!\!\cdots\!\int\exp\{2\pi i\ S((\eta+\kappa)^k\tau)\}d\xi_1\cdots d\xi_n+O(N^{-a})\ .$$

We change the variables of integration from $(\xi_1, \cdots, \xi_n)$ to $(u_1, \cdots, u_n)$. Then

$$\frac{\partial(u_1, \cdots, u_n)}{\partial(\xi_1, \cdots, \xi_n)}=2^{-r_2}\sqrt{D}\,N(\mathfrak{a}_1\mathfrak{m})\ .$$

Hence

$$\begin{aligned}(10.5)\qquad &\exp\{2\pi i\,S((\lambda+\kappa)^k\tau)\}\\ &=\frac{2^{r_2}}{\sqrt{D}\,N(\mathfrak{a}_1\mathfrak{m})}\int\!\cdots\!\int_{E_\lambda}\exp\{2\pi i\ S((\eta+\kappa)^k)\}du_1\cdots du_n+O(N^{-a})\ ,\end{aligned}$$

where $E_\lambda$ is the set of $(u_1, \cdots, u_n)$ such that

$$(u_1, \cdots, u_n)=x(\lambda+\kappa)+\sum_{i=1}^{n} t_i x(\rho_i)\ , \qquad 0\leqq t_i<1\ (i=1, \cdots, n)\ .$$

Summing up both sides of (10.5) over the $\lambda$ such that

$$x(\lambda+\kappa)\in V\ , \qquad \lambda\in\mathfrak{a}_1\mathfrak{m}\ ,$$

and noting that the number of such $\lambda$ is $O(N^n\cdot N(\mathfrak{a}_1\mathfrak{m})^{-1})$, we have

$$\begin{aligned}(10.6)\qquad &\sum_{\substack{x(\lambda+\kappa)\in V\\ \lambda\in\mathfrak{a}_1\mathfrak{m}}}\exp\{2\pi i\ S((\lambda+\kappa)^k\tau)\}\\ &=\frac{2^{r_2}}{\sqrt{D}\,N(\mathfrak{a}_1\mathfrak{m})}\sum_{\substack{x(\lambda+\kappa)\in V\\ \lambda\in\mathfrak{a}_1\mathfrak{m}}}\int\!\cdots\!\int_{E_\lambda}\exp\{2\pi i\ S((\eta+\kappa)^k\tau)\}du_1\cdots du_n\\ &\quad+O(N^{n-a}N(\mathfrak{a}_1\mathfrak{m})^{-1})\end{aligned}$$

$$=\frac{2^{r_2}}{\sqrt{D}N(\mathfrak{a}_1\mathfrak{m})}\int_V\cdots\int\exp\{2\pi i\, S((\eta+\kappa)^k\tau)\}du_1\cdots du_n$$
$$+O(N^{n-1}N(\mathfrak{a}_1\mathfrak{m})^{-1}d)+O(N^{n-a}N(\mathfrak{a}_1\mathfrak{m})^{-1}) ,$$

where $d$ is the diameter of $E_\lambda$. Putting (10.6) into (10.4) and noting that $d=O(\theta)=O(T)=O(N^{1-a})$, we have theorem at once.

Now we quote two lemmas from [8] and a lemma from [5].

LEMMA 10.1. *We have*

$$(10.7)\quad S(z;N)^{2^{k-1}}\ll N^{n(2^{k-1}-1)}$$
$$+N^{n(2^{k-1}-k)}\sum_{\lambda_1,\cdots,\lambda_{k-1}}\left|\sum_{\lambda}\exp\{2\pi iS(k!\,\lambda_1\cdots\lambda_{k-1}\lambda z)\}\right| ,$$

*where* $\lambda, \lambda_1, \cdots, \lambda_{k-1}$ *run through the non-zero elements of* $\mathfrak{m}$ *such that*

$$x(\lambda+\lambda_{k_1}+\cdots+\lambda_{k_g})\in V_1=V-x(\mu)$$

*for all* $1\leqq k_1<\cdots<k_g\leqq k-1$, $V_1$ *being the translation of* $V$ *by* $-x(\mu)$.

PROOF. ([8], p. 331, (58)).

In (10.7), if $\lambda_1, \cdots, \lambda_{k-1}$ are fixed, then $\lambda$ runs through the numbers such that

$$x(\lambda)\in P=P(\lambda_1,\cdots,\lambda_{k-1})=\bigcap_{1\leqq k_1<\cdots<k_g\leqq k-1}(V_1-x(\lambda_{k_1}+\cdots+\lambda_{k_g})) .$$

Putting $\nu=k!\,\lambda_1\cdots\lambda_{k-1}$, we have

LEMMA 10.2. *If we define the sum*

$$S(P)=\sum_{\substack{x(\lambda)\in P\\ \lambda\in\mathfrak{m}}}\exp(2\pi i\, S(\lambda\nu z)) ,$$

*then*

$$(10.8)\qquad S(P)\ll N^{n-1}\min_{1\leqq i\leqq n}(N, \|S(\mu_i\nu z)\|^{-1}) ,$$

*where* $\mu_1, \cdots, \mu_n$ *is the basis of* $\mathfrak{m}$, $\|x\|$ *is the distance between* $x$ *and the nearest rational integers.*

PROOF. ([8], p. 332, (64)).

LEMMA 10.3. *Let* $z$ *be a point in* $B^0$, *which is defined by the Farey division with respect to* $(H, T)$. *Let* $Q$ *be an* $n$*-dimensional cube*

$$Q=\{(x_1,\cdots,x_n)\mid |x_i|\leqq W\ (i=1,\cdots,n)\}$$

*and define*

$$L=\sum_{x(\nu)\in Q}\min(U,\ ||S(\mu_i\nu z)||^{-1})\ ,$$

*where the sum is taken over the integers* $\nu$ *such that* $x(\nu)\in Q$ *and* $\mu_1, \cdots, \mu_n$ *is the basis of* $\mathfrak{O}$. *Then we have*

$$L\ll W^n U\left(\frac{1}{T}+\frac{1}{W}+\frac{H\log H}{WU}+\frac{\log H}{U}\right).$$

PROOF. ([5], Theorem 3.3).

THEOREM 10.2. *If* $k>1$ *and* $z\in B^0$, *then*

$$S(z;\ N)^{2^{k-1}}\ll N^{n(2^{k-1}-1)}+N^{n\cdot 2^{k-1}+\varepsilon+a-1}\ ,$$

*where* $\varepsilon$ *is arbitrary small positive constant.*

PROOF. Combining Lemmas 10.1 and 10.2, we have

$$\text{(10.9)}\qquad \begin{aligned} S(z;\ N)^{2^{k-1}}&\ll N^{n(2^{k-1}-1)}\\ &+N^{n(2^{k-1}-k)+n-1}\sum_{\lambda_1,\cdots,\lambda_{k-1}}\min_{1\leqq i\leqq n}(N,\ ||S(\mu_i\nu z)||^{-1})\ , \end{aligned}$$

where $\lambda_1, \cdots, \lambda_{k-1}$ run through all non-zero integers such that $|\lambda_i|\leqq cN$ $(i=1, \cdots, k-1)$, $\nu=k!\ \lambda_1\cdots\lambda_{k-1}$ and $\mu_1, \cdots, \mu_n$ is a basis of $\mathfrak{O}$. The value of the last sum in (10.9) does not exceed

$$\text{(10.10)}\qquad \sum_{|\nu|\leqq cN^{k-1}}\min_{1\leqq i\leqq n}(N,\ ||S(\mu_i\nu z)||^{-1})\sum_{\lambda_1,\cdots,\lambda_{k-1}}^{*}1\ ,$$

where $\sum^*1$ is the number of $(\lambda_1, \cdots, \lambda_{k-1})$ such that

$$\nu=k!\ \lambda_1\cdots\lambda_{k-1}\ ,$$
$$|\lambda_i|\leqq cN\qquad (i=1, \cdots, k-1)\ .$$

Since

$$\sum{}^*1\ll N^{\varepsilon}N(\nu)^{\varepsilon}\ ,$$

we have

$$\begin{aligned} S(z;\ N)^{2^{k-1}}&\ll N^{n(2^{k-1}-1)}\\ &+N^{n(2^{k-1}-k)+n-1+\varepsilon}\sum_{|\nu|\leqq cN^{k-1}}\min_{1\leqq i\leqq n}(N,\ ||S(\mu_i\nu z)||^{-1})\ , \end{aligned}$$

where the sum is taken over the integers $\nu$ such that $|\nu|\leqq cN^{k-1}$. Applying Lemma 10.3 to the last sum, we have

$$\begin{aligned} &\sum\min_{1\leqq i\leqq n}(N,\ ||S(\mu_i\nu z)||^{-1})\\ &\quad\ll N^{n(k-1)+1}\left(\frac{1}{T}+\frac{1}{N^{k-1}}+\frac{H\log H}{N^k}+\frac{\log H}{N}\right)\ll N^{n(k-1)+a}\log N \end{aligned}$$

and the proof is completed.

## § 11. Estimation of $H(z)$ on $E_3$.

Consider $S(z; N)$ in previous paragraph with $N=XM^{1/n}$, $X$ being a constant which will be determined later.

THEOREM 11.1. *If* $z \in E_3$, *then*

$$\operatorname{Re} S(z; N) - S(0; N) \leqq -cN^n .$$

PROOF. If $k=1$, then we have, by (10.1),

$$S(z; N) \ll N^{n-1} \min_{1 \leqq i \leqq n} (\|S(\mu_i z)\|^{-1}) , \tag{11.1}$$

where $\mu_1, \cdots, \mu_n$ is a basis of $\mathfrak{m}$. Let $\rho_1, \cdots, \rho_n$ be a basis of $(\mathfrak{m}\mathfrak{d})^{-1}$ such that

$$S(\rho_i \mu_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad (1 \leqq i, j \leqq n) .$$

We write $S(\mu_i z) = a_i + d_i$ $(i=1, \cdots, n)$ with rational integers $a_i$ and $-1/2 \leqq d_i < 1/2$ $(i=1, \cdots, n)$ and put

$$\alpha = \sum_{i=1}^{n} a_i \rho_i , \qquad \tau = \sum_{i=1}^{n} d_i \rho_i .$$

Then $z = \alpha + \tau$ and

$$\|S(\mu_i z)\| = |d_i| \qquad (i=1, \cdots, n) .$$

If $\alpha \to \mathfrak{a}$, then $\mathfrak{m} \subset \mathfrak{a}$, which implies that $\alpha \in A_0 \cup A_1$. Since $z \in E_3$, there exists an index $l$ for which

$$|\tau_l|^{-1} \leqq \frac{2\pi}{C_0 y_l} \ll M^{1/n} .$$

Since

$$\min_{1 \leqq i \leqq n} (\|S(\mu_i z)\|^{-1}) = \min_{1 \leqq i \leqq n} (|d_i|^{-1}) \ll \min_{1 \leqq j \leqq n} (|\tau_j|^{-1}) \ll M^{1/n} ,$$

we have by (11.1) $S(z; N) \ll N^{n-1} M^{1/n}$, or

$$|S(z; N)| \leqq cX^{-1} N^n .$$

Since

$$S(0; N) = \frac{(2\pi)^{r_2} N^n}{\sqrt{D}\, N(\mathfrak{m})} + O(N^{n-1})$$

([5], Lemma 3.2), taking $X$ sufficiently large, we have

$$\operatorname{Re} S(z; N) - S(0; N) \leqq |S(z; N)| - S(0; N) \leqq -cN^n .$$

If $k>1$, then we consider the Farey division of $E$ into $B^0$ and $B_\gamma$ $(\gamma \in \Gamma)$ with $N = XM^{1/n}$.

If $z \in B^0$, then by Theorem 10.2

$$S(z; N) \ll N^{n-\delta} \qquad (\delta = (1-a+\varepsilon)/2^{k-1}) .$$

Hence theorem is evident.

If $z \in B_\gamma$ with $\gamma \to \mathfrak{a}$, then by (10.3)

$$\operatorname{Re} S(z; N) \leqq \frac{2^{r_2}}{\sqrt{D}\, N(\mathfrak{m})} |G(\gamma) I_1| + O(N^{n-a}) , \tag{11.2}$$

where

$$I_1 = \int \underset{V}{\cdots} \int \exp\{2\pi i\, S(\eta^k \tau)\} du_1 \cdots du_n . \tag{11.3}$$

Now we see that there exists a constant $b_0<1$ such that $|G(\gamma)|<b_0$ for all $\gamma \notin A_0 \cup A_1$. In fact, it follows from Theorem 2.1 that there exists a number $N_0$ such that $|G(\gamma)| \leqq 1/2$ for $\gamma$ with $\gamma \to \mathfrak{a}$, $N(\mathfrak{a}) \geqq N_0$. On the other hand, by the definitions of $A_0$ and $A_1$,

$$\delta = \max_{\substack{N(\mathfrak{a}) \leqq N_0 \\ \gamma \notin A_0 \cup A_1, \gamma \to \mathfrak{a}}} |G(\gamma)| < 1 .$$

Hence $|G(\gamma)| \leqq b_0 = \max(\delta, 1/2) < 1$ for all $\gamma \notin A_0 \cup A_1$ and

$$\operatorname{Re} S(z; N) \leqq \frac{b_0 2^{r_2}}{\sqrt{D}\, N(\mathfrak{m})} |I_1| + O(N^{n-a})$$

for $\gamma \notin A_0 \cup A_1$, which gives

$$\operatorname{Re} S(z; N) \leqq \frac{b_0+1}{2} S(0; N) \tag{11.4}$$

for sufficiently large $N$, since $|I_1| \leqq \pi^{r_2} N^n$. Thus we have

$$\operatorname{Re} S(z; N) - S(0; N) \leqq -\frac{b_0}{2} S(0; N) \leqq -cN^n .$$

Finally assume that $\gamma \in A_0 \cup A_1$. We put $z = \gamma + \tau$. Then there exists an index $l$ for which

$$|\tau_l| \geqq C_0 y_l \geqq cM^{-k/n} .$$

Suppose $K^{(l)}$ is real. Without the loss of generality we may assume that $l=1$, $\tau_1>0$. (11.3) is written as follows:

$$I_1=\int\cdots\int e^{2\pi i S_1}du_2\cdots du_n\int_0^N e^{2\pi i u_1^k \tau_1}du_1 ,$$

where $S(\eta^k\tau)=S_1+u_1^k\tau_1$. Hence

$$|I_1|\leqq\int_{-N}^{N}\cdots\int_{-N}^{N} du_2\cdots du_n \left|\int_0^N e^{2\pi i u_1^k \tau_1}du_1\right| .$$

Since

$$\left|\int_0^N e^{2\pi i u^k \tau_1}du\right|\leqq\frac{4}{\tau_1^{1/k}} ,$$

we have

(11.5) $$|I_1|\ll N^{n-1}\tau_1^{-1/k}\leqq cX^{-1}N^n .$$

Suppose $K^{(l)}$ is imaginary. In this case, we may assume $l=n$ and $|\tau_n|\geqq cM^{-k/n}$. We write in $I_1$

$$S(\eta^k\tau)=S_2+\tau_{r+1}(u_{r+1}+iu_n)^k+\tau_n(u_{r+1}-iu_n)^k ,$$

and

$$I_1=\int\cdots\int e^{2\pi i S_2}du_1\cdots du_r du_{r+2}\cdots du_{n-1}$$
$$\times\iint_{u_{r+1}^2+u_n^2\leqq N^2}\exp[4\pi i\ \mathrm{Re}\{\tau_{r+1}(u_{r+1}+iu_n)^k\}]du_{r+1}du_n .$$

Put

$$u_{r+1}+iu_n=Ue^{i\theta} ,\qquad \tau_{r+1}=Re^{i\varphi} ,$$

then

(11.6) $$|I_1|\leqq\int_{-N}^{N}\cdots\int_{-N}^{N}du_1\cdots du_r du_{r+2}\cdots du_{n-1}$$
$$\times\left|\int_0^N dU\int_0^{2\pi}U\exp\{4\pi i U^k R\cos(k\theta+\varphi)\}d\theta\right| .$$

The last double integral is

$$\int_0^N dU\int_0^{2\pi}\exp\{4\pi i U^k R\cos(k\theta+\varphi)\}d\theta=2\pi\int_0^N UJ_0(4\pi RU^k)dU .$$

If $N\leqq R^{-1/k}$, then

$$\left|\int_0^N UJ_0(4\pi RU^k)dU\right|\leqq\int_0^N UdU\leqq R^{-2/k} .$$

If $N>R^{-1/k}$, then we divide the integral by $U=R^{-1/k}$:

$$\int_0^N UJ_0(4\pi RU^k)dU=\int_0^{R^{-1/k}}+\int_{R^{-1/k}}^N .$$

It will be sufficient to consider the second integral in the right hand side:

$$\int_{R^{-1/k}}^{U} UJ_0(4\pi RU^k)dU=\frac{R^{-2/k}}{k}\int_1^{RN^k} t^{2/k-1}J_0(4\pi t)dt\ .$$

Applying the second mean-value theorem to the last integral, we have

$$\int_1^{RN^k} t^{2/k-1}J_0(4\pi t)dt=\int_1^{\xi} J_0(4\pi t)dt=\frac{1}{4\pi}\int_{4\pi}^{4\pi\xi} J_0(u)du \qquad (1<\xi<RN^k)\ .$$

This is bounded. In fact,

$$\int_0^z J_0(u)du=\frac{1}{\pi}\int_0^z du\int_0^\pi \cos(u\sin\theta)d\theta \tag{11.7}$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin(z\sin\theta)}{\sin\theta}d\theta=\frac{2}{\pi}\left\{\int_0^{\pi/4}+\int_{\pi/4}^{\pi/2}\right\}$$

and we see

$$\int_{\pi/4}^{\pi/2}\frac{\sin(z\sin\theta)}{\sin\theta}d\theta\leqq\int_{\pi/4}^{\pi/2}\frac{d\theta}{\sin\theta}\leqq\frac{\sqrt{2}}{4}\pi\ ,$$

$$\int_0^{\pi/4}\frac{\sin(z\sin\theta)}{\sin\theta}d\theta=\int_0^{1/\sqrt{2}}\frac{\sin zt}{t\sqrt{1-t^2}}dt$$

$$=\int_0^{1/\sqrt{2}}\frac{\sin zt}{t}(1+O(t^2))dt=\int_0^{1/\sqrt{2}}\frac{\sin zt}{t}dt+O(1)\ .$$

Hence (11.7) is bounded. Therefore we have

$$|I_1|\leqq cN^{n-2}|\tau_n|^{-2/k}\leqq cX^{-2}N^n\ . \tag{11.8}$$

Collecting the results (11.2), (11.5) and (11.8) and taking $X$ sufficiently large, we have

$$\mathrm{Re}\, S(z;N)-S(0;N)\leqq -cN^n$$

for large $N$. Thus the proof is completed.

THEOREM 11.2. *If* $z\in E_3$, *then*

$$|H(z)|\leqq\exp(-cM)\ .$$

PROOF. From (8.1) we can derive the inequality

$$\log|H(z)|\leqq\mathrm{Re}\sum_{\substack{x(\nu)\in V\\ \nu\equiv\mu(m)}} e(\nu;y,z)-\sum_{\substack{x(\nu)\in V\\ \nu\equiv\mu(m)}} e(\nu;y,0)\ . \tag{11.9}$$

If $x(\nu)\in V$, then

$$S(|\nu|^k y) \leqq nN^k C_0 M^{-k/n} = nC_0 X^k .$$

Hence the right hand side of (11.9) does not exceed

$$e^{-nC_0X^k}\{\mathrm{Re}\, S(z;\, N) - S(0;\, N)\} .$$

Since $X$ is a constant, we have by Theorem 11.1

$$\log |H(z)| \leqq c(\mathrm{Re}\, S(z;\, N) - S(0;\, N)) \leqq -cN^n \leqq -cM .$$

and our theorem is proved.

## § 12. Asymptotic formula for the partition function.

Now we are in a position to prove the asymptotic formula for the partition function $A(\nu;\, y)$. We put

$$J_j = \int_{\phi(E_1(\gamma_j))} H(z) e^{-2\pi i S(\nu z)} dx_1 \cdots dx_n .$$

Let $\tau = z - \gamma_j$ and define $w_q$, $w_p$ as in the previous paragraphs. Then, writing $z$ instead of $z - \gamma_j$, we have

$$\begin{aligned}
(12.1) \quad J_j &= e^{-2\pi i S(\nu\gamma_j)} \int_{\phi(E_1(0))} H(z) e^{-2\pi i S(\nu z)} dx_1 \cdots dx_n \\
&= e^{-2\pi i S(\nu\gamma_j)} \frac{2^{r_2}\sqrt{D}}{(2\pi)^n} y_1 \cdots y_n \\
&\quad \times \int_W H(z) \exp\left(-i\frac{M}{k} \sum_{q=1}^{r_1} w_q\right) \prod_{q=1}^{r_1} dw_q \\
&\quad \times \prod_{p=r_1+1}^{r_1+r_2} w_p \exp\{-2i\sqrt{M} w_p \cos(\theta_p + \varphi_p)\} d\theta_p dw_p ,
\end{aligned}$$

where $\theta_p = \arg z_p$, $\varphi_p = \arg \nu^{(p)} (p = r_1+1, \cdots, r_1+r_2)$ and the domain of integration $W$ is defined as follows;

$$W; \quad \begin{aligned}
&|w_q| \leqq M^{-\Delta} && (q = 1, \cdots, r_1) , \\
&0 \leqq w_p \leqq M^{-\Delta} && (p = r_1+1, \cdots, r_1+r_2) , \\
&0 \leqq \theta_p < 2\pi && (p = r_1+1, \cdots, r_1+r_2) .
\end{aligned}$$

We denote by $J^*$ the integral over $W$ in (12.1). Then we have by Theorem 8.1

$$\begin{aligned}
(12.2) \quad J^* &= \int_W e^{-MQ} \prod_{q=1}^{r_1} dw_q \prod_{p=r_1+1}^{r_1+r_2} w_p \\
&\quad \times \exp\left\{-\frac{2+k}{2k^2} M w_p^2 - 2i\sqrt{M} w_p \cos(\theta_p + \varphi_p)\right\} d\theta_p dw_p
\end{aligned}$$

$$+O(M^{-c})\int_W e^{-MQ}\prod_{q=1}^{r_1}dw_q\prod_{p=r_1+1}^{r_1+r_2}w_p\exp\Big(-\frac{2+k}{2k^2}Mw_p^2\Big)d\theta_p dw_p$$
$$=J_1^*+O(M^{-c})J_2^* .$$

Since $Q$ is a positive definite quadratic form, there exists a positive constant $a$ such that $Q\geqq a(w_1^2+\cdots+w_{r_1}^2)$. Hence

$$J_2^*\ll\Big(\int_0^\infty e^{-aMw^2}dw\Big)^{r_1}\Big(\int_0^\infty w\exp\Big(-\frac{2+k}{2k^2}Mw^2\Big)dw\Big)^{r_2}\ll M^{-n/2} . \tag{12.3}$$

As for $J_1^*$, we write

$$J_1^*=L_0\cdot L_1^{r_2} , \tag{12.4}$$

where

$$L_0=\int_{-M^{-\Delta}}^{M^{-\Delta}}\cdots\int_{-M^{-\Delta}}^{M^{-\Delta}}e^{-MQ}dw_1\cdots dw_{r_1} ,$$
$$L_1=\int_0^{2\pi}d\theta\int_0^{M^{-\Delta}}w\exp\Big\{-\frac{2+k}{2k^2}Mw^2-2i\sqrt{M}w\cos\theta\Big\}dw .$$

First we write

$$L_0=\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty e^{-MQ}dw_1\cdots dw_{r_1}+\Big\{\int_{-M^{-\Delta}}^{M^{-\Delta}}\cdots\int_{-M^{-\Delta}}^{M^{-\Delta}}-\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty\Big\}e^{-MQ}dw_1\cdots dw_{r_1} .$$

The second term in this right hand side is estimated as follows:

$$\ll\int_{M^{-\Delta}}^\infty e^{-aMw^2}dw\Big(\int_0^\infty e^{-aMw^2}dw\Big)^{r_1-1}\ll M^{-r_1/2}\exp(-cM^{1-2\Delta}) .$$

By a simple calculation we have

$$\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty e^{-MQ}dw_1\cdots dw_{r_1}=|Q|^{-1/2}\pi^{r_1/2}M^{-r_1/2} .$$

Hence

$$L_0=|Q|^{-1/2}\pi^{r_1/2}M^{-r_1/2}+O(M^{-r_1/2-c}) . \tag{12.5}$$

As for $L_1$, we have

$$L_1=2\pi\int_0^{M^{-\Delta}}w\exp\Big(-\frac{2+k}{2k^2}Mw^2\Big)J_0(2\sqrt{M}w)dw$$
$$=\frac{2\pi}{M}\int_0^{M^{1/2-\Delta}}t\exp\Big(-\frac{2+k}{2k^2}t^2\Big)J_0(2t)dt$$
$$=\frac{2\pi}{M}\Big\{\int_0^\infty-\int_{M^{1/2-\Delta}}^\infty\Big\}t\exp\Big(-\frac{2+k}{2k^2}t^2\Big)J_0(2t)dt .$$

By a formula for Bessel function we have

$$\int_0^\infty t\exp\Big(-\frac{2+k}{2k^2}t^2\Big)J_0(2t)dt=\frac{k^2}{2+k}\exp\Big(-\frac{2k^2}{2+k}\Big)$$

([9], p. 394) and

$$\int_{M^{1/2-\Delta}}^\infty t\exp\Big(-\frac{2+k}{2k^2}t^2\Big)J_0(2t)dt\ll\int_{M^{1/2-\Delta}}^\infty te^{-ct^2}dt\ll\exp(-cM^{1-2\Delta})\,.$$

Hence

$$L_1=\frac{2\pi}{M}\frac{k^2}{2+k}\exp\Big(-\frac{2k^2}{2+k}\Big)+O(\exp(-cM^{1-2\Delta}))\,. \tag{12.6}$$

Collecting the results from (12.1) to (12.6), we have

$$\begin{aligned}J_j&=e^{-2\pi iS(\nu\gamma_j)}\frac{2^{r_2}\sqrt{D}}{(2\pi)^n}y_1\cdots y_n\frac{2^{r_2}\pi^{n/2}k^{2r_2}}{(2+k)^{r_2}M^{n/2}}\cdot\frac{2^{r_1/2}k^{(r_1+1)/2}}{(k+r_1)^{1/2}}\exp\Big(-\frac{2k^2r_2}{2+k}\Big)\\&\quad\times(1+O(M^{-\epsilon}))\\&=e^{-2\pi iS(\nu\gamma_j)}\frac{\sqrt{D}\,M^{r_1/2}k^{2r_2-r_1/2+1/2}}{2^{r_1/2}\pi^{n/2}(2+k)^{r_2}(k+r_1)^{1/2}N(\nu)}\exp\Big(-\frac{2k^2r_2}{2+k}\Big)\times(1+O(M^{-\epsilon}))\,.\end{aligned}$$

Now assume that $\nu\in\mathfrak{m}_0^k$, then

$$\sum_{j=1}^f e^{-2\pi iS(\nu\gamma_j)}=N(\mathfrak{m}_0^k)\,.$$

Hence we have

$$\begin{aligned}&\sum_{j=1}^f\int_{\phi(E_1(\gamma_j))}H(z)e^{-2\pi iS(\nu z)}dx_1\cdots dx_n\\&=\frac{\sqrt{D}\,M^{r_1/2}k^{2r_2-r_1/2+1/2}N(\mathfrak{m}_0)^k}{2^{r_1/2}\pi^{n/2}(2+k)^{r_2}(k+r_1)^{1/2}N(\nu)}\exp\Big(-\frac{2k^2r_2}{2+k}\Big)\times(1+O(M^{-\epsilon}))\,.\end{aligned} \tag{12.7}$$

On the other hand, we have by Theorems 8.1, 9.1 and 11.2

$$\Big\{\sum_{j=1}^g\int_{\phi(E_1(\sigma_j))}+\int_{\phi(E_2)}+\int_{\phi(E_3)}\Big\}H(z)e^{-2\pi iS(\nu z)}dx_1\cdots dx_n=O(\exp(-cM^{\epsilon}))\,. \tag{12.8}$$

Since

$$f(y;0)=\exp(M+R(y))(1+O(M^{-\epsilon}))$$

(we write $R(y)$ instead of $R(y;0)$), we have by (12.7) and (12.8)

$$A(\nu;y)\sim\frac{D^{1/2}M^{r_1/2}k^{2r_2-r_1/2+1/2}N(\mathfrak{m}_0)^k}{2^{r_1/2}\pi^{n/2}(2+k)^{r_2}(k+r_1)^{1/2}N(\nu)}\exp\Big(-\frac{2k^2r_2}{2+k}+M+R(y)\Big)$$

as $N(\nu)\to\infty$. Thus we have completed the proof of our Main Theorem.
As special cases, we have

COROLLARY 1. *If $K$ is totally real, then*

$$A(\nu;\, y)=e^{-(n/k)M}P(\nu)\ ,$$

*where $P(\nu)$ is the number of the partitions of $\nu$ into $k$-th powers of totally positive integers congruent to $\mu$ mod $\mathfrak{m}$ and we have*

$$P(\nu)\sim\frac{D^{1/2}k^{(1-n)/2}M^{n/2}N(\mathfrak{m}_0)^k}{(2\pi)^{n/2}(n+k)^{1/2}N(\nu)}\exp\left(\left(1+\frac{n}{k}\right)M+R(y)\right),$$

$$M=\left\{\Gamma\left(1+\frac{1}{k}\right)^n\zeta\left(1+\frac{n}{k}\right)k^{n/k}D^{-1/2}N(\nu)^{1/k}N(\mathfrak{m})^{-1}\right\}^{k/(n+k)}.$$

COROLLARY 2. *Let $P(N)$ be the number of the partitions of $N$ into the $k$-th powers of integers which are congruent to $a$ mod $m$, where $0<a\leqq m$ and $(a,m)=1$. Then we have*

$$P(N)\sim\Gamma\left(\frac{a}{m}\right)^k\frac{m^{k(a/m-1/2)}k^{1/2}d^{a/m}}{(1+k)^{1/2+a/m}(2\pi)^{(1+k)/2}}N^{ba/m-a/m-1/2}e^{dN^b}\ ,$$

*where*

$$b=\frac{1}{1+k}\ ,\quad d=(k+1)\left\{\frac{1}{km}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right\}^{k/(1+k)}.$$

PROOF. By Corollary 1,

$$P(N)\sim\frac{1}{N}\left(\frac{M}{2\pi(1+k)}\right)^{1/2}\exp\left(\frac{k+1}{k}M+R(y)\right),$$

where

$$M=\left\{\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)k^{1/k}N^{1/k}m^{-1}\right\}^{k/(k+1)}=\frac{dk}{1+k}N^b\ ,$$

$$y=M/kN\ ,$$

and $R(y)$ is the residue of

$$\frac{1}{k}\frac{\Gamma(s/k)}{y^{s/k}}\frac{1}{m^s}\cdot\zeta\left(s,\frac{a}{m}\right)\zeta\left(1+\frac{s}{k}\right)$$

at $s=0$, where $\zeta(s, a/m)$ is the Hurwitz zeta function. For small $s$ we have the expansions;

$$m^{-s}y^{-s/k}=1-\frac{s}{k}\log(ym^{1/k})+\cdots\ ,$$

$$\Gamma\left(\frac{s}{k}\right)=\frac{k}{s}-C+\cdots\,,$$

$$\zeta\left(1+\frac{s}{k}\right)=\frac{k}{s}+C+\cdots\,,$$

$$\zeta\left(s,\frac{a}{m}\right)=\frac{1}{2}-\frac{a}{m}+\left(\log\Gamma\left(\frac{a}{m}\right)-\frac{1}{2}\log 2\pi\right)s+\cdots \qquad \text{([10], p. 271)},$$

where $C$ is Euler's constant. Hence

$$R(y)=-\left(\frac{1}{2}-\frac{a}{m}\right)\log(ym^k)+k\left(\log\Gamma\left(\frac{a}{m}\right)-\frac{1}{2}\log 2\pi\right)$$

and consequently

$$\frac{1}{N}\left(\frac{M}{2\pi(1+k)}\right)^{1/2}e^{R(y)}=\Gamma\left(\frac{a}{m}\right)^k(2\pi)^{-k/2}\left(\frac{Mm^k}{kN}\right)^{a/m-1/2}\frac{1}{N}\left(\frac{M}{2\pi(1+k)}\right)^{1/2}$$

$$=\Gamma\left(\frac{a}{m}\right)^k\frac{M^{a/m}}{(2\pi)^{(1+k)/2}}\frac{m^{k(a/m-1/2)}}{k^{a/m-1/2}(1+k)^{1/2}}N^{-1/2-a/m}\,.$$

Thus we have Corollary 2.

## § 13. Estimation of $R(y)$.

In § 6, $R(y)$ was defined to be the sum of residues:

$$R(y)=\sum_{\lambda}\sum_{\sigma=0}\mathrm{Res}\Psi_\lambda(s;\,y,\,0)\,.$$

We shall prove in this paragraph that

$$R(y)\ll(\log N(\nu))^{r_1+r_2}\,.$$

It follows from Cauchy's theorem that

$$R(y)=\sum_{\lambda}\frac{1}{2\pi i}\int_{C_\lambda}\Psi_\lambda(s;\,y,\,0)ds\,,$$

where $C_\lambda$ is the simple closed curve lying in the strip $|\sigma|\leqq\varepsilon(<1/4k)$ and including the poles of $\Psi_\lambda(s;\,y,\,0)$ on the line $\sigma=0$. Since $z_1=\cdots=z_n=0$, we can write

$$R(y)=\sum_{\{m\}}\frac{1}{2\pi i}\int_{C_\lambda}\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q+1)}{y_q^{s+iV_q}}\prod_{p=r_1+1}^{r_1+r_2}\frac{\Gamma(2s+2iV_p+1)}{(2y_p)^{2s+2iV_p}}$$

$$\times\phi(s)2^{-r_2}Z_\lambda(ks;\,\mathfrak{m},\,\mu)\frac{1}{s}\prod_{q=1}^{r+1}\frac{1}{s+iV_q}ds\,,$$

where the sum is over all rational integers $m_1, \cdots, m_r$ and

$$\phi(s)=s\zeta(1+ns)\ ,$$
$$V_q=\frac{2\pi}{e_q k}\sum_{j=1}^{r} E_q^{(j)} m_j \qquad (q=1, \cdots, r+1)\ .$$

The poles of $\Psi_\lambda(s; y, 0)$ inside $C_\lambda$ are at $s=0, -iV_1, \cdots, -iV_{r+1}$. Let $it_1, \cdots, it_m$ be the distinct poles among them and define the curve or the sum of curves in $s$-plane by

$$C'_\lambda=\{s|\min_{1\le j\le n}(|s-it_j|)=\varepsilon\}\ .$$

Then we can replace $C_\lambda$ by $C'_\lambda$ and moreover we can write

$$\int_{C'_\lambda}\Psi_\lambda ds=\sum_{j=1}^{m}\int_{C_{\lambda,j}}\Psi_\lambda ds\ ,$$

where $C_{\lambda,j}$ is the part of $C'_\lambda$ defined by

$$C_{\lambda,j}=\{s|\min_{1\le l\le m}(|s-it_l|)=|s-it_j|\} \qquad (j=1, \cdots, m)\ .$$

On $C_{\lambda,j}$, we put $s=it_j+\varepsilon e^{i\theta}$. Then, using the results in §§ 5, 6, we have the following estimations:

$$\prod_{q=1}^{r_1}\frac{\Gamma(s+iV_q+1)}{y_q^{s+iV_q}}\prod_{p=r_1+1}^{r_1+r_2}\frac{\Gamma(2s+2iV_p+1)}{(2y_p)^{2s+2iV_p}}$$
$$\ll M^{\varepsilon k}\prod_{q=1}^{r+1}(1+|t+V_q|)\exp\left(-\frac{\pi}{2}\sum_{q=1}^{r+1}e_q|t+V_q|\right)\ ,$$
$$\phi(s)\ll(1+|t|)^c\ll\prod_{q=1}^{r+1}(1+|V_q|)^c\ ,$$
$$Z_\lambda(ks;\mathrm{m},\mu)\ll\prod_{q=1}^{r+1}(1+|t+V_q|)^c\ll\prod_{q=1}^{r+1}(1+|V_q|)^c\ .$$

Hence

$$\int_{C_{\lambda,j}}\Psi_\lambda ds\ll M^{\varepsilon k}\varepsilon^{-r-1}\prod_{q=1}^{r+1}(1+|V_q|)^c\exp\left(-\frac{\pi}{2}\sum_{q=1}^{r+1}e_q|t+V_q|\right)\ .$$

We consider the square of $\sum e_q|t+V_q|$:

$$\left(\sum_{q=1}^{r+1}e_q|t+V_q|\right)^2\ge\sum_{q=1}^{r+1}e_q^2(t+V_q)^2$$
$$=(r_1+4r_2)t^2+2t\sum_{q=1}^{r+1}e_q^2V_q+\sum_{q=1}^{r+1}e_q^2V_q^2\ .$$

Since this last quadratic polynomial in $t$ takes the minimum value for

$t=-\sum_{q=1}^{r+1} e_q^2 V_q/(r_1+4r_2)$,

$$(\sum_{q=1}^{r+1} e_q|t+V_q|)^2 \geqq \sum_{q=1}^{r+1} e_q^2 V_q^2 - (\sum_{q=1}^{r+1} e_q^2 V_q)^2/(r_1+4r_2) .$$

Since $\sum_{q=1}^{r+1} e_q V_q=0$, we have

$$(\sum_{q=1}^{r+1} e_q^2 V_q)^2 = (\sum_{q=r_1+1}^{r_1+r_2} e_q V_q)^2 \leqq r_2 \sum_{q=1}^{r+1} e_q^2 V_q^2 .$$

Hence

$$(\sum_{q=1}^{r+1} e_q|t+V_q|)^2 \geqq \left(1-\frac{r_2}{r_1+4r_2}\right)\sum_{q=1}^{r+1} e_q^2 V_q^2$$

$$\geqq \frac{1}{2n}\sum_{q=1}^{r+1}|V_q|$$

and

$$\int_{C_{\lambda j}} \Psi_\lambda ds \ll M^{\varepsilon k} \varepsilon^{-r-1} \prod_{q=1}^{r+1}(1+|V_q|)^c \exp\left(-\frac{\pi}{4n}\sum_{q=1}^{r+1}|V_q|\right) .$$

On summing up the right hand side over $j$ and $\{m\}$, we have, in the same manner as in the proof of Lemma 5.1,

$$R(y) \ll M^{\varepsilon k} \varepsilon^{-r-1} .$$

Since this estimation is uniform in $\varepsilon$, we can put $\varepsilon=1/\log M$ and obtain the desired result

$$R(y) \ll (\log M)^{r+1} .$$

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