TOKYO J. MATH.
VOL. 2, NO. 2, 1979

Expansive Automorphisms of Locally Compact Solvable Groups

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Introduction

Let G be a locally compact group and σ be a bicontinuous automorphism of G. We call the automorphism σ *expansive* if there exists an open neighborhood U of the identity e in G such that $x\in U$ and $x\neq e$ imply $\sigma^n(x)\notin U$ for some integer n. Obviously, σ is expansive if and only if $\bigcap_{i=-\infty}^{\infty}\sigma^i(U)=\{e\}$ holds.

The structures of expansive automorphisms and of compact groups admitting them have been investigated by several authors, including Eisenberg [4, 5], Wu [12], Lam [8], Lawton [9] and Dateyama and the present author [2]. However the structure of locally compact groups which admit expansive automorphisms are yet unknown, except in special cases. For example, if a locally compact almost maximal group admits an expansive automorphism, then it is abelian [8]. But there exists a locally compact connected nilpotent group which admits an expansive automorphism [1].

It will be interesting to investigate what kind of locally compact connected groups admit expansive automorphisms. Our aim is to discuss this problem. However the nilpotent case has been already examined in [1].

Throughout this paper, all subgroups of the group G are closed subgroups and all automorphisms are onto and bicontinuous. The restriction and the factor of an automorphism will be denoted by the same symbols if there is no risk of confusion.

§1. Main results.

We shall show the following Theorems 1 and 2 which are main results of this paper.

Received January 16, 1979

THEOREM 1. *Let G be a locally compact connected group and σ be an automorphism of G. Assume that G is solvable. If σ is expansive, then G is nilpotent.*

It is known (Lemma 4.2 of [7]) that there exists in G the maximum compact normal subgroup K such that G/K is an analytic group. Since K is maximal normal in G, it is strictly invariant with respect to every automorphism. If $\sigma: G\to G$ is an expansive automorphism, then $\sigma: K\to K$ is also expansive. Hence K is metrizable (cf. see [3]), whence so is G.

It is unknown yet whether every factor automorphism is expansive whenever an automorphism $\sigma: G\to G$ is expansive. However, in case G is compact, it is proved in [2] that if an automorphism is expansive, then every factor automorphism is also expansive. In the locally compact case we have

THEOREM 2. *Let G be a locally compact connected solvable group and K be as above. If $\sigma: G\to G$ is an expansive automorphism, then the factor automorphism $\sigma: G/K\to G/K$ is also expansive.*

For the proof of Theorems 1 and 2, we need a structure theorem of a locally compact connected group (Theorem 13 of [7]). Since G is locally compact connected and solvable, it has maximal compact subgroups, and all such subgroups are connected and are conjugate to each other. Let A denote one of them. Then G contains subgroups $H_1, \cdots, H_r$ all isomorphic to the vector group and such that any element $g\in G$ can be split uniquely and continuously in the form

$$(1)\qquad g=h_1\cdots h_r a\ ,\qquad h_i\in H_i\ ,\qquad a\in A\ .$$

In particular, the space of G is the direct product of the compact space of A and that of $H_1\times\cdots\times H_r$, which is homeomorphic to the r-dimensional Euclidean space. It is obvious that the subgroup K is contained in A. Since K is a normal abelian subgroup of G, it is central in G (see Theorem 4 of [7]).

PROOF OF THEOREM 1. Using Theorem 2 and the following Lemma 1, we obtain easily Theorem 1. Indeed, under the notations of Theorem 2, G/K is analytic and $\sigma: G/K\to G/K$ is expansive. Since G is solvable, G/K is also solvable and hence it is nilpotent by the following Lemma 1. Let K_0 be the connected component of the identity of K, then K_0 is normal and by Lemma 2.2 of [7] it is abelian. Thus K_0 is central in G (Theorem 4 of [7]). Since K/K_0 is totally disconnected and normal, K/K_0 is also central in G/K_0. Therefore G is nilpotent.

Let X be an analytic group and σ be an automorphism of X. We denote by $\mathscr{L}(X)$ the Lie algebra of X and by $d\sigma$ the differential of σ. The exponential map $\mathscr{L}(X)\to X$ is a real analytic diffeomorphism of some open connected neighborhood of the zero vector in $\mathscr{L}(X)$ onto an open connected neighborhood of the identity in X, and $\sigma\circ\exp=\exp\circ d\sigma$ holds.

LEMMA 1. *With the above notations, if X is solvable and σ is expansive, then X is nilpotent.*

PROOF. Since $\mathscr{L}(X)$ is a finite-dimensional vector space, it follows from [4] that $d\sigma: \mathscr{L}(X)\to\mathscr{L}(X)$ is expansive if and only if all the eigenvalues of $d\sigma$ are off the unit circle.

It will be enough to show that $\mathscr{L}(X)$ is nilpotent to get the conclusion of the lemma. For the proof we shall use the technique in [6] (p. 151). Let $\mathscr{L}(X)=\sum_{\alpha\in\Lambda}\mathfrak{G}^{\alpha}$ be the decomposition of $\mathscr{L}(X)$ into generalized eigenspaces with respect to $d\sigma$, that is, Λ is a suitable indexing set and

$$\mathfrak{G}^{\alpha}=\{Y\in\mathscr{L}(X): f_{\alpha}^{r}(d\sigma)Y=0 \text{ for some } r>0\}$$

where f_{α}, $\alpha\in\Lambda$, are irreducible polynomials with real coefficients. It is easy to see that for any $\alpha\in\Lambda$ all the eigenvalues of $d\sigma_{|\mathfrak{G}^{\alpha}}$ have the same absolute value, say, λ_{α}. We now put $\mathfrak{G}^{+}=\sum_{\lambda_{\alpha}>1}\mathfrak{G}^{\alpha}$ and $\mathfrak{G}^{-}=\sum_{\lambda_{\alpha}<1}\mathfrak{G}^{\alpha}$. Then it is easy to see that $\mathfrak{G}^{+}$ and $\mathfrak{G}^{-}$ are nilpotent Lie subalgebras of $\mathscr{L}(X)$. Since $\mathscr{L}(X)$ is solvable, it follows from Lie's theorem that the nilradical of $\mathscr{L}(X)$ contains $\mathfrak{G}^{+}$ and $\mathfrak{G}^{-}$. On the other hand, since all the eigenvalues of $d\sigma$ are off the unit circle, we get $\mathscr{L}(X)=\mathfrak{G}^{+}+\mathfrak{G}^{-}$, and hence X is nilpotent.

We shall show Theorem 2 in the remainder of this section.

PROOF OF THEOREM 2. We consider the following cases: (i) K is totally disconnected, (ii) K is not so.

We first show Case (i). Since G is a locally compact connected metrizable group and K is totally disconnected, G is locally the direct product of K and L where L is some local Lie group (cf. pp. 182~183 of [10]). Let W be an expansive neighborhood for (G,σ). Then $L\cap W$ is open in L and it is a local Lie subgroup of L. Since K is totally disconnected, σ induces a local automorphism of $L\cap W$. Thus there is an open neighborhood U of $L\cap W$ such that $\sigma^{-1}(U)$ and $\sigma(U)$ are contained in $L\cap W$. Since $U_1=\sigma^{-1}(U)\cap U\cap\sigma(U)$ is open in L, U_1K is also open in G and

$$\bigcap_{i=-1}^{1} \sigma^i(U_1)K = \left\{\bigcap_{i=-2}^{2} \sigma^i(U)\right\}K \qquad (\text{since } \sigma^{-1}(U_1),\ \sigma(U_1)\subset U_1)$$
$$= \left\{\bigcap_{i=-1}^{1} \sigma^i(U_1)\right\}K .$$

By induction we now have

$$\bigcap_{-\infty}^{\infty} \sigma^i(U_1)K = \left\{\bigcap_{-\infty}^{\infty} \sigma^i(U_1)\right\}K$$
$$= K \qquad (\text{since } U_1 \subset W) .$$

Since U_1 is open in L and U_1K is also open in G, $\dot{U}=\{xK: x \in U_1\}$ is an open neighborhood of the identity of G/K such that $\bigcap_{-\infty}^{\infty} \sigma^i(\dot{U})=\{K\}$. This implies that $\sigma: G/K \to G/K$ is expansive.

It remains to show Case (ii). To do this, we shall prepare the following

LEMMA 2. *Let A and K be as above. Then there exists a subgroup K_1 such that $A=KK_1$ and $N=K\cap K_1$ is totally disconnected.*

PROOF. As before we denote by K_o the connected component of the identity of K. Let A^* denote the character group of A and K_o^* the annihilator of K_o in A^*, then K_o^* is the character group of the factor group A/K_o. On the other hand, A^*/K_o^* is the character group of K_o. Since K_o is known to be finite-dimensional ([9]), A^*/K_o^* must be torsion free and rank $(A^*/K_o^*)=\dim (K_o)<\infty$. Hence we can find a subgroup K_1^* of A^* such that $K_1^* \cap K_o^*=\{1\}$ and rank $(K_1^*)=$ rank (A^*/K_o^*). From those, we have

$$\begin{aligned}\text{rank}\,(A^*) &= \text{rank}\,(A^*/K_o^*)+\text{rank}\,(K_o^*)\\ &= \text{rank}\,(K_1^*)+\text{rank}\,(K_o^*)\\ &= \text{rank}\,(K_1^* \times K_2^*)\ ,\end{aligned}$$

so that $A^*/(K_o^* \times K_1^*)$ is a torsion group. Take the annihilator of K_o^* and K_1^* in A respectively. Then we have at once the conclusion of the lemma.

Since $A/K\subset G/K$ and G/K is analytic, A/K is a torus and by Lemma 2 so is K_1/N where $N=K\cap K_1$. Thus K_1 is locally the direct product of N and L where L is some local Lie group. So we have that LK is open in A. We write $H=(\prod_{i=1}^{r} H_i)L$, where each H_i is the subgroup in the splitting (1). Since HK is homeomorphic to the product space $H\times K$ with the product topology, H has an open subset V of H such that $\sigma^{-1}(V)$ and $\sigma(V)$ are contained in HK. It is easy to verify that σ

induces open maps σ_1 and σ_1^{-1}, each from V into H, such that $\sigma_1(V)K=\sigma(V)K$ and $\sigma_1^{-1}(V)K=\sigma^{-1}(V)K$.

Let W be an expansive neighborhood for (G, σ). Then $V_1=W\cap V$ is open in H, and $W_1=\sigma_1^{-1}(V_1)\cap V_1\cap\sigma_1(V_1)$ is also open in H. Hence W_1K is open in G. We now have

$$\bigcap_{i=-1}^{1}\sigma^i(W_1)K=\left\{\bigcap_{i=-1}^{1}\sigma_1^i(W_1)\right\}K=\left\{\bigcap_{i=-1}^{1}\sigma^i(W_1)\right\}K\,,$$

and by induction

$$\bigcap_{i=-\infty}^{\infty}\sigma^i(W_1)K=\left\{\bigcap_{i=-\infty}^{\infty}\sigma^i(W_1)\right\}K=K\,.$$

Since W_1K is open in G, $\dot{W}=\{xK: x\in W_1\}$ is an open neighborhood of the identity of G/K such that $\bigcap_{i=-\infty}^{\infty}\sigma^i(\dot{W})=\{K\}$. Therefore $\sigma: G/K\to G/K$ is expansive.

The proof of Theorem 1 is completed.

References

[1] N. Aoki, The structure of a locally compact nilpotent group which admit expansive automorphisms, preprint.

[2] N. Aoki and M. Dateyama, The relationship between algebraic numbers and expansiveness of automorphisms on compact abelian groups, to appear.

[3] B. F. Brayant, On expansive homeomorphisms, Pacific J. Math., **10** (1960), 1163–1167.

[4] M. Eisenberg, Expansive transformation semi-groups of endomorphisms, Fund. Math., **59** (1966), 313–321.

[5] M. Eisenberg, A note on positively expansive endomorphisms, Math. Scand., **19** (1966), 217–218.

[6] S. G. Dani, Spectrum of an affine transformation, Duke Math. J., **44** (1977), 129–155.

[7] K. Iwasawa, On some type of topological groups, Ann. of Math., **50** (1949), 507–558.

[8] Ping-F. Lam, On expansive transformation groups, Trans. Amer. Math. Soc., **150** (1970), 131–138.

[9] W. Lawton, The structure of compact connected groups which admits an expansive automorphism, Recent Advances in Topological Dynamics, Lecture notes in Math., **318**, Springer-Verlarg, 1973.

[10] D. Montgomery and L. Zippin, Topological Transformation Groups, Interscience Publ. Inc., New York, 1966.

[11] H. Omori, Infinite-dimensional Lie groups, Lecture notes in Math., **427**, Springer-Verlarg, 1974.

[12] T. S. Wu, Expansive automorphisms in compact groups, Math. Scand., **18** (1966), 23–24.

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