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Solutions of $x''=t^{\alpha\lambda-2}x^{1+\alpha}$ with Movable Singularity

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Introduction

As in the previous paper [1], we consider here a second order non-linear differential equation

$$(1) \qquad x''=t^{\alpha\lambda-2}x^{1+\alpha}\ , \qquad '=d/dt\ , \qquad \alpha>0\ , \qquad \alpha\lambda>1\ ,$$

in a domain

$$G: \qquad 0<t<\infty\ , \qquad 0\leqq x<\infty\ .$$

As we restrict ourselves entirely within the real domain, any real power of a nonnegative-valued variable should be regarded as representing its nonnegative-valued branch. So, for example,

$$t^{\alpha\lambda-2}>0\ , \qquad x^{1+\alpha}\geqq 0$$

in G.

The solutions of (1) to be considered here are those which satisfy the "initial condition"

$$\lim_{t\to 0} x=a\ , \qquad \lim_{t\to 0} x'=b\ , \qquad 0<a<\infty\ , \qquad |b|<\infty\ .$$

Such solutions will be denoted by $\phi(t, a, b)$. The object of this paper is to show that each $\phi(t, a, b)$ has, in general, a movable singularity and to obtain the explicit expression of $\phi(t, a, b)$ valid in the vicinity of its movable singularity.

To do this, we have to make use of some of the results obtained in [1]. This section is devoted to the brief description of them.

The equation (1) has a solution

$$x=\psi(t)=[\lambda(\lambda+1)]^{1/\alpha}t^{-\lambda}\ .$$

For any solution $x(t)$ of (1), let us define a function $y(t)$ by

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$$x(t)=\psi(t)[y(t)]^{1/\alpha} .$$

Then a function $z(y)$ defined by

$$y=y(t) , \qquad z=ty'(t) , \qquad y>0 ,$$

satisfies a following differential equation:

$$(2) \qquad \frac{dz}{dy}=\frac{-\lambda(\lambda+1)\alpha^2y^2+(2\lambda+1)\alpha yz-(1-\alpha)z^2+\lambda(\lambda+1)\alpha^2y^3}{\alpha yz} .$$

Conversely let $z(y)$ be a solution of (2) and $y(t)$ be any solution of

$$ty'=z(y) , \quad y>0 .$$

Then $x=\psi(t)[y(t)]^{1/\alpha}$ is a solution of (1). Therefore every solution of (1) will bring forth a solution of (2) and, since $y(t)$ contains an arbitrary constant, every solution of (2) will bring forth a one-parameter family of solutions of (1).

If we notice that

$$(3) \qquad y=[\lambda(\lambda+1)]^{-1}t^{\alpha\lambda}x^\alpha , \qquad ty'=[\lambda(\lambda+1)]^{-1}(\alpha\lambda t^{\alpha\lambda}x^\alpha+\alpha t^{\alpha\lambda+1}x^{\alpha-1}x'] ,$$

and hence

$$(4) \qquad z/y=ty'/y=\alpha tx'/x+\alpha\lambda ,$$

it is obvious that a solution $x=\phi(t, a, b)$ of (1) will give rise to a solution $z=z(y)$ of (2) such that

$$\lim_{y\to 0} z(y)=0 , \qquad \lim_{y\to 0} z(y)/y=\alpha\lambda .$$

As was proved in [1], such a solution can be expressed explicitly by a following double power series in y and $y^{1/\alpha\lambda}$ absolutely convergent in the neighbourhood of $y=0$:

$$(5) \qquad z=z(y, C)=\alpha\lambda y+y\sum_{m+n>0} v_{mn}y^m(Cy^{1/\alpha\lambda})^n , \qquad v_{01}=1 .$$

Here the value of a constant C is determined by the initial values a and b.

To make clear the dependence of C on a and b, we notice that, since $\alpha\lambda>1$, $Cy^{1/\alpha\lambda}$ is the term of the lowest degree in the double power series

$$\sum_{m+n>0} v_{mn}y^m(Cy^{1/\alpha\lambda})^n , \qquad v_{01}=1 .$$

Consequently

$$C=\lim_{y\to 0} y^{-1/\alpha\lambda}\left(\frac{z}{y}-\alpha\lambda\right)=\lim_{y\to 0} y^{-1/\alpha\lambda}\left(\frac{ty'}{y}-\alpha\lambda\right).$$

Then from (3) and (4) we get

$$C=\lim_{t\to 0}\,[\lambda(\lambda+1)]^{1/\alpha\lambda}\frac{\alpha\phi'(t,a,b)}{[\phi(t,a,b)]^{1+1/\lambda}}=[\lambda(\lambda+1)]^{1/\alpha\lambda}\frac{\alpha b}{a^{1+1/\lambda}}.$$

Let us consider a following dynamical system

$$(6)\quad \begin{aligned} \frac{dy}{ds}&=\alpha yz\,,\\ \frac{dz}{ds}&=-\lambda(\lambda+1)\alpha^2y^2+(2\lambda+1)\alpha yz-(1-\alpha)z^2+\lambda(\lambda+1)\alpha^2y^3\,, \end{aligned}$$

associated with the equation (2). A solution curve of (2) in (y, z)-plane represents an orbit (or union of several orbits) of (6).

As one can observe easily, $y=z=0$ and $y=1, z=0$ are the critical points of (6), and the solution $z=z(y, C)$ of (2) corresponding to a solution $\phi(t, a, b)$ of (1) represents an orbit of (6) which tends to $y=z=0$ as $s\to-\infty$ having a straight line $z=\alpha\lambda y$ as its tangent at $y=z=0$. As this is true for every a and b, there exist infinitely many such orbits, and as was proved in [1], one of them tends to $y=1, z=0$ as $s\to\infty$. Since the critical point $y=1, z=0$ is a saddle point, the phase portrait near this orbit will look like Figure 1.

Let $\hat{C}$ be the value of C which corresponds to this particular orbit. Then, as was proved in [1], every solution $y=\hat{y}(t)$ of

$$ty'=z(y, \hat{C})$$

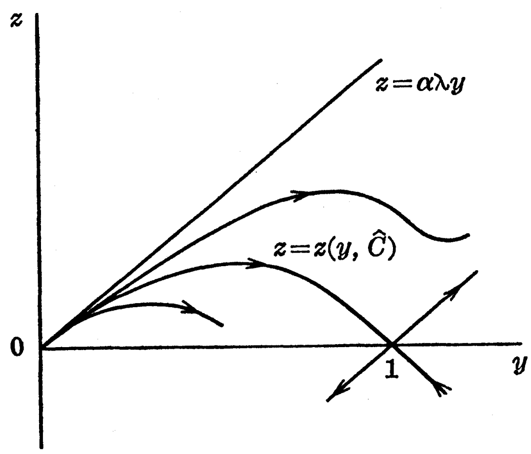

FIGURE 1

will bring forth a bounded solution of (1). Exactly to say,

$$x=\psi(t)[\hat{y}(t)]^{1/\alpha}$$

is a solution of (1) which is defined and bounded for $0<t<\infty$ together with its derivative.

In other words, if we define $\hat{b}(a)$ by

$$\hat{C}=[\lambda(\lambda+1)]^{1/\alpha\lambda}\frac{\alpha\hat{b}(a)}{a^{1+1/\lambda}},$$

then $x=\phi(t, a, \hat{b}(a))$ is a bounded solution of (1) for every a. Hence such particular solution $\phi(t, a, \hat{b}(a))$ does not have any singularity within $0<t<\infty$. In what follows we shall show that, if $b\neq\hat{b}(a)$, $\phi(t, a, b)$ will generally have a movable singularity somewhere in $0<t<\infty$.

§ 1. The explicit expression of $\phi(t, a, b)$ at $t=0$.

Before discussing about the movable singularity, we shall give here the explicit analytical expression of $\phi(t, a, b)$ valid in the neighbourhood of a fixed singularity $t=0$. Although this has already been done in [1], we shall study it again in more detail and make clear how $\phi(t, a, b)$ depends on its initial values a and b.

The solution $y(t)$ of

$$ty'=z(y, C)$$

is given implicitly by

$$\int\frac{dy}{z(y, C)}=\int\frac{dt}{t}=\log t+\text{const}.$$

Since $z(y, C)$ is given by (5):

$$z(y, C)=\alpha\lambda y+y\sum_{m+n>0}v_{mn}y^m(Cy^{1/\alpha\lambda})^n,$$

termwise integration after taking its inverse will yield

$$\int\frac{dy}{z(y, C)}=\frac{1}{\alpha\lambda}(\log y+\sum_{m+n>0}\hat{v}_{mn}y^m(Cy^{1/\alpha\lambda})^n)=\log t+\text{const}.$$

Multiplying $\alpha\lambda$ and taking the exponentials of both sides, we have

$$(Bt)^{\alpha\lambda}=y(1+\sum_{m+n>0}c_{mn}y^m(Cy^{1/\alpha\lambda})^n)$$

where B is an arbitrary positive constant. To obtain the explicit expression of $y(t)$ from this implicit one, we need the following lemma.

LEMMA 1. *Let η be a function of ζ defined implicitly by*

$$\zeta=\eta(1+\sum_{m+n>0}\gamma_{mn}\eta^m(\eta^\mu[h\log\eta+C])^n)\,,\qquad \mu>0\,, \tag{7}$$

where h and C are arbitrary constants and $h=0$ whenever μ is not an integer, and the power series in η and $\eta^\mu[h\log\eta+C]$ in the right-hand member is absolutely convergent in the neighbourhood of $\eta=0$, $\eta^\mu[h\log\eta+C]=0$. Then we have

$$\eta=\zeta(1+\sum_{m+n>0}\hat{\gamma}_{mn}\zeta^m(\zeta^\mu[h\log\zeta+C])^n)\,.$$

Here the double power series in the right-hand member is absolutely convergent in the neighbourhood of $\zeta=0$, $\zeta^\mu[h\log\zeta+C]=0$.

PROOF. This lemma is due to R. A. Smith [2]. We shall sketch his proof here.

From the given relation (7), we get

$$\zeta^\mu=\eta^\mu(1+A_1)\,,$$
$$h\log\zeta+C=h\log\eta+C+A_2\,,$$

where $A_k(k=1,2)$ is a double power series in η and $\eta^\mu[h\log\eta+C]$ lacking constant term. Thus we have

$$\zeta^\mu[h\log\zeta+C]=\eta^\mu[h\log\eta+C](1+A_1)+\eta^\mu(A_2+A_1A_2)\,.$$

If we notice that $h=0$ when μ is not an integer, the right-hand side of the above equality is a double power series in η and $\eta^\mu[h\log\eta+C]$ even when μ is not an integer and the only first-degree term is $\eta^\mu[h\log\eta+C]$. Therefore if we put

$$\eta^\mu[h\log\eta+C]=\xi\,,\qquad \zeta^\mu[h\log\zeta+C]=\sigma\,,$$

we have

$$\zeta=\eta(1+\sum_{m+n>0}\gamma_{mn}\eta^m\xi^n)\,,\qquad \sigma=\xi+\sum_{m+n>1}\delta_{mn}\eta^m\xi^n\,.$$

Since the right-hand sides are holomorphic functions of η and ξ in the neighbourhood of $\eta=\xi=0$, and

$$\frac{\partial(\zeta,\sigma)}{\partial(\eta,\xi)}=1$$

at $\eta=\xi=0$, η and ξ are holomorphic functions of ζ and σ in the neighbourhood of $\zeta=\sigma=0$. Also $\zeta=0$ implies $\eta=0$. Hence we have

$$\eta=\zeta(1+\sum_{m+n>0}\hat{\gamma}_{mn}\zeta^m\sigma^n)$$

in the neighbourhood of $\zeta=\sigma=0$. This proves Lemma 1.

In order to apply Lemma 1 to our problem, we have only to put

$$\eta=y,\ \zeta=(Bt)^{\alpha\lambda},\ \mu=1/\alpha\lambda,\ h=0,\ \gamma_{mn}=c_{mn}$$

in (7). Then we immediately get

$$y=(Bt)^{\alpha\lambda}(1+\sum_{m+n>0}\hat{c}_{mn}(Bt)^{\alpha\lambda m}(CBt)^n)\ .$$

Hence $y^{1/\alpha}$ can be expressed as

$$y^{1/\alpha}=B^\lambda t^\lambda(1+\sum_{m+n>0}\gamma_{mn}(Bt)^{\alpha\lambda m}(CBt)^n)\ .$$

Inserting it into

$$\phi(t,\ a,\ b)=[\lambda(\lambda+1)]^{1/\alpha}t^{-\lambda}y^{1/\alpha}$$

we obtain

$$\phi(t,\ a,\ b)=[\lambda(\lambda+1)]^{1/\alpha}B^\lambda(1+\sum_{m+n>0}\gamma_{mn}(Bt)^{\alpha\lambda m}(CBt)^n)\ . \tag{8}$$

The initial condition

$$\lim_{t\to 0}\phi(t,\ a,\ b)=a$$

implies

$$[\lambda(\lambda+1)]^{1/\alpha}B^\lambda=a \quad \text{or} \quad B=a^{1/\lambda}[\lambda(\lambda+1)]^{-1/\alpha\lambda}\ .$$

Also we already know that

$$C=[\lambda(\lambda+1)]^{1/\alpha\lambda}\frac{\alpha b}{a^{1+1/\lambda}}\ .$$

Inserting these values of B and C into (8), we finally get

$$\phi(t,\ a,\ b)=a\left(1+\sum_{m+n>0}\gamma_{mn}\left(\frac{a^\alpha}{\lambda(\lambda+1)}t^{\alpha\lambda}\right)^m\left(\frac{\alpha b}{a}t\right)^n\right)\ . \tag{9}$$

Here one will easily observe that

$$\gamma_{01}=1/\alpha$$

since $\alpha\lambda>1$ and $\phi'(t, a, b)\to b$ as $t\to 0$. Moreover it is not difficult to show that

$$\gamma_{0n}=0,\ n=2, 3, \cdots .$$

§ 2. The solution $\phi(t, a, b)$ with $b>\hat{b}(a)$.

To study the solutions $\phi(t, a, b)$ other than $\phi(t, a, \hat{b}(a))$, we begin with the case $b>\hat{b}(a)$.

Let $z=z(y, C)$ be a solution of (2) brought forth by $\phi(t, a, b)$ with $b>\hat{b}(a)$. As $b>\hat{b}(a)$ implies $C>\hat{C}$, the expression (5) will give us

$$z(y, C)>z(y, \hat{C})$$

if y is sufficiently small. Due to the uniqueness of the solution of (2), this inequality holds good as long as both solutions are defined and holomorphic.

LEMMA 2. *The solution $z(y, C)(C>\hat{C})$ of (2) is defined for $0<y<\infty$ and*

1) $\lim_{y\to\infty} z(y, C)=\infty$, 2) $\lim_{y\to\infty} y^{-1}z(y, C)=\infty$.

PROOF. As one will observe from Figure 1, $z(y, C)$ cannot tend to 0 as $y\to 1$ if $C>\hat{C}$. Consequently, to show that $z(y, C)$ is defined for $0<y<\infty$, it is sufficient to show that $z(y, C)$ does not tend to ∞ as y tends to a finite value.

Assume that

$$\lim_{y\to\eta} z(y, C)=\infty , \qquad 0<\eta<\infty ,$$

to derive a contradiction. By putting $1/z=\zeta$, (2) will become

$$\frac{d\zeta}{dy}=\lambda(\lambda+1)\alpha y\zeta^3-(2\lambda+1)\zeta^2+\frac{1-\alpha}{\alpha}\frac{\zeta}{y}-\lambda(\lambda+1)\alpha y^2\zeta^3 , \tag{10}$$

and $\zeta=1/z(y, C)$ is a solution of (10) such that $\zeta=0$ for $y=\eta$. Since the right-hand side of (10) is holomorphic at $y=\eta$, $\zeta=0$, such a solution must be unique. However, as $\zeta=0$ is also such a solution, this is obviously a contradiction.

Thus $z(y, C)$ is defined and positive for $0<y<\infty$. Therefore the orbit of (6) represented by $z=z(y, C)$ crosses the line $y=1$ for some positive value of s. This means that, at some $\tau>0$, we have

$$\phi(\tau, a, b)=\psi(\tau) .$$

Then, as was proved in [1], the following inequality holds for $t>\tau$:

$$\phi(t, a, b)>\psi(t)+(\phi'(\tau, a, b)-\psi'(\tau))(t-\tau) ,$$
$$\phi'(\tau, a, b)>\psi'(\tau) .$$

(In [1], the above inequality was proved for the bounded solution $\phi(t, a, \hat{b}(a))$. However, as one can easily see, the proof is valid if $\phi(t, a, b)<\psi(t)$ for $t<\tau$ and $\phi(\tau, a, b)=\psi(\tau)$.)

This inequality implies that there exists an $\omega(0<\omega\leqq\infty)$ such that

(11) $$\lim_{t\to\omega}\phi(t, a, b)=\infty .$$

Since

$$\phi''=t^{\alpha\lambda-2}\phi^{1+\alpha}>0 ,$$

ϕ' is a nondecreasing function. So it follows from (11) that

(12) $$\phi'(t, a, b)>0$$

if t is sufficiently close to ω. Then, from (3), (4), (11) and (12), it follows that

$$\lim_{t\to\omega}y(t)=\infty , \qquad \lim_{y\to\infty}z(y, C)=\lim_{t\to\omega}ty'(t)=\infty .$$

Thus we have proved 1).

To prove 2), we put $y^{-1}z=u$. Then u satisfies a differential equation

(13) $$\frac{du}{dy}=\frac{-\lambda(\lambda+1)\alpha^2+(2\lambda+1)\alpha u-u^2+\lambda(\lambda+1)\alpha^2 y}{\alpha y u}$$

and $u=y^{-1}z(y, C)$ is a solution of (13) defined for $0<y<\infty$. Also u is positive for $0<y<\infty$. Suppose that there exists $k>0$ snch that $0\leqq u\leqq k$ for $0<y<\infty$. Then if $K>0$ is chosen sufficiently large

$$-\lambda(\lambda+1)\alpha^2+(2\lambda+1)\alpha u-u^2>-K$$

and the inequality

$$\frac{du}{dy}>\frac{-K+\lambda(\lambda+1)\alpha^2 y}{\alpha y u}$$

holds for $0<y<\infty$. However the solution of

$$\frac{dU}{dy}=\frac{-K+\lambda(\lambda+1)\alpha^2 y}{\alpha y U}$$

is given by

$$U^2/2=-(K/\alpha)\log y+\lambda(\lambda+1)\alpha y+\text{const}$$

and hence $U\to\infty$ as $y\to\infty$. Therefore $u\to\infty$ as $y\to\infty$ in contradiction with our assumption $0\leqq u\leqq k$.

Therefore u is not bounded as $y\to\infty$ and we have

$$\limsup_{y\to\infty} u=\limsup_{y\to\infty} y^{-1}z(y, C)=\infty .$$

Consequently we can find a sequence $\{y_n\}$ such that

$$y_1<y_2<\cdots<y_n<\cdots , \qquad \lim y_n=\infty ,$$
$$u(y_1)<u(y_2)<\cdots<u(y_n)<\cdots , \qquad \lim u(y_n)=\infty ,$$
$$\left.\frac{du}{dy}\right|_{y=y_n}>0 , \quad n=1, 2, \cdots .$$

Let us denote by $f(y, u)$ the right-hand side of (13) and consider a function

$$f(y, u(y_n))=\frac{1}{\alpha y}\frac{-\lambda(\lambda+1)\alpha^2+(2\lambda+1)\alpha u(y_n)-u(y_n)^2}{u(y_n)}+\frac{\lambda(\lambda+1)\alpha}{u(y_n)} .$$

Since

$$\lim_{u\to\infty}\frac{-\lambda(\lambda+1)\alpha^2+(2\lambda+1)\alpha u-u^2}{u}=-\infty$$

and $u(y_n)\to\infty$ as $n\to\infty$, we have

$$\frac{-\lambda(\lambda+1)\alpha^2+(2\lambda+1)\alpha u(y_n)-u(y_n)^2}{u(y_n)}<0$$

if n is sufficiently large. Hence $f(y, u(y_n))$ is an increasing function of y for large n. Therefore if $y>y_n$,

$$f(y, u(y_n))>f(y_n, u(y_n))=\left.\frac{du}{dy}\right|_{y=y_n}>0 .$$

From this we easily get

$$u(y)>u(y_n)$$

if $y>y_n$. Indeed, if not, there exists $\hat{y}>y_n$ such that

$$u(y)>u(y_n) , \qquad \hat{y}>y>y_n ,$$
$$u(\hat{y})=u(y_n)$$

since $du/dy>0$ at $y=y_n$. This implies

$$\left.\frac{du}{dy}\right|_{y=\hat{y}}=f(\hat{y}, u(\hat{y}))=f(\hat{y}, u(y_n))<0$$

in contradiction with the above inequality.

This being valid for every large n, we get

$$\lim_{y\to\infty} u(y)=\infty .$$

Thus we have proved 2).

As an immediate consequence of Lemma 2, we get

LEMMA 3. *Let $\phi(t, a, b)$ be a solution of* (1) *with $b>\hat{b}(a)$. Then*

$$\lim_{t\to\omega}\frac{t\phi'(t, a, b)}{\phi(t, a, b)}=\infty , \qquad \lim_{t\to\omega}\phi'(t, a, b)=\infty ,$$

where ω is a positive number or ∞ which appears in (11).

PROOF. Let $z(y, C)$ be a solution of (2) corresponding to $\phi(t, a, b)$. Then from 2) of Lemma 2, we have

$$\lim_{y\to\infty} z(y, C)/y=\lim_{t\to\omega} ty'/y=\infty .$$

So, from (4), we get

$$\lim_{t\to\omega}\frac{t\phi'(t, a, b)}{\phi(t, a, b)}=\infty ,$$

which is the first assertion to be proved.

As $\phi(t, a, b)\to\infty$ as $t\to\omega$, the second assertion is obvious if $\omega<\infty$. So suppose that $\omega=\infty$. Then, for any given $T>0$, there exists $M>0$ such that

$$\phi(t, a, b)>M \quad \text{for} \quad t\geqq T .$$

Therefore

$$\phi''(t, a, b)=t^{\alpha\lambda-2}[\phi(t, a, b)]^{1+\alpha}>M^{1+\alpha}t^{\alpha\lambda-2} , \qquad t\geqq T .$$

Integrating both sides of this inequality from T to $t>T$, we get

$$\phi'(t, a, b)-\phi'(T, a, b)>\frac{M^{1+\alpha}}{\alpha\lambda-1}(t^{\alpha\lambda-1}-T^{\alpha\lambda-1}) .$$

As $\alpha\lambda>1$, the right-hand member tends to ∞ as $t\to\omega=\infty$. Thus we get the second relation

$$\lim_{t\to\omega}\phi'(t, a, b)=\infty .$$

LEMMA 4. *Let* $\phi(t, a, b)$ *be a solution of* (1) *with* $b>\hat{b}(a)$ *and* $z(y, C)$ *be a corresponding solution of* (2). *Then*

$$\lim_{y\to\infty} y^{-3/2}z(y, C)=\alpha\sqrt{\frac{2\lambda(\lambda+1)}{\alpha+2}} .$$

PROOF. Since the relation (3) will give

$$y^{-3/2}z(y, C)=[\lambda(\lambda+1)]^{1/2}\{\alpha\lambda t^{-\alpha\lambda/2}[\phi(t, a, b)]^{-\alpha/2} + \alpha t^{1-\alpha\lambda/2}[\phi(t, a, b)]^{-1-\alpha/2}\phi'(t, a, b)\} , \tag{14}$$

and we already know that

$$\lim_{t\to\omega} y=\infty ,$$

what we have to show is that the right-hand side of (14) tends to $\alpha\sqrt{2\lambda(\lambda+1)/(\alpha+2)}$ as $t\to\omega$. However, as

$$\lim_{t\to\omega} t^{-\alpha\lambda/2}[\phi(t, a, b)]^{-\alpha/2}=0$$

by (11), all we need is to show that

$$\lim_{t\to\omega} t^{1-\alpha\lambda/2}\phi^{-1-\alpha/2}\phi'=\sqrt{2/(\alpha+2)} ,$$

or, what is the same thing, that

$$\lim_{t\to\omega} t^{2-\alpha\lambda}\phi^{-\alpha-2}\phi'^2=2/(\alpha+2) .$$

(i) The case when $\alpha\lambda\geqq 2$.

In this case

$$t^{-\alpha\lambda+2}\phi^{-\alpha-2}\phi'^2=\phi'^2/(t^{\alpha\lambda-2}\phi^{\alpha+2})$$

takes the form ∞/∞ as $t\to\omega$ from Lemmas 2 and 3. So, to apply l'Hospital's theorem, we consider

$$\lim_{t\to\omega}\left[\frac{d}{dt}(\phi'^2)\Big/\frac{d}{dt}(t^{\alpha\lambda-2}\phi^{\alpha+2})\right] .$$

Then since

$$\frac{d}{dt}(\phi'^2)=2\phi'\phi''=2t^{\alpha\lambda-2}\phi^{1+\alpha}\phi' ,$$

$$\frac{d}{dt}(t^{\alpha\lambda-2}\phi^{\alpha+2})=(\alpha\lambda-2)t^{\alpha\lambda-3}\phi^{\alpha+2}+(\alpha+2)t^{\alpha\lambda-2}\phi^{\alpha+1}\phi' ,$$

we have

$$\lim_{t\to\omega}\left[\frac{d}{dt}(\phi'^2)\Big/\frac{d}{dt}(t^{\alpha\lambda-2}\phi^{\alpha+2})\right]$$
$$=\lim_{t\to\omega}\left[1\Big/\left(\frac{\alpha\lambda-2}{2}\frac{\phi}{t\phi'}+\frac{\alpha+2}{2}\right)\right].$$

As $\phi/t\phi'\to 0$ as $t\to\omega$ by Lemma 3, we obtain the required result.

ii) The case when $\alpha\lambda<2$.

In this case we write

$$t^{-\alpha\lambda+2}\phi^{-\alpha-2}\phi'^2=(t^{2-\alpha\lambda}\phi'^2)/\phi^{\alpha+2} .$$

Then the limit of the right-hand side as $t\to\omega$ is again of the form ∞/∞. Differentiating the numerator and the denominator and passing to the limit, we get

$$\lim_{t\to\omega}\left[\frac{d}{dt}(t^{2-\alpha\lambda}\phi'^2)\Big/\frac{d}{dt}(\phi^{\alpha+2})\right]$$
$$=\lim_{t\to\omega}[((2-\alpha\lambda)t^{1-\alpha\lambda}\phi'^2+2t^{2-\alpha\lambda}\phi'\phi'')/(\alpha+2)\phi^{\alpha+1}\phi']$$
$$=\lim_{t\to\omega}[((2-\alpha\lambda)t^{1-\alpha\lambda}\phi'^2+2\phi^{1+\alpha}\phi')/(\alpha+2)\phi^{\alpha+1}\phi']$$
$$=\lim_{t\to\omega}\frac{2-\alpha\lambda}{\alpha+2}\frac{\phi'}{t^{\alpha\lambda-1}\phi^{\alpha+1}}+\frac{2}{\alpha+2} .$$

Since $\alpha\lambda>1$, the first term is of the form ∞/∞. So, to apply l'Hospital's theorem again, we consider the limit

$$\lim_{t\to\omega}\left[\phi''\Big/\frac{d}{dt}(t^{\alpha\lambda-1}\phi^{\alpha+1})\right]$$
$$=\lim_{t\to\omega}[t^{\alpha\lambda-2}\phi^{1+\alpha}/((\alpha\lambda-1)t^{\alpha\lambda-2}\phi^{\alpha+1}+(\alpha+1)t^{\alpha\lambda-1}\phi^{\alpha}\phi')]$$
$$=\lim_{t\to\omega}[1/((\alpha\lambda-1)+(\alpha+1)t\phi'/\phi)] .$$

As $t\phi'/\phi\to\infty$ by Lemma 3, this limit is equal to zero. Thus we have obtained

$$\lim_{t\to\omega}[t^{2-\alpha\lambda}\phi'^2/\phi^{\alpha+2}]=2/(\alpha+2) .$$

§ 3. Explicit construction of the solution $\phi(t, a, b)$ with $b>\hat{b}(a)$.

From Lemma 4 just proved, we now know that $z(y, C)$ corresponding to $\phi(t, a, b)$ $(b>\hat{b}(a))$ is a solution of (2) such that

$$\lim_{y\to\infty} y^{-3/2}z=\alpha\sqrt{\frac{2\lambda(\lambda+1)}{\alpha+2}}\,.$$

In view of this fact, we put

$$y^{-1/2}=\eta\,,\qquad z^{-1}=\eta^3\Big(\frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}}+u\Big)$$

and transform the equation (2) into the following one:

$$\eta\frac{du}{d\eta}=\frac{(2\lambda+1)(\alpha+2)}{\lambda(\lambda+1)\alpha^2}\,\eta+\Big(2+\frac{4}{\alpha}\Big)u+\cdots \tag{15}$$

where the unwritten part is a polynomial of η and u starting with the terms of the second degree. What we need is the solution of (15) which tends to zero as $\eta\to 0$. Since $\eta=0$ is a singularity of Briot-Bouquet type and $2+4/\alpha>0$, such solution can be expressed as

$$u=\sum_{m+n>0} u_{mn}\eta^m(A\eta^{2+4/\alpha})^n$$

if $2+4/\alpha$ is not an integer, and as

$$u=\sum_{m+n>0} u_{mn}\eta^m[\eta^{2+4/\alpha}(c_1\log\eta+A)]^n$$

if $2+4/\alpha$ is an integer. Here A is an arbitrary constant and c_1 is a constant (which might be zero). Thus we get

$$(z(y, C))^{-1}=y^{-3/2}\Big(\frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}}+\sum_{m+n>0} u_{mn}y^{-m/2}(Ay^{-1-2/\alpha})^n\Big) \tag{16}$$

if $4/\alpha$ is not an integer, and

$$(z(y, C))^{-1}=y^{-3/2}\Big(\frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}}+\sum_{m+n>0} u_{mn}y^{-m/2}[y^{-1-2/\alpha}(c\log y+A)]^n\Big) \tag{16'}$$

otherwise $(c=-c_1/2)$.

The solution $y(t)$ of

$$ty'=z(y, C)$$

is then obtained from

$$\int_{y_0}^{y}\frac{dy}{z(y,C)}=\int_{t_0}^{t}\frac{dt}{t}=\log\frac{t}{t_0}\,,\qquad y_0=y(t_0)\,.$$

To carry out the integration on the left-hand side explicitly, we return to the variable η and replace the expression (16) or (16′) by

$$z(y,C)^{-1}=\eta^3F(\eta)\,,$$

$$F(\eta)=\frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}}+\sum_{m+n>0}u_{mn}\eta^m(A\eta^{2+4/\alpha})^n \tag{17}$$

or

$$F(\eta)=\frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}}+\sum_{m+n>0}u_{mn}\eta^m[\eta^{2+4/\alpha}(c_1\log\eta+A)]^n\,. \tag{17′}$$

Then we get

$$\int_{y_0}^{y}z(y,C)^{-1}dy=-2\int_{\eta_0}^{\eta}F(\eta)d\eta\,,\qquad \eta=y^{-1/2}\,,\qquad \eta_0=y_0^{-1/2}\,.$$

Since $y\to\infty$ as $t\to\omega$, we have the equality

$$\lim_{\eta\to 0}\int_{\eta_0}^{\eta}F(\eta)d\eta=-\frac{1}{2}\lim_{t\to\omega}\log\frac{t}{t_0}\,.$$

Since $F(\eta)$ is bounded as $\eta\to 0$, the left-hand side of the above equality has a finite (negative) value. This implies the finiteness of ω. In other words, there exists a finite positive number ω such that

$$\lim_{t\to\omega}\phi(t,a,b)=\infty\,.$$

Hence $\phi(t,a,b)(b>\hat{b}(a))$ has a movable singularity at $t=\omega$.

ω being finite, $y(t)$ is given by

$$\int_0^{y^{-1/2}}F(\eta)d\eta=-\frac{1}{2}\log\frac{t}{\omega}\,.$$

Now let us put

$$\frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}}=\gamma\,,\qquad 2+\frac{4}{\alpha}=\mu\,.$$

Then termwise integration of (17) will give

$$\int_0^{\eta}F(\eta)d\eta=\gamma\eta[1+\sum_{m+n>0}\gamma_{mn}\eta^m(A\eta^{\mu})^n]=-\frac{1}{2}\log\frac{t}{\omega}\,. \tag{18}$$

Also, if $F(\eta)$ is of the form (17′), we get

$$\int_0^\eta F(\eta)d\eta=\gamma\eta[1+\sum_{m+n>0}\gamma_{mn}\eta^m(\eta^\mu[c_1\log\eta+A])^n]=-\frac{1}{2}\log\frac{t}{\omega}\ . \tag{18′}$$

This can be done by termwise integration and rearrangement of the integrated series noticing that, in this case, μ is a positive integer. Such rearrangement is justified by the absolute convergence of the series (cf. [2], p. 309).

To obtain the explicit expression of $y(t)$, we have to solve (18) or (18′) with respect to η and then put $\eta=y^{-1/2}$. This can be done with the aid of Lemma 1. In fact, (18) and (18′) can be written as

$$\eta[1+\sum_{m+n>0}\gamma_{mn}\eta^m(\eta^\mu[c_1\log\eta+A])^n]=\tau\ ,\qquad \tau=-\frac{1}{2\gamma}\log\frac{t}{\omega}\ ,$$

where $c_1=0$ if μ is not an integer. Thus Lemma 1 can be applied directly and we get

$$\eta=y^{-1/2}=\tau[1+\sum_{m+n>0}\hat{\gamma}_{mn}\tau^m(\tau^\mu[c_1\log\tau+A])^n]\ .$$

First let us suppose that $\mu=2+4/\alpha$ is not an integer. Then since $c_1=0$ in this case, we have

$$y^{-1/2}=\tau[1+\sum_{m+n>0}\hat{\gamma}_{mn}\tau^m(A\tau^\mu)^n]\ .$$

Hence $y^{-1/2}$ is a holomorphic function of τ and τ^μ in the neighbourhood of $\tau=0$. On the other hand,

$$\tau=-\frac{1}{2\gamma}\log\frac{t}{\omega}$$

is a holomorphic function of t in the neighbourhood of $t=\omega$ and admits a Taylor expansion

$$\tau=\frac{1}{2\gamma\omega}(\omega-t)\left(1+\frac{1}{2\omega}(\omega-t)+\cdots\right)\ .$$

Therefore $y^{-1/2}$ is a holomorphic function of $\omega-t$ and $(\omega-t)^\mu$ in the neighbourhood of $t=\omega$. Hence it can be expressed as a double power series in $\omega-t$ and $(\omega-t)^\mu$ in the following way:

$$y^{-1/2}=\frac{1}{2\gamma\omega}(\omega-t)[1+\sum_{m+n>0}a_{mn}(\omega-t)^m(\omega-t)^{\mu n}]\ .$$

From this we obtain

$$y^{1/\alpha}=(2\gamma\omega)^{2/\alpha}(\omega-t)^{-2/\alpha}(1+\sum_{m+n>0}\hat{a}_{mn}(\omega-t)^m(\omega-t)^{\mu n})$$
$$=\left(\frac{2(\alpha+2)\omega^2}{\alpha^2\lambda(\lambda+1)}\right)^{1/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{m+n>0}\hat{a}_{mn}(\omega-t)^m((\omega-t)^{2+4/\alpha})^n]\ .$$

Next suppose that $\mu=2+4/\alpha$ is an integer. In this case, $c_1\neq 0$ and

$$y^{-1/2}=\tau[1+\sum_{m+n>0}\hat{\gamma}_{mn}\tau^m(\tau^\mu[c_1\log\tau+A])^n]\ .$$

This expression shows that $y^{-1/2}$ is a holomorphic function of τ and $\tau^\mu[c_1\log\tau+A]$ in the neighborhood of $\tau=0$, $\tau^\mu[c_1\log\tau+A]=0$. Since

$$\tau=\frac{1}{2\gamma\omega}(\omega-t)\left(1+\frac{1}{2\omega}(\omega-t)+\cdots\right),$$

we have

$$\tau^\mu(c_1\log\tau+A)=\left(\frac{1}{2\gamma\omega}\right)^\mu(\omega-t)^\mu(1+\cdots)$$
$$\times[c_1\log(\omega-t)-c_1\log 2\gamma\omega+A+\sum_{r=1}^{\infty}c_r(\omega-t)^r]$$
$$=(\omega-t)^\mu\log(\omega-t)B_1(t)+B_2(t)$$

where $B_1(t)$ and $B_2(t)$ are power series of $\omega-t$ absolutely convergent in the neighbourhood of $t=\omega$. Therefore $y^{-1/2}$ can be expressed as

$$y^{-1/2}=\frac{1}{2\gamma\omega}(\omega-t)[1+\sum_{m+n>0}\tilde{\gamma}_{mn}(\omega-t)^m((\omega-t)^\mu\log(\omega-t))^n]\ .$$

From this we obtain

$$y^{1/\alpha}=(2\gamma\omega)^{2/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{m+n>0}\delta_{mn}(\omega-t)^m((\omega-t)^\mu\log(\omega-t))^n]\ .$$

Since μ is a positive integer, we can rearrange the above expression into a form

$$y^{1/\alpha}=(2\gamma\omega)^{2/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{k>0}^{\infty}S_k(t)]\ ,$$
$$S_k(t)=\sum_{m+n\mu=k}\delta_{mn}(\omega-t)^m((\omega-t)^\mu\log(\omega-t))^n$$
$$=(\omega-t)^k\sum_{m+n\mu=k}\delta_{mn}(\log(\omega-t))^n\ .$$

Since $m+n\mu=k$ implies $n=(k-m)/\mu$, $\sum_{m+n\mu=k}\delta_{mn}(\log(\omega-t))^n$ is a polynomial of $\log(\omega-t)$ whose degree is at most $[k/\mu]$ where [ ] is the Gauss' symbol. Hence we can write

$$y^{1/\alpha}=(2\gamma\omega)^{2/\alpha}(w-t)^{-2/\alpha}[1+\sum_{m>0}(\omega-t)^m q_m(\log(\omega-t))]$$
$$=\left(\frac{2(\alpha+2)\omega^2}{\alpha^2\lambda(\lambda+1)}\right)^{1/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{m>0}(\omega-t)^m q_m(\log(\omega-t))]$$

where $q_m(\xi)$ is a polynomial of ξ whose degree is at most $[m/\mu]$.

Inserting these expressions of $y^{1/\alpha}$ into

$$\phi(t, a, b)=[\lambda(\lambda+1)]^{1/\alpha}t^{-\lambda}y^{1/\alpha}$$

and noticing that

$$t^{-\lambda}=\omega^{-\lambda}\left(1+\frac{\lambda}{\omega}(\omega-t)+\cdots\right),$$

in the neighbourhood of $t=\omega$, we obtain the following expression of $\phi(t, a, b)$ which is valid in the neighbourhood of its movable singularity $t=\omega$:

$$\phi(t, a, b)=\left(\frac{2(\alpha+2)}{\alpha^2\omega^{\alpha\lambda-2}}\right)^{1/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{m+n>0}c_{mn}(\omega-t)^m((\omega-t)^{2+4/\alpha})^n],$$
$$\text{if } 4/\alpha\neq\text{integer},$$

$$\phi(t, a, b)=\left(\frac{2(\alpha+2)}{\alpha^2\omega^{\alpha\lambda-2}}\right)^{1/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{m>0}(\omega-t)^m p_m(\log(\omega-t))],$$
$$\text{if } 4/\alpha=\text{integer},$$

where $p_m(\xi)$ is a polynomial of ξ whose degree is at most $[m/\mu]=[m\alpha/(2\alpha+4)]$.

§ 4. The solution curve of $z(y, C)$ with $C<\hat{C}$.

To study the behaviour of $\phi(t, a, b)$ with $b<\hat{b}(a)$, we must examine the corresponding orbit of the system (6) in detail.

As we already know, the system has (0, 0) and (1, 0) as critical points and (1, 0) is a saddle point. There exist infinitely many orbits tending to (0, 0) as $s\to-\infty$ with the common tangent $z=\alpha\lambda y$ at (0, 0). Among them, one and only one orbit tends to (1, 0) as $s\to\infty$ which is represented by a curve $z=z(y, \hat{C})$.

Also, as we can easily observe from (6), z-axis is an invariant set of the system. If $\alpha=1$, it consists entirely of critical points. Other-

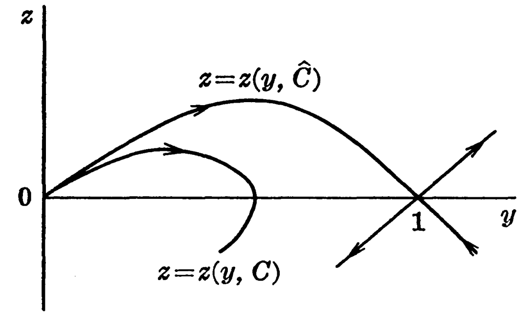

FIGURE 2

wise it consists of three orbits one of which is a critical point (0, 0).

Let us consider an orbit corresponding to $\phi(t, a, b)$ with $b<\hat{b}(a)$. It is represented by a curve $z=z(y, C)$ with $C<\hat{C}$. Hence it is located below the curve $z=z(y, \hat{C})$. As one can observe from Figure 2, this orbit crosses y-axis somewhere between 0 and 1 and goes into the region

$$D: \quad 0<y<1\,, \qquad z<0\,.$$

Since $dy/ds=\alpha yz<0$ in D, the point $(y(s), z(s))$ on the orbit moves leftwards as s increases as long as it stays in D.

Now on the segment $0<y<1$ on y-axis, we have

$$\frac{dz}{ds}=\lambda(\lambda+1)\alpha^2(y^3-y^2)<0\,.$$

Consequently $(y(s), z(s))$ can never leave D and it keeps on moving to the left as $s\to\infty$. Thus we have the following alternative.

(i) $(y(s), z(s))$ tends to $(y_\infty, -\infty)$ as $s\to\infty$ where $0\leqq y_\infty<1$.

(ii) $(y(s), z(s))$ tends to a critical point other than (1, 0) as $s\to\infty$.

In the case (i), we have to have

$$\lim_{s\to\infty}\frac{dy}{ds}=\lim_{y\to y_\infty}\alpha yz(y, C)=0\,.$$

Since $z\to-\infty$ as $s\to\infty$, this is possible only when $y_\infty=0$. Hence, in the case (i),

$$\lim_{s\to\infty} y(s)=0\,.$$

In the case (ii), we also have

$$\lim_{s\to\infty} y(s)=0$$

because every critical point other than (1, 0) lies on z-axis. (Detailed investigation shows that when $\alpha<1$, the case (i) takes place and when $\alpha\geqq 1$, the case (ii) takes place.)

As $y=0$ implies $\phi(t, a, b)=0$, $\phi(t, a, b)$ tends to zero as $t\to\omega(0<\omega\leqq\infty)$. If $\omega=\infty$, then $\phi(t, a, b)$ is a bounded solution of (1). This is however absurd because the boundedness of the solution implies $b=\hat{b}(a)$. Hence ω must be finite.

Since $\phi(t, a, b)>0$ for $0<t<\omega$ and $\phi(\omega, a, b)=0$,

$$\phi'(\omega, a, b)\leqq 0 .$$

As $\phi(\omega, a, b)=\phi'(\omega, a, b)=0$ implies $\phi(t, a, b)\equiv 0$ because of the uniqueness of the solution, we must have

$$\phi'(\omega, a, b)<0 .$$

Thus it follows that

$$\lim_{t\to\omega}\frac{t\phi'}{\phi}=-\infty .$$

From this and (4):

$$z/y=ty'/y=\alpha tx'/x+\alpha\lambda ,$$

we get

$$\lim_{s\to\infty}\frac{z(s)}{y(s)}=\lim_{y\to 0}\frac{z(y, C)}{y}=-\infty .$$

§5. The solution $\phi(t, a, b)$ with $b<\hat{b}(a)$.

The transformation

$$z\to w=yz^{-1}$$

will change (2) into

$$y\frac{dw}{dy}=\frac{1}{\alpha}w-(2\lambda+1)w^2+\lambda(\lambda+1)\alpha w^3-\lambda(\lambda+1)\alpha yw^3 . \tag{19}$$

Then from what we have shown in the preceding section,

$$w=y(z(y, C))^{-1} , \qquad C<\hat{C}$$

is a solution of (19) such that

$$\lim_{y\to 0} w=0 .$$

As $y=0$ is a Briot-Bouquet type singularity and $1/\alpha>0$ and also the right-hand side of (19) is divisible by w, such solutions are given by

$$w=\sum_{m+n>0} w_{mn}y^m(By^{1/\alpha})^n , \qquad w_{01}=1 , \tag{20}$$

even if $1/\alpha$ is an integer. Here B is an arbitrary constant.

If $1/\alpha$ is not an integer, it is known that (19) has one and only one holomorphic solution which tends to 0 as $y\to 0$. This solution is obtained by putting $B=0$ in (20). Hence it is given by

$$w=\sum_{m=1}^{\infty} w_{m0}y^m .$$

However $w\equiv 0$ is also such a solution. Therefore

$$w_{m0}=0 , \qquad m=1, 2, \cdots .$$

Consequently $By^{1/\alpha}$ is the term of the lowest degree in the expression (20).

If $1/\alpha$ is an integer, say $N(>0)$, then the double power series in (20) can be rearranged into a single power series

$$\sum_{k=1}^{\infty} \alpha_k y^k .$$

As the right-hand member of (19) is divisible by w, it can easily be proved that

$$\alpha_k=0 , \qquad k<N$$

and α_N is arbitrary. So $\alpha_N y^N=\alpha_N y^{1/\alpha}$ is again the term of the lowest degree.

Thus, in both cases, we can write

$$\begin{aligned} w&=y(z(y, C))^{-1} \\ &=By^{1/\alpha}[1+\sum_{m+n>0} w_{mn}y^m(By^{1/\alpha})^n] . \end{aligned}$$

Hence we get

$$(z(y, C))^{-1}=By^{1/\alpha-1}[1+\sum_{m+n>0} w_{mn}y^m(By^{1/\alpha})^n] .$$

$z(y, C)$ being negative for sufficiently small y, B must be negative. So we shall put $B=-1/\Gamma(\Gamma>0)$ hereafter.

The solution $y(t)$ of $ty'=z(y, C)$ such that

$$\lim_{t\to\omega} y(t)=0$$

is then obtained from

$$\int_0^y (z(y, C))^{-1}dy=\int_\omega^t \frac{dt}{t}=\log\frac{t}{\omega} .$$

Carrying out the termwise integration of the left-hand side and dividing both sides by $-\alpha$, we obtain

$$\Gamma^{-1}y^{1/\alpha}[1+\sum_{m+n>0}\beta_{mn}y^m(\Gamma^{-1}y^{1/\alpha})^n]=-\frac{1}{\alpha}\log\frac{t}{\omega} .$$

If we put $\Gamma^{-1}y^{1/\alpha}=\zeta$, $y=(\Gamma\zeta)^\alpha$ in the above equality and then apply Lemma 1, we immediately have

$$y^{1/\alpha}=-\frac{\Gamma}{\alpha}\log\frac{t}{\omega}[1+\sum_{m+n>0}\hat{\beta}_{mn}\left(-\frac{1}{\alpha}\log\frac{t}{\omega}\right)^m\left(-\frac{\Gamma}{\alpha}\log\frac{t}{\omega}\right)^{\alpha n}]$$

which shows that $y^{1/\alpha}$ is a holomorphic function of $-\log(t/\omega)$ and $(-\log(t/\omega))^\alpha$ when t is sufficiently close to ω. Applying the same argument as was used at the end of §3, $y^{1/\alpha}$ can be expressed as a double power series in $\omega-t$ and $(\omega-t)^\alpha$ in the following form:

$$y^{1/\alpha}=\frac{\Gamma}{\alpha\omega}(\omega-t)[1+\sum_{m+n>0}\tilde{\beta}_{mn}(\omega-t)^m(\omega-t)^{\alpha n}] .$$

Substituting it into

$$\phi(t, a, b)=[\lambda(\lambda+1)]^{1/\alpha}t^{-\lambda}y^{1/\alpha}$$

and noticing that

$$t^{-1}=\omega^{-\lambda}\left(1+\frac{\lambda}{\omega}(\omega-t)+\cdots\right)$$

in the neighbourhood of $t=\omega$, we get the following expression of $\phi(t, a, b)$ valid in the neighbourhood of $t=\omega$:

$$\phi(t, a, b)=A(\omega-t)[1+\sum_{m+n>0}b_{mn}(\omega-t)^m(\omega-t)^{\alpha n}] ,$$

$$A=\frac{\Gamma(\lambda(\lambda+1))^{1/\alpha}}{\alpha\omega^{\lambda+1}} .$$

This expression shows that $t=\omega$ is a movable branch point of the solution if α is not an integer.

§ 6. Summary of the results obtained.

Summarizing the results obtained so far, together with those mentioned in [1], we get the following theorem.

THEOREM. *Let $x=\phi(t, a, b)$ be a solution of the differential equation*

$$x''=t^{\alpha\lambda-2}x^{1+\alpha}\ , \qquad \alpha>0\ , \qquad \alpha\lambda>1\ ,$$

such that

$$\lim_{t\to 0} x=a\ , \qquad \lim_{t\to 0} x'=b\ , \qquad 0<a<\infty\ , \quad |b|<\infty\ .$$

Such a solution actually exists and has following properties.

1) *$\phi(t, a, b)$ admits following double power series expression in the neighbourhood of $t=0$:*

$$\phi(t, a, b)=a\Big(1+\sum_{m+n>0}\gamma_{mn}\Big(\frac{a^{\alpha}}{\lambda(\lambda+1)}\ t^{\alpha\lambda}\Big)^{m}\Big(\frac{\alpha b}{a}\,t\Big)^{n}\Big)\ .$$

2) *For each a $(0<a<\infty)$, there exists one and only one value $\hat{b}(a)$ of b such that $x=\phi(t, a, \hat{b}(a))$ is defined and bounded for $0<t<\infty$ together with its derivative. In the neighbourhood of $t=\infty$, $\phi(t, a, \hat{b}(a))$ can be expressed in the following form:*

$$\phi(t, a, \hat{b}(a))=[\lambda(\lambda+1)]^{1/\alpha}t^{-\lambda}(1+\sum_{n>0}c_{n}t^{n\mu/\alpha})\ ,$$

where μ is a negative eigenvalue of a matrix

$$\begin{pmatrix} 0 & \alpha \\ \lambda(\lambda+1)\alpha^{2} & (2\lambda+1)\alpha \end{pmatrix}\ .$$

3) *If $b>\hat{b}(a)$, $\phi(t, a, b)$ has a movable singularity at $t=\omega(0<\omega<\infty)$ and*

$$\lim_{t\to\omega}\phi(t, a, b)=\infty\ .$$

In the neighbourhood of $t=\omega$, $\phi(t, a, b)$ can be expressed as

$$\phi(t, a, b)=\Big(\frac{2(\alpha+2)}{\alpha^{2}\omega^{\alpha\lambda-2}}\Big)^{1/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{m+n>0}c_{mn}(\omega-t)^{m}((\omega-t)^{2+4/\alpha})^{n}]\ ,$$

if $4/\alpha$ is not an integer, and as

$$\phi(t, a, b)=\left(\frac{2(\alpha+2)}{\alpha^2\omega^{\alpha\lambda-2}}\right)^{1/\alpha}(\omega-t)^{-2/\alpha}[1+\sum_{m>0}(\omega-t)^m p_m(\log(\omega-t))] ,$$

$p_m(\xi)$: a polynomial of ξ whose degree is at most $[m\alpha/(2\alpha+4)]$,

if $4/\alpha$ is an integer. Here ω naturally depends on a and b.

4) *If $b<\hat{b}(a)$, then*

$$\lim_{t\to\omega}\phi(t, a, b)=0$$

for some finite positive ω, and in the neighbourhood of $t=\omega$, $\phi(t, a, b)$ is expressed in the following form:

$$\phi(t, a, b)=A(\omega-t)[1+\sum_{m+n>0} b_{mn}(\omega-t)^m(\omega-t)^{\alpha n}] .$$

Here A and ω depend on a and b, and $t=\omega$ is a movable branch point of the solution unless α is an integer.

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References

[1] T. SAITO, On bounded solutions of $x''=t^{\beta}x^{1+\alpha}$, Tokyo J. Math., vol. **1**, no. 1 (1978), 57–75.

[2] R. A. SMITH, Singularities of solutions of certain plane autonomous systems, Proc. Roy. Soc. Edinburgh, Sect. A, vol. **72**, 4 (1973/74), 307–315.

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