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On Unimodal Linear Transformations and Chaos I

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Introduction

Recently, much work has been done to investigate for a parametrized family of continuous mappings on an interval how the structure of orbits for such maps changes as the parameters vary. These investigations are motivated by the desire to analyze chaotic phenomena frequently observed in nature, in studying how the behavior of dynamical systems (appearing in various models such as Lorentz models, ecological models or models describing chemical reactions and so on) is influenced by the change in characteristic parameters and turns into a turbulent state.

For the case of one-dimensional systems, attempts have been made to characterize how the change in parameter values gives rise to the existence of unstable non-periodic orbits; such characterization is done by describing the nature of periodic points that appear; for instance, by the appearance of periodic points of period 3 [5]. If the mapping in question possesses an invariant measure, the existence of unstable non-periodic orbits suggests that the mapping may have ergodic or mixing properties. Furthermore, it is possible to measure the "size" of the set of points having non-periodic orbits, in terms of the "size" of the support of the invariant measure. It seems to be also natural to explain the appearance of the so-called "window" phenomena, observing that in such cases the invariant measure is not absolutely continuous with respect to the Lebesgue measure on the interval.

In this paper, we concentrate ourselves on the study of unimodal linear transformations on the unit interval [0, 1], which are the simplest one-dimensional models. Let us define a family $\{f_\mu\}$ of mappings of the unit interval [0, 1] in the following way, where the parameter μ is determined by a pair of real numbers (a, b):

$f_\mu(x)$ is linear with the slope a on the sub-interval $[0, c]$ for some $c(0<c<1)$, linear with slope $-b$ on the sub-interval $[c, 1]$, and is

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continuous at c.

In this paper, we shall deal only with the case $a=b$. The case $a\neq b$ will be treated in the subsequent paper. The detailed consideration of the orbit structure for the mapping $f_{(a,b)}$ with $a\neq b$ can be done, more or less, by extending the methods done for the case $a=b$; in this sense, the case $a=b$ contains most of the essential features. However, for some pair (a, b) with $a\neq b$, in sharp contrast to the case $a=b$, the so-called "window" phenomena will take place. Namely, there exists a unique periodic orbit, and for all the points x in $[0, 1]$, except for a set of Lebesgue measure 0, $\{f^n_{(a,b)}(x)\}$ converges to this periodic orbit. On the other hand, there exists a periodic point of period 3 for $f_{(a,b)}$, and consequently there are uncountably many points (constituting a set of Lebesgue measure 0, nevertheless) whose orbits under $f_{(a,b)}$ are all non-periodic and unstable.

The mappings $f_{(a,a)}$ which we shall consider in this paper are defined explicitly as follows:

$$f_{(a,a)}(x) = \begin{cases} ax & \text{for } x\in\left[0, \frac{1}{2}\right] \\ -ax+a & \text{for } x\in\left(\frac{1}{2}, 1\right], \end{cases}$$

where $0<a\leqq 2$. We note that in case $a<1$, the orbit under $f_{(a,a)}$ of every point $x\in[0, 1]$ will converge to the unique fixed point 0, while if $a=1$, every point $x\in[0, 1/2]$ is a fixed point of $f_{(a,a)}$, and furthermore $f_{(a,a)}[1/2, 1]=[0, 1/2]$. Thus, when $a\leqq 1$, the transformation $f_{(a,a)}$ has a trivial structure. Consequently, we shall discuss from now on only the case for $1<a\leqq 2$. We see that in each such case, $f_{(a,a)}$ has a unique fixed point other than 0, and this fixed point is unstable. Since $a>1$, $f_{(a,a)}(1/2)=a/2>1/2$, and $f_{(a,a)}(a/2)=a(1-a/2)<1/2$, it is easy to see that $f_{(a,a)}(a/2, 1]=[0, a(1-a/2))$ and that $f_{(a,a)}(0, a(1-a/2))\supset(0, a(1-a/2))$. Furthermore, for each $x\in(0, a(1-a/2))$, there exists an integer N (depending on x) such that $f^n_{(a,a)}(x)\in[a(1-a/2), a/2]$ for all $n\geqq N$ and $f^m_{(a,a)}(x)<f^{m+1}_{(a,a)}(x)$ for $0\leqq m<N$. We also see that $f_{(a,a)}[a(1-a/2), a/2]=[a(1-a/2), a/2]$. Thus, the set $(0, a(1-a/2))\cup(a/2, 1)$ is transient for $f_{(a,a)}$, while the set $[a(1-a/2), a/2]$ is absorbing. Consequently, as far as asymptotic behavior of $f_{(a,a)}$ is concerned, it is enough to consider the restriction of $f_{(a,a)}$ to the subset $[a(1-a/2), a/2]$. It is easy to see that this restriction of $f_{(a,a)}$ is isomorphic to the map of $[0, 1]$ onto $[0, 1]$ which we denote from now on by f_a, defined as follows:

$$f_a(x)=\begin{cases} ax+2-a & \text{for } x\in\left[0,\,1-\dfrac{1}{a}\right) \\ -ax+a & \text{for } x\in\left[1-\dfrac{1}{a},\,1\right], \end{cases}$$

where $1<a\leqq 2$.

By the results of Lasota and Yorke [4], and Li and Yorke [6], we know that there exists a unique ergodic invariant measure for each such f_a which is absolutely continuous with respect to the Lebesgue measure, and that the density function of the invariant measure is equal almost everywhere to some function of bounded variation. Furthermore, if $a>\sqrt{2}$, then Theorems 1 and 2 of Bowen [1] imply that f_a is a weak Bernoulli transformation.

This paper contains two main theorems. In the first theorem, the density of the invariant measure mentioned above is determined explicitly. In the second, a representation of f_a by means of a symbolic dynamical system is obtained.

In a subsequent paper, explicit determination of the density of the invariant measure will be made for the transformations $f_{(a,b)}$ with $a\neq b$, and a detailed analysis of the "window" and "islands" phenomena will be presented. Furthermore comparison with the case for $a=b$ will be made carefully by utilizing the representation of $f_{(a,b)}$ by a symbolic dynamical system.

In concluding these introductory remarks, we would like to thank Professors Masaya Yamaguchi, Yuji Ito, Yoichiro Takahashi and Toshio Niwa for their interest on the problem and their valuable advices.

§1. Some definitions and notations.

In part I, we consider the transformation f_a on [0, 1] for $1<a\leqq 2$ defined by

$$f_a(x)=\begin{cases} ax+2-a & \text{for } x\in\left[0,\,1-\dfrac{1}{a}\right] \\ -ax+a & \text{for } x\in\left[1-\dfrac{1}{a},\,1\right]. \end{cases} \tag{1}$$

Let us say f_a to be of Markov type if for some natural number n

$$f_a^n(0)=0 \quad\text{and}\quad f_a^k(0)\neq 0 \quad\text{for}\quad 1\leqq k\leqq n-1\,, \tag{2}$$

that is, 0 is a periodic point of f_a with period n. We say a Markov type transformation f_a to be of even type (resp. odd type) if the number s defined by

$$(3) \qquad s=\#\left\{k;\ 1\leqq k\leqq n-3,\ f_a^k(0)>1-\frac{1}{a}\right\}$$

is even (resp. odd).

To represent f_a by a symbolic dynamical system, it is convenient to define the generator (the fundamental partition) of f_a separately as follows: In the case of odd Markov type, define the partition $\{I_0, I_1\}$ of the interval [0, 1] by

$$(4) \qquad I_0=\left[0,1-\frac{1}{a}\right] \quad \text{and} \quad I_1=\left(1-\frac{1}{a},1\right],$$

and call it the fundamental partition of f_a; in the case of even Markov type or of non-Markov type, define $\{I_0, I_1\}$ by

$$(5) \qquad I_0=\left[0,1-\frac{1}{a}\right) \quad \text{and} \quad I_1=\left(1-\frac{1}{a},1\right].$$

We will show in §3 that these partitions give the generator for the respective cases.

Denote by Ω the cartesian product $\{0, 1\}^{N^*}$, where N^* means the set of non-negative integers. We consider theproduct topology defined from the discrete topology on the coordinate space $\{0, 1\}$. For an element ω of Ω, $\omega(n)$ means the n-th coordinate of ω. If we write

$$(6) \qquad \varepsilon_a(x)=\begin{cases}0 & \text{for } x\in I_0\\ 1 & \text{for } x\in I_1,\end{cases}$$

and

$$(7) \qquad \omega_a^x(n)=\varepsilon_a(f_a^n(x)),$$

we have an element ω_a^x of Ω for every $x\in[0,1]$. Denote by π_a the mapping from [0, 1] to Ω which maps x to ω_a^x. Then it is easy to show that the following commutative relation holds:

$$(8) \qquad \sigma\circ\pi_a(x)=\pi_a\circ f_a(x) \quad \text{for every} \quad x\in[0,1],$$

where σ is the shift operator on Ω.

Define $S(k, \omega)$, for a non-negative integer k and $\omega\in\Omega$, by

$$(9)\qquad S(0,\omega)=1$$

$$S(k,\omega)=\begin{cases}1 & \text{if } \sum_{j=1}^{k-1}\omega(j) \text{ is even}\\ -1 & \text{if } \sum_{j=1}^{k-1}\omega(j) \text{ is odd, for } k\geqq 1,\end{cases}$$

and write simply $S(k, x)$ for $S(k, \omega_a^x)$. If we write

$$(10)\qquad \alpha(x)=1-a^{-1}(1+(-1)^{\varepsilon_a(x)}),$$

then it follows from (1) that

$$(11)\qquad x=\alpha(x)+a^{-1}(-1)^{\varepsilon_a(x)}f_a(x).$$

By using (11) successively, we obtain

$$(12)\qquad \begin{aligned} x&=\sum_{k=0}^{n-1}a^{-k}S(k,x)\alpha(f_a^k(x))+a^{-n}S(n,x)f_a^k(x)\\ &=\sum_{k=0}^{\infty}a^{-k}S(k,x)\alpha(f_a^k(x)),\end{aligned}$$

and therefore, in view of the following relations

$$(13)\qquad \alpha(f_a^k(x))=1-a^{-1}(1+(-1)^{\omega_a^x(k)}),$$

$$(14)\qquad S(k,x)(-1)^{\omega_a^x(k)}=S(k+1,x),$$

we have the following identity:

$$(15)\qquad x=1-\sum_{k=0}^{\infty}a^{-(k+1)}S(k,x).$$

We call (15) the f_a-expansion of x. This relation also shows that the fundamental partition $\{I_0, I_1\}$ is the generator of f_a.

§2. The invariant measure and the periodic orbits.

In this section, we derive the density function of the invariant measure for f_a, and then determine the support of this density function and the type of the periodic orbits of f_a.

Define a function $h_a(x)$ on $[0, 1]$ by

$$(16)\qquad h_a(x)=\sum_{n=0}^{\infty}a^{-n}S(n,0)1_{[f_a^n(0),1]}(x),$$

where 1_A is the indicator function of A. It can be shown that h_a is a function of bounded variations. Note that $h_a(x)$ can be written as

$$h_a(x)=\sum_{n:S(n,0)=1} a^{-n}1_{[f_a^n(0),1]}(x)-\sum_{n:S(n,0)=-1} a^{-n}1_{[f_a^n(0),1]}(x)\ . \tag{17}$$

THEOREM 2.1. (i) *$h_a(x)$ is the density function of the invariant measure for f_a for $1<a\leqq 2$.*

(ii) *If $\sqrt{2}<a$, then $h_a(x)$ is positive for every $x\in[0, 1]$.*

PROOF. To show (i), it is enough to show

$$h_a(x)=a^{-1}h_a\left(\frac{1}{a}x-\frac{2-a}{a}\right)1_{[2-a,1]}(x)+a^{-1}h_a\left(-\frac{1}{a}x+1\right)\quad \text{a.e. } x\in[0, 1]\ , \tag{18}$$

$$h_a(x)\geqq 0\ . \tag{19}$$

First we consider the case $x\in[0, 2-a)$. The right hand side of (18) is

$$\begin{aligned} a^{-1}h_a\left(-\frac{1}{a}x+1\right)&=\sum_{n=0}^{\infty} a^{-(n+1)}S(n, 0)1_{[f_a^n(0),1]}\left(-\frac{1}{a}x+1\right) \\ &=\sum_{n:\omega_a^0(n)=0} a^{-(n+1)}S(n, 0)1_{[f_a^n(0),1]}\left(-\frac{1}{a}x+1\right) \\ &\quad+\sum_{n:\omega_a^0(n)=1} a^{-(n+1)}S(n, 0)1_{[f_a^n(0),1]}\left(-\frac{1}{a}x+1\right)\ . \end{aligned} \tag{20}$$

If $\omega_a^0(n)=0$, then $f_a^n(0)<1-1/a<-(1/a)x+1$, so the first term of the right hand side of (20) is equal to $\sum_{n:\omega_a^0(n)=0} a^{-(n+1)}S(n, 0)$. The second term is equal to

$$\sum_{n:\omega_a^0(n)=1} a^{-(n+1)}S(n, 0)-\sum_{n:\omega_a^0(n)=1} a^{-(n+1)}S(n, 0)1_{[f_a^{n+1}(0),1]}(x)\ , \tag{21}$$

so, the left hand side of (20) equals

$$\begin{aligned} &\sum_{n=0}^{\infty} a^{-(n+1)}S(n, 0)+\sum_{n:\omega_a^0(n)=1} a^{-(n+1)}S(n+1, 0)1_{[f_a^{n+1}(0),1]}(x) \\ &=\sum_{n=0}^{\infty} a^{-(n+1)}S(n, 0)+\sum_{n=1}^{\infty} a^{-n}S(n, 0)1_{[f_a^n(0),1]}(x)\ , \end{aligned} \tag{22}$$

here the last equality follows from the fact that $\omega_a^0(n)=0$ implies $f_a^{n+1}(0)>2-a>x$. So, from (15), we get

$$\begin{aligned} a^{-1}h_a\left(-\frac{1}{a}+1\right)&=1+\sum_{n=1}^{\infty} a^{-n}S(n, 0)1_{[f_a^n(0),1]}(x) \\ &=h_a(x)\ . \end{aligned} \tag{23}$$

In the case $x\in[2-a, 1]$, we have analogously

$$a^{-1}h_a\left(-\frac{1}{a}x+1\right)=1+\sum_{n:\omega_a^0(n)=1} a^{-(n+1)}S(n+1, 0)1_{[f_a^{n+1}(0),1]}(x) , \tag{24}$$

and also we have

$$a^{-1}h_a\left(\frac{1}{a}x-\frac{2-a}{a}\right)=\sum_{n=0}^{\infty} a^{-(n+1)}S(n, 0)1_{[f_a^n(0),1]}\left(\frac{1}{a}x-\frac{2-a}{a}\right) . \tag{25}$$

Note that if $\omega_a^0(n)=1$ then $f_a^n(0)>2-a>(1/a)x-(2-a)/a$. Then we get

$$\begin{aligned} a^{-1}h_a\left(\frac{1}{a}x-\frac{2-a}{a}\right) &= \sum_{n:\omega_a^0(x)=0} a^{-(n+1)}S(n, 0)1_{[f_a^n(0),1]}\left(\frac{1}{a}x-\frac{2-a}{a}\right) \\ &= \sum_{n:\omega_a^0(n)=0} a^{-(n+1)}S(n+1, 0)1_{[f_a^{n+1}(0),1]}(x) , \end{aligned}$$

and so we have

$$\begin{aligned} & a^{-1}h_a\left(\frac{1}{a}x-\frac{2-a}{a}\right)+a^{-1}h_a\left(-\frac{1}{a}x+1\right) \\ & =\sum_{n=0}^{\infty} a^{-n}S(n, 0)1_{[f_a^n(0),1]}(x)=h_a(x) . \end{aligned} \tag{26}$$

From (15) and the formula $\sum_{n=0}^{\infty} a^{-n}=a/(a-1)$, it follows that

$$\sum_{n:S(n,0)=-1} a^{-n}=\frac{2a-a^2}{2(a-1)} , \tag{27}$$

so, if $a>\sqrt{2}$, we get

$$h_a(x)\geqq 1-\sum_{n:S(n,0)=-1} a^{-n}>0 \text{ for every } x\in[0, 1] . \tag{28}$$

The relation $h_a(x) \geqq 0$ in the case of $1<a\leqq\sqrt{2}$ will be given in Theorem 2.3.

Let α_k be the maximal root of the equation $t^{2k+1}-2t^{2k-1}-1=0$. Then it is easy to show that $2>\alpha_k>\alpha_{k+1}$ and $\lim_{k\to\infty}\alpha_k=\sqrt{2}$.

THEOREM 2.2. (1) *f_a has a periodic point with period $2k+1$ if and only if $a\geqq\alpha_k$.*

(2) *f_a has a periodic point with period $2^m(2k+1)$ if and only if $a\geqq\alpha_k^{1/2^m}$.*

PROOF. (1) If $a=\alpha_k$, then 0 is a periodic point with period $2k+1$, that is, $\omega_a^0=\dot{0}11\cdots\dot{1}$. If $a>\alpha_k$, then $\omega_a^0<\dot{0}11\cdots\dot{1}$, and so there exists some $x\in[0, 1]$ such that $\omega_a^x=\dot{0}11\cdots\dot{1}$, if we use the assertion of Theorem 3.1. On the other hand, if $a<\alpha_k$ then the periodic orbit which lies in the interval $[(a^2-2)/a(1+a), 1]$ has only even periods, so, if f_a has a periodic point with period $2k+1$, it lies in $(0, (a^2-2)/a(1+a))$. However it contradicts to the fact that any $x\in(0, (a^2-2)/a(1+a))$ satisfies $f_a^{2k+1}x>a^{2k+1}x$. The second part of the theorem will follow from Theorem 2.3.

Let $2^{1/2^{m+1}}<a\leqq 2^{1/2^m}$ and define the intervals A_k^m for $0\leqq k\leqq 2^m-1$ by

$$A_k^m=\begin{cases}[f_a^{2^m+k}(1), f_a^k(1)] & \text{if} \quad S(k, 1)=1\\ [f_a^k(1), f_a^{2^m+k}(1)] & \text{if} \quad S(k, 1)=-1 .\end{cases} \tag{29}$$

THEOREM 2.3. *Let* $2^{1/2^{m+1}}<a\leqq 2^{1/2^m}$. *Then we have*

(i) f_a *maps* A_k^m *onto* A_{k+1}^m *homeomorphically for* $0\leqq k\leqq 2^m-2$, *and* $A_{2^m-1}^m$ *onto* A_0^m.

(ii) $f_a^{2^m}|A_k^m$ *is isomorphic to* $f_{a^{2^m}}$ *for every* $0\leqq k\leqq 2^m-1$, *where* $f_a^{2^m}|A_k^m$ *is the restriction of* $f_a^{2^m}$ *to* A_k^m.

(iii) $\bigcup_{k=0}^{2^m-1}A_k^m$ *is the support of the density function* $h_a(x)$.

To prove this theorem, we prepare several lemmas.

LEMMA 2.1. *If* $a\leqq\sqrt{2}$, *then* $0<1-(1/a)<f_a^3(1)<a/(a+1)<f_a^2(1)<1$. *If we write* $A_0=[f_a^2(1), 1]$ *and* $A_1=[0, f_a^3(1)]$, *then it follows that* $f_aA_0=A_1$ *and* $f_aA_1=A_0$.

LEMMA 2.2. $(A_1, f_a^2|A_1)$ *is isomorphic to* $([0, 1], f_{a^2})$.

PROOF. Define a map φ of A_1 onto $[0, 1]$ by

$$\varphi(x)=\frac{f_a^3(1)-x}{f_a^3(1)}\ , \tag{30}$$

then it is easy to check that $\varphi\circ f_a^2\circ\varphi^{-1}=f_{a^2}$.

LEMMA 2.3. $S_a(2n, x)=S_{a^2}(n, \varphi(x))$ *for every* $x\in A_1$.

PROOF. It follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned}\omega_a^x(2k)&=1-\omega_{a^2}^{\varphi(x)}(k)\\ \omega_a^x(2k+1)&=1\ ,\end{aligned} \tag{31}$$

so we can calculate $S_a(2n, x)$ by

$$S_a(2n, x)=(-1)^{\sum_{k=0}^{2n-1}\omega_a^x(k)}$$

$$=(-1)^n(-1)^{\sum_{k=0}^{n-1}\omega_a^x(2k)}$$

$$=(-1)^{2n}(-1)^{\sum_{k=0}^{n-1}\omega_{a^2}^{\varphi(x)}(k)}$$

$$=S_{a^2}(n,\ \varphi(x))\ .$$

LEMMA 2.4. *If* $a<\sqrt{2}$, *then*

$$h_a(x)=\begin{cases} a^{-2}h_{a^2}(\varphi(x)) & \text{if } x\in A_1 \\ 0 & \text{if } f_a^3(1)<x<f_a^2(1) \\ a^{-1}h_{a^2}(\psi(x)) & \text{if } x\in A_0\ , \end{cases} \tag{32}$$

where φ *is the one given by* (30) *and* ψ *is a map of* A_0 *onto* [0, 1] *defined by*

$$\psi(x)=\frac{x-f_a^2(1)}{1-f_a^2(1)}\ . \tag{33}$$

PROOF. It follows from Lemma 2.1 that $f_a^{2n}(0)\in A_1$ and $f_a^{2n+1}(0)\in A_0$. So, for $f_a^3(1)<x<f_a^2(1)$, we get

$$\begin{aligned} h_a(x)&=\sum_{n=0}^{\infty} a^{-2n}S_a(2n,\ 0) \\ &=\sum_{n=0}^{\infty}(a^2)^{-n}{}_2S_{a^2}(n,\ 1) \\ &=0\ . \end{aligned} \tag{34}$$

And for $x\in A_1$, we get

$$\begin{aligned} h_a(x)&=\sum_{n=0}^{\infty} a^{-2n}S_a(2n,\ 0)1_{[f_a^{2n}(0),1]}(x) \\ &=\sum_{n=0}^{\infty} a^{-2n}S_a(2n,\ 0)-\sum_{n=1}^{\infty} a^{-2n}S_a(2n,\ 0)1_{[o,f_a^{2n}(0)]}(x)\ . \end{aligned} \tag{35}$$

The first term is equal to 0 as in (34). From Lemma 2.2 it follows

$$\varphi\circ f_a^{2n}(0)=f_{a^2}^{n}\circ\varphi(0)=f_{a^2}^{n-1}(0)\ , \tag{36}$$

so, if we notice that φ is monotone decreasing, then the second term of (35) is equal to

$$\sum_{n=0}^{\infty} a^{-2n}S_{a^2}(n-1,\ 0)1_{[f_{a^2}^{n-1}(0),1]}(\varphi(x)) \tag{37}$$

$$=a^{-2}\sum_{n=0}^{\infty}(a^2)^{-n}S_{a^2}(n, 0)1_{[f^n_{a^2}(0),1]}(\varphi(x))$$

$$=a^{-2}h_{a^2}(\varphi(x)) .$$

For $x \in A_0$, from (18) it follows

$$h_a(x)=ah_a(f_a(x)) =a^{-1}h_{a^2}(\varphi\circ f_a(x)) =a^{-1}h_{a^2}(\psi(x)) . \tag{38}$$

PROOF OF THEOREM 2.3. To prove Theorem 2.3, it is sufficient to apply Lemma 2.4 repeatedly.

§3. Representation of f_a to subshift.

In this section we consider the representation of f_a by a symbolic dynamical system more precisely (cf. [2]). Let us define an ordered relation in the space Ω. For $\omega, \omega' \in \Omega$, we write $\omega<\omega'$ if there exists a natural number n such that

$$\omega(k)=\omega'(k) \quad \text{for} \quad 0\leqq k\leqq n-1 \tag{39}$$

and

$$\begin{aligned} &\omega(n)<\omega'(n) \quad &\text{if} \quad S(n, \omega)=1 \\ &\omega(n)>\omega'(n) \quad &\text{if} \quad S(n, \omega)=-1 . \end{aligned} \tag{40}$$

Define a function ρ on Ω by

$$\rho(\omega)=1-\sum_{n=0}^{\infty}S(n, \omega)a^{-(n+1)} . \tag{41}$$

It is easy to see that ρ is continuous, and from (15) it follows $\rho\circ\pi_a(x)=x$. By using the relation

$$S(n, \omega)S(k, \sigma^n\omega)=S(n+k, \omega) , \tag{42}$$

we get

$$S(n, \omega)\rho(\sigma^n\omega)=S(n, \omega)-\sum_{k=0}^{\infty}a^{-(k+1)}S(n+k, \omega) . \tag{43}$$

Denote by Y_a the set $\pi_a[0, 1]$ and let X_a be the closure of Y_a. Then it is clear that ρ maps Y_a onto [0, 1], and X_a onto [0, 1].

LEMMA 3.1. (i) *For any* $x, x' \in [0, 1]$, $x<x'$ *implies* $\pi_a(x)<\pi_a(x')$.

(ii) *For any* $\omega, \omega' \in Y_a$, $\omega < \omega'$ *implies* $\rho(\omega) < \rho(\omega')$.
(iii) *For any* $\omega, \omega' \in X_a$, $\omega < \omega'$ *implies* $\rho(\omega) \leqq \rho(\omega')$.

PROOF. These follow easily from the definition of the ordered relation and the continuity of ρ.

LEMMA 3.2. *For any* $1 < a \leqq 2$,

$$\lim_{x \downarrow 0} \pi_a(x) = \omega_a^0 \quad \textit{and} \quad \lim_{x \uparrow 1} \pi_a(x) = \omega_a^1 , \tag{44}$$

and so, for any $\omega \in X_a$, *it follows that*

$$\begin{aligned} &\rho(\omega) = 0 \quad \textit{if and only if} \quad \omega = \omega_a^0 \\ &\rho(\omega) = 1 \quad \textit{if and only if} \quad \omega = \omega_a^1 . \end{aligned} \tag{45}$$

PROOF. The existence of limits follows from Lemma 3.1. In order to prove $\lim_{x \downarrow 0} \pi_a(x) = \omega_a^0$, we consider the following several cases:

Case (i): The case of non-Markov type.
In this case, we have

$$f^n(0) \neq 1 - \frac{1}{a} \quad \text{for every} \quad n . \tag{46}$$

So, for any natural number N, ther exists a $\delta > 0$ such that for every $x \in (0, \delta)$ and every $n \leqq N$,

$$|f_a^n(0) - f_a^n(x)| < \left| f_a^n(0) - \left(1 - \frac{1}{a}\right) \right| , \tag{47}$$

which means that $\omega_a^x(n) = \omega_a^0(n)$. So we have $\lim_{x \downarrow 0} \pi_a(x) = \omega_a^0$.

Case (ii): The case of even Markov type (with period n).
In this case we have

$$f_a^k(0) \neq 1 - \frac{1}{a} \quad \text{for every} \quad k \leqq n-3 , \tag{48}$$

$$f_a^{n-2}(0) = 1 - \frac{1}{a} \quad \text{and} \quad S(n-2, 0) = 1 . \tag{49}$$

From (48), there exists a $\delta > 0$ such that for every $x \in (0, \delta)$ and every $k \leqq n-3$,

$$|f_a^k(0) - f_a^k(x)| < \left| f_a^k(0) - \left(1 - \frac{1}{a}\right) \right| , \tag{50}$$

which means $\omega_a^0(k) = \omega_a^x(k)$. But, from (49) it follows

(51) $$f_a^{n-2}(x)>1-\frac{1}{a}\ ,$$

which means $\omega_a^0(n-2)=\omega_a^x(n-2)$, by the definition of the generator $\{I_0, I_1\}$. In the same manner, we can show that, for any natural number m, there exists a $\delta'>0$ such that $\omega_a^x(k)=\omega_a^0(k)$ for every $x\in(0,\delta')$ and every $k\leqq mn-2$, so we have $\lim_{x\downarrow 0}\pi_a(x)=\omega_a^0$.

Case (iii): The case of odd Markov type.

We can prove that $\lim_{x\downarrow 0}\pi_a(x)=\omega_a^0$ in the same manner as in the Case (ii).

If we notice that $\sigma\omega_a^1=\omega_a^0$, we also get $\lim_{x\uparrow 1}\pi_a(x)=\omega_a^1$.

LEMMA 3.3. (i) *If $\omega_a^{x-}\equiv\sup_{y<x}\pi_a(y)\neq\omega_a^x$, then there exists a natural number n such that*

(52) $$\omega_a^{x-}(k)=\omega_a^x(k)\qquad \textit{for}\quad 0\leqq k\leqq n-1\ ,$$

(53) $$\omega_a^{x-}(n)\neq\omega_a^x(n)$$

and

(54) $$\sigma^{n+1}\omega_a^{x-}=\sigma^{n+1}\omega_a^x=\omega_a^1\ .$$

(ii) *The similar assertion holds for $\omega_a^{x+}\equiv\inf_{y>x}\pi_a(y)$.*

PROOF. We only prove the assertion (i), since the assertion (ii) can be shown in the same manner. To simplify the notations, we write ω^- for ω_a^{x-} and ω for ω_a^x. Suppose (52) and (53) are satisfied; then from (15) we get

(55) $$\begin{aligned}x&=\rho(\omega^-)\\&=1-\sum_{k=0}^{n}a^{-(k+1)}S(k,\omega^-)-a^{-(n+2)}S(n+1,\omega^-)-\sum_{k=n+2}^{\infty}a^{-(k+1)}S(k,\omega^-)\ ,\end{aligned}$$

(56) $$\begin{aligned}x&=\rho(\omega)\\&=1-\sum_{k=0}^{n}a^{-(k+1)}S(k,\omega)-a^{-(n+2)}S(n+1,\omega)-\sum_{k=n+2}^{\infty}a^{-(k+1)}S(k,\omega)\ .\end{aligned}$$

From (43) and (52) it follows that

(57) $$\begin{aligned}&S(n+1,\omega^-)-S(n+1,\omega)\\&\quad=-S(n+2,\omega^-)(1-\rho(\sigma^{n+2}\omega^-))+S(n+2,\omega)(1-\rho(\sigma^{n+2}\omega))\ .\end{aligned}$$

From (53), the left hand side of (57) is equal to 2 or -2, so we get

(58) $$\omega^{-}(n+1)=\omega(n+1)=1 \quad\text{and}\quad \rho(\sigma^{n+2}\omega^{-})=\rho(\sigma^{n+2}\omega)=0\ ,$$

which implies (54), by virtue of Lemma 3.2.

PROPOSITION 3.1. (i) *As a map from* X_a *to* $[0, 1]$, ρ *is one-to-one except for a countably many points of* X_a.

(ii) *For any* $x\in[0, 1]$, $\rho^{-1}(x)$ *contains at most two points.*

PROOF. If we notice that

(59) $$\rho^{-1}(x)=\{\omega\in X_a;\ \omega_a^{x-}\leqq\omega\leqq\omega_a^{x+}\}\ ,$$

then it is easy to prove the assertions, by using Lemma 3.3.

Let W^n be the space $\{0, 1\}^{N_n}$, when $N_n=\{0, 1, 2, \cdots, n-1\}$, and consider the ordered relation on W^n by restricting the order $<$ on Ω. The shift operator σ can be considered as a mapping from W^k into W^{k-1} for each k. For $\omega\in\Omega$, denote by $\omega[0, n)$ the natural projection of ω on W^n. Namely,

(60) $$\omega[0, n)=(\omega(0), \omega(1), \cdots, \omega(n-1))\ .$$

LEMMA 3.4. *Let* Y_a^n *be the subset of* W^n *defined by*

(61) $$Y_a^n=\{u\in W^n;\ u=\omega_a^x[0, n)\ \textit{for some}\ x\in[0, 1]\}\ ,$$

then we have

$(62)_n$ $$Y_a^n=\{u\in W^n;\ \sigma^k u\geqq\omega_a^0[0, n-k)\ \textit{for every}\ 0\leqq k<n\}\ .$$

PROOF. Let us prove the assertion by induction on n. In the case of $n=1$, we have $Y_a^1=\{0, 1\}$ and $(62)_1$ is evident. Assume that $(62)_n$ holds. Let $u\in W^{n+1}$ satisfy

(63) $$\sigma^k u\geqq\omega_a^0[0, n+1-k)\quad\text{for every}\quad 0\leqq k<n+1\ ;$$

then $\sigma u\in Y_a^n$. If $u(0)=1$ and $\sigma u\neq\omega_a^1[0, n)$, then $\sigma u=\omega_a^x[0, n)$ for some $x\in[0, 1)$. It then follows that $u=\omega_a^{x'}[0, n+1)$ for $x'=-(1/a)x+1$. If $u(0)=1$ and $\sigma u=\omega_a^1[0, n)$, then, by using Lemma 3.2, we get $\sigma u=\omega_a^x[0, n)$ for some $x\in[0, 1)$, and thus we reach the same conclusion. If $u(0)=0$, then $\sigma u\geqq\sigma\omega_a^0[0, n)$, which means that $\sigma u=\omega_a^x[0, n)$ for some $x\in[2-a, 1)$. It follows that $u=\omega_a^{x'}[0, n+1)$ for $x'=(1/a)x-(2-a)/a$. Thus we get

(64) $$Y_a^{n+1}\supset\{u\in W^{n+1};\ \sigma^k u\geqq\omega_a^0[0, n+1-k)\quad\text{for every}\quad 0\leqq k<n+1\}\ .$$

The converse inclusion is evident, and we have $(62)_{n+1}$.

THEOREM 3.1 (cf. [2]). *One can characterize* X_a *as follows:*

$$X_a=\{\omega \in \Omega;\ \sigma^k\omega \geqq \omega_a^0 \quad \textit{for every} \quad k\geqq 0\}\ . \tag{65}$$

PROOF. It is easy to see that every $\omega \in X_a$ satisfies the condition in the right-hand side of (65). And the converse inclusion follows from Lemma 3.4.

In the remainder of this section, we derive a decomposition of Y_a^n for the sake of the next section. Let W_0^n and W_1^n be defined by

$$\begin{aligned} W_0^n&=\{u \in Y_a^n;\ \bar{u} \in Y_a^n \quad \text{and} \quad \bar{u}>u\}\ ,\\ W_1^n&=\{u \in Y_a^n;\ \bar{u} \in Y_a^n \quad \text{and} \quad \bar{u}<u\}\ , \end{aligned} \tag{66}$$

where we denote by $\bar{u}$ the element of W^n obtained from u by changing the last coordinate to 1(resp. 0) if it is 0(resp. 1). We denote the connection of the words u and v by $u\cdot v$. Namely, $u\cdot v=(u_1, \cdots, u_n, v_1, \cdots, v_m)$ where $u=(u_1, \cdots, u_n)$ and $v=(v_1, \cdots, v_m)$ (the number m may be infinite).

LEMMA 3.5. *If* $u \in W_0^n \cup W_1^n$, *then* $u\cdot\omega_a^1 \in X_a$.

PROOF. Let us prove by induction on n. It is evident for $n=1$. Assume that the lemma holds for n. Let $u \in W_0^{n+1} \cup W_1^{n+1}$. Then $\sigma u \in W_0^n \cup W_1^n$ and so $\sigma u\cdot\omega_a^1 \in X_a$. From Theorem 3.1, it is enough to prove that $u\cdot\omega_a^1 \geqq \omega_a^0$. In the case when $u>\omega_a^0[0, n+1)$, it is clear. If $u=\omega_a^0[0, n+1)$, we have $S(n, u\cdot\omega_a^1)=1$ by the assumption $u \in W_0^{n+1} \cup W_1^{n+1}$, and so $\sigma^n\omega_a^0 \leqq \omega_a^1$ implies $u\cdot\omega_a^1 \geqq \omega_a^0$.

PROPOSITION 3.2. *We can decompose* Y_a^n *as follows*:

$$\begin{aligned} Y_a^n&=\bigcup_{k=0}^{n} W_0^{n-k}\cdot\omega_a^1[0, k)\\ &=\bigcup_{k=0}^{n-1} W_1^{n-k}\cdot\omega_a^1[0, k) \cup \{\omega_a^0[0, n)\}\ , \end{aligned} \tag{$67)_n$}$$

where we denote

$$W_i^{n-k}\cdot\omega_a^1[0, k)=\{u\cdot\omega_a^1[0, k);\ u \in W_i^{n-k}\}\ . \tag{68}$$

PROOF. We prove only the first decomposition by induction on n. Noting that $W_0^1=\{0\}$ and $W_1^1=\{1\}$, it is evident for $n=1$. Assume that $(67)_n$ holds. Let $u \in Y_a^{n+1}$ satisfy $u \notin W_0^{n+1}$ By the induction hypothesis, we get, for some k and some $u' \in W_0^{n-k}$,

$$u[0, n)=u'\cdot\omega_a^1[0, k)\ , \tag{69}$$

then we can show in the following way that $u=u'\cdot\omega_a^1[0, k+1)$ holds: By using Lemma 3.5, we get $u'\cdot\omega_a^1[0, k+1) \in Y_a^{n+1}$. So, if $u\neq u'\cdot\omega_a^1[0, k+1)$, then we have $u<u'\cdot\omega_a^1[0, k+1)$, since an element of W_0^n has even number

of 1's among its coodinates. But this contradicts the assumption $u \notin W_0^{n+1}$. One can show the other decomposition in the same manner.

§4. Topological entropy.

The measure theoretic entropy of f_a with respect to the invariant measure given in §2 is clearly $\log a$. In this section we show that the topological entropy of f_a is also $\log a$ (cf. [3] and [7]).

If we denote by N_a^n the number of elements in Y_a^n, then the topological entropy $h^*(f_a)$ can be given by

(70) $$h^*(f_a)=\lim_{n\to\infty}\frac{1}{n}\log N_a^n .$$

We denote by $[u]$ for $u\in W^n$ the sub-interval of $[0, 1]$ determined by u. Namely,

(71) $$[u]=\{x\in[0, 1];\ \omega_a^x[0, n)=u\} .$$

If we denote by m the Lebesgue measure defined on $[0, 1]$, then we can show easily that $m([u])\leqq a^{-n}$. It follows that $N_a^n\geqq a^n$, which implies $h^*(f_a)\geqq\log a$. In the remainder of this section, we show the opposite inequality $h^*(f_a)\leqq\log a$.

LEMMA 4.1. (i) *If* $u\in Y_a^n$ *is represented as*

(72) $$u=u_0\cdot\omega_a^1[0, k)\quad \textit{for some}\quad 0\leqq k\leqq n\quad \textit{and some}\quad u_0\in W_0^{n-k}$$

and

(73) $$u=u_1\cdot\omega_a^1[0, j]\quad \textit{for some}\quad 0\leqq j\leqq n-1\quad \textit{and some}\quad u_1\in W_1^{n-j} ,$$

then we get

(74) $$m([u])=a^{-n}\{S(k, 1)f_a^k(1)+S(j, 1)f_a^j(1)\} .$$

(ii) *If* $u\in Y_a^n$ *is represented as* (72) *and, instead of* (73),

(75) $$u=\omega_a^0[0, n) ,$$

then we get

(76) $$m([u])=a^{-n}\{S(k, 1)f_a^k(1)+S(n+1, 1)f_a^{n+1}(1)\} .$$

PROOF. Let us show that, from (72) and (73) it follows that

(77) $$\overline{\pi([u])}=\{\omega\in X_a;\ u_1\cdot\omega_a^1\leqq\omega\leqq u_0\cdot\omega_a^1\} .$$

If $\omega \in X_a$ satisfies $u_1 \cdot \omega_a^1 \leqq \omega \leqq u_0 \cdot \omega_a^1$, then $\omega[0, n) = u$, from which it follows that $\omega \in \overline{\pi([u])}$. On the other hand, if $\omega \in \overline{\pi([u])}$, then $\omega = u \cdot \omega'$ for some $\omega' \in X_a$. By using Theorem 3.1, it follows that

$$\sigma^{n-k}\omega \leqq \omega_a^1 \quad \text{and} \quad \sigma^{n-j}\omega \leqq \omega_a^1 . \tag{78}$$

Noting that $u_0(u_1)$ has even (odd, respectively) number of 1's among its coordinates, we obtain that

$$u_1 \cdot \omega_a^1 \leqq \omega \leqq u_0 \cdot \omega_a^1 , \tag{79}$$

which completes the proof of (77).

From (77) we have

$$\begin{aligned} m([u]) &= \rho(u_0 \cdot \omega_a^1) - \rho(u_1 \cdot \omega_a^1) , \\ &= \rho(u \cdot \sigma^k \omega_a^1) - \rho(u \cdot \sigma^j \omega_a^1) . \end{aligned} \tag{80}$$

And by the definition of ρ and (43), we have

$$\rho(u \cdot \sigma^k \omega_a^1) = 1 - \sum_{i=0}^{n-1} a^{-i} S(i, u \cdot \sigma^k \omega_a^1) + a^{-n} S(n, u \cdot \sigma^k \omega_a^1)(\rho(\sigma^k \omega_a^1) - 1) \tag{81}$$

and

$$\rho(u \cdot \sigma^j \omega_a^1) = 1 - \sum_{i=0}^{n-1} a^{-i} S(i, u \cdot \sigma^j \omega_a^1) + a^{-n} S(n, u \cdot \sigma^j \omega_a^1)(\rho(\sigma^j \omega_a^1) - 1) . \tag{82}$$

From (72) and (73) it follows that

$$S(i, u \cdot \sigma^k \omega_a^1) = S(i, u \cdot \sigma^j \omega_a^1) \quad \text{for} \quad 0 \leqq i \leqq n , \tag{83}$$

$$S(n, u \cdot \sigma^k \omega_a^1) = S(n-k, u \cdot \sigma^k \omega_a^1) S(k, \omega_a^1) = S(k, \omega_a^1) \tag{84}$$

and

$$S(n, u \cdot \sigma^j \omega_a^1) = S(n-j, u \cdot \sigma^j \omega_a^1) S(j, \omega_a^1) = -S(j, \omega_a^1) . \tag{85}$$

So, from (80) we get

$$m([u]) = a^{-n}(S(k, \omega_a^1)\rho(\sigma^k \omega_a^1) + S(j, \omega_a^1)\rho(\sigma^j \omega_a^1)) , \tag{86}$$

which completes the proof of (74). A similar argument shows (76).

Denote by $N_{a,0}^n(N_{a,1}^n)$ the number of elements in $W_0^n(W_1^n$, respectively). We can easily show that

$$N_{a\,0}^n = N_{a,1}^n \tag{87}$$

and

$$N_a^n = \sum_{k=0}^{n} N_{a,0}^k . \tag{88}$$

LEMMA 4.2. *If $a > \sqrt{2}$, then, for some constant C, we get*

$$N_{a,0}^n \leqq Ca^n \quad \textit{for every} \quad n \geqq 0 . \tag{89}$$

PROOF. From the relation $\sum_{u \in Y_a^n} m([u]) = 1$, we obtain

$$1 = a^{-n}\left(2\sum_{k=0}^{n-1} S(k, 1) f_a^k(1) N_{a,0}^{n-k} + S(n, 1) f_a^n(1) + S(n+1, 1) f_a^{n+1}(1)\right) , \tag{90}$$

using (74), (76) and (87). In order to show (89), we consider the following two cases:

Case (i): $(\sqrt{5}+1)/2 < a \leqq 2$.
In this case we have $S(0, 1) = 1$ and $f_a(1) = 0$, so from (90) we have

$$1 \geqq a^{-n}\left(2N_{a,0}^n - 2\sum_{k=2}^{n} N_{a,0}^{n-k}\right) , \tag{91}$$

from which it follows that

$$N_{a,0}^n \leqq \frac{1}{2} a^n + \sum_{k=0}^{n-2} N_{a,0}^k . \tag{92}$$

So, if we pick a constant C satisfying

$$C \geqq \frac{1}{2\left(1 - \dfrac{1}{a(a-1)}\right)} , \tag{93}$$

we can show inductively that $N_{a,0}^n \leqq Ca^n$ holds for each n. Here we use the inequality $1/a(a-1) < 1$, which follows from $a > (\sqrt{5}+1)/2$. More precisely, if $N_{a,0}^k \leqq Ca^k$ for every $k < n$, then it follows that

$$\begin{aligned} N_{a,0}^n &\leqq \frac{1}{2} a^n + C\frac{a^{n-1}}{a-1} \\ &= Ca^n\left(\frac{1}{2C} + \frac{1}{a(a-1)}\right) \leqq Ca^n . \end{aligned} \tag{94}$$

Case (ii): $\sqrt{2} < a \leqq (\sqrt{5}+1)/2$.
In this case we have

$$S(0, 1) = S(3, 1) = 1 , \quad S(1, 1) = S(2, 1) = S(4, 1) = -1 , \tag{95}$$

$$f_a^4(1) < f_a^3(1) < \frac{a}{a+1} < f_a^3(1) .$$

As in the Case (i), it follows from (90) that

$$N_{a,0}^n \leqq \frac{1}{2}a^n + \frac{a}{a+1}(N_{a,0}^{n-2} - N_{a,0}^{n-3} + N_{a,0}^{n-4}) + \sum_{k=0}^{n-5} N_{a,0}^k \\ \leqq \frac{1}{2}a^n + \frac{a}{a+1}(N_{a,0}^{n-3} + N_{a,0}^{n-4}) + \sum_{k=0}^{n-5} N_{a,0}^k , \tag{96}$$

by virtue of relation $N_{a,0}^{n-2} \leqq 2N_{a,0}^{n-3}$.

So, if we pick a constant C satisfying

$$C > \frac{1}{2\left(1 - \frac{a^2 - a + 1}{a^4(a-1)}\right)} , \tag{97}$$

we can show $N_{a,0}^n \leqq Ca^n$ inductively.

From this lemma and (88) we get

$$h^*(f_a) \leqq \log a \quad \text{for every} \quad \sqrt{2} < a \leqq 2 . \tag{98}$$

But by using Theorem 2.3, the same assertion follows also for $1 < a \leqq \sqrt{2}$. Consequently we have

THEOREM 4.1. *$h^*(f_a) = \log a$ for $1 < a \leqq 2$.*

From this theorem and Theorem 3.1, we have

COROLLARY 4.1. *If $a < a'$, then $\omega_a^0 > \omega_{a'}^0$.*

References

[1] R. BOWEN, Bernoulli maps of the intervals, Israel J. Math., **28** (1977), 161-168.

[2] SH. ITO and Y. TAKAHASHI, Markov subshifts and realization of β-expansions, J. Math. Soc. Japan, **26** (1974), 33-55.

[3] L. JONKER and D. RAND, A lower bound for the entropy of the unit interval, preprint.

[4] A. LASOTA and J. A. YORKE, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc., **186** (1973), 481-488.

[5] T. Y. LI and J. A. YORKE, Period three implies chaos, Amer, Math. Monthly, **82** (1975), 985-992.

[6] T. Y. LI and J. A. YORKE, Ergodic transformation from an interval into itself, Trans. Amer. Math. Soc., **235** (1978), 183-192.

[7] J. MILNOR and W. THURSTON, On iterated maps of the interval I, preprint.

[8] P. STEFAN, A theorem of Sarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line, Comm. Math. Phys., **54** (1977), 237-248.

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