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A Numerical Investigation on Cumulative Sum of the Liouville Function

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Let $\lambda(n)$ be the Liouville function defined as $\lambda(n)=(-1)^{\nu}$, where ν is the number of prime factors of a positive integer n, multiple factors being counted according to their multiplicity. Thus $\lambda(1)=1$, $\lambda(2)=-1$, $\lambda(3)=-1$, $\lambda(4)=1$, $\lambda(5)=-1$, $\lambda(6)=1$, $\lambda(7)=-1$, $\lambda(8)=-1$, $\lambda(9)=1$, $\lambda(10)=1, \cdots$.

We put

$$L(x)=\sum_{n=1}^{x} \lambda(n) .$$

In this paper we assume x to be a positive integer. Thus $L(1)=1$, $L(2)=0$, $L(3)=-1$, $L(4)=0$, $L(5)=-1$, $L(6)=0$, $L(7)=-1$, $L(8)=-2$, $L(9)=-1$, $L(10)=0, \cdots$.

The object of this note is to report some numerical results obtained by the author on $L(x)$, $x\leqq 10^9$, especially on how $L(x)$ changes its sign as x increases from 1 to 10^9.

For convenience we divide the integers $1-10^9$ into subregions each consisting of 10000 consecutive integers.

1−10000. In this region, $L(x)=0$ only for $x=2$, 4, 6, 10, 16, 26, 40, 96, 586; $L(x)>0$ only for $x=1$.

10001−906150000. Always $L(x)<0$.

906150001−906160000. $L(x)=0$ for 54 values of x, the first of which is 906150256; $L(x)>0$ for 1529 values of x, the first of which is 906150257.

906160001−906180000. Always $L(x)<0$

906180001−906190000. $L(x)=0$ for 16 values of x; $L(x)>0$ for 9612 values of x.

906190001−906200000. $L(x)=0$ for 75 values of x; $L(x)>0$ for 7784 values of x.

906200001−906210000. $L(x)=0$ for 22 values of x; $L(x)>0$ for 9643 values of x.

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906210001–906470000. Always $L(x)>0$.

906470001–906480000. $L(x)=0$ for 15 values of x; $L(x)>0$ for 9882 values of x.

906480001–906490000. $L(x)=0$ for 61 values of x; the last of which is 906488080; $L(x)>0$ for 6976 values of x; the last of which is 906488079.

Thus $L(x)<0$ from $x=587$ to $x=906150255$, and from $x=906488081$ to $x=10^9$.

$L(x)$ takes sometimes negative values, and sometimes equals zero from $x=2$ to $x=586$. $L(x)$ takes sometimes positive, sometimes negative values, and sometimes equals zero from $x=906150256$ to $x=906488080$.

This time we divide the integers $1-10^9$ into subregions each consisting of 10^8 consecutive integers. We give a short list of minimum and maximum values of $L(x)$ for each of the subregions.

min$=-10443$	max$=1$	$(x\leqq 10^8)$	$L(10^8)=-3884$
min$=-17847$	max$=-1497$	$(10^8<x\leqq 2\cdot 10^8)$	$L(2\cdot 10^8)=-11126$
min$=-19647$	max$=-2262$	$(2\cdot 10^8<x\leqq 3\cdot 10^8)$	$L(3\cdot 10^8)=-16648$
min$=-19496$	max$=-2016$	$(3\cdot 10^8<x\leqq 4\cdot 10^8)$	$L(4\cdot 10^8)=-11200$
min$=-28531$	max$=-9240$	$(4\cdot 10^8<x\leqq 5\cdot 10^8)$	$L(5\cdot 10^8)=-18804$
min$=-19836$	max$=-2842$	$(5\cdot 10^8<x\leqq 6\cdot 10^8)$	$L(6\cdot 10^8)=-15350$
min$=-28731$	max$=-12250$	$(6\cdot 10^8<x\leqq 7\cdot 10^8)$	$L(7\cdot 10^8)=-25384$
min$=-29736$	max$=-12701$	$(7\cdot 10^8<x\leqq 8\cdot 10^8)$	$L(8\cdot 10^8)=-19292$
min$=-20870$	max$=-3158$	$(8\cdot 10^8<x\leqq 9\cdot 10^8)$	$L(9\cdot 10^8)=-4630$
min$=-27756$	max$=829$	$(9\cdot 10^8<x\leqq 10^9)$	$L(10^9)=-25216$

Comments on literature. Lehman [2] found that $L(906180359)=1$, but he could not ascertain whether $x=906180359$ is the integral value of x next to 1 for which $L(x)>0$ or not.

Comments on theoretical results. Tanaka [3] contains a proof that $L(x)-B\sqrt{x}$ changes its sign infinitely often as x tends to infinity, where $B=1/\zeta(1/2)=-0.68477$. But it is an open problem until the present whether $L(x)$ itself changes its sign infinitely often or not. In connection with this problem, see Ingham [1].

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References

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