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Eigenvalues of the Laplacian of Warped Product

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Introduction

Let (M, g) be a compact connected Riemannian manifold with metric tensor g and $\triangle$ be the Laplacian acting on differentiable functions on M, that is,

$$\triangle\varphi = -\sum g^{ji}\nabla_j\nabla_i\varphi\ .$$

Let $\mathrm{Spec}\,(M, g)=\{0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\cdots\}$ be the set of eigenvalues of $\triangle$, where each eigenvalue is repeated as many times as its multiplicity.

Let (B, g) and (F, h) be Riemannian manifolds and $f>0$ a differentiable function on B. Consider the product manifold $B\times F$ with its projections $\pi: B\times F\to B$ and $\eta: B\times F\to F$. The warped product $M=B\times_f F$ is the manifold $B\times F$ furnished with the Riemannian structure $\bar{g}$ such that

$$\bar{g}(X, Y)=g(\pi_*X, \pi_*Y)+f^2(\pi m)h(\eta_*X, \eta_*Y)$$

for tangent vectors $X, Y\in T_mM$. If $f\equiv 1$, then $B\times_f F$ is nothing but the ordinary Riemannian product.

Making use of the warped product, N. Ejiri [3] constructed examples of compact, connected, non-flat and irreducible Riemannian manifolds which are isospectral but non-isometric. He proves that if $\mathrm{Spec}\,(M, h)=\mathrm{Spec}\,(M', h')$ then $\mathrm{Spec}\,(B\times_f M)=\mathrm{Spec}\,(B\times_f M')$, where f is a positive differentiable function on B.

In this paper we show the converse.

THEOREM 1. *Let (B, g), (M, h), and (M', h') be compact connected Riemannian manifolds and $f>0$ a differentiable function on B. If* $\mathrm{Spec}\,(B\times_f M)=\mathrm{Spec}\,(B\times_f M')$ *then* $\mathrm{Spec}\,(M, h)=\mathrm{Spec}\,(M', h')$.

Let (B, g) and (F, h) be compact connected Riemannian manifolds and $f>0$ a differentiable function on B. When we study the spectrum of

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the warped product $B\times_f F$, we introduce a differential operator $L_{(f;\lambda_i)}$ acting on differentiable functions on B, which is defined by $L_{(f;\lambda_i)}=\triangle^B-(n/f)\operatorname{grad} f+\lambda_i/f^2$, where $n=\dim F$ and λ_i is the i-th eigenvalue of the Laplacian of (F, h) (cf. Ejiri [3] or section 1). We denote the least eigenvalue of the operator $L_{(f;\lambda_i)}$ by $\mu_1(f;\lambda_i)$. About the relation between $\mu_1(f;\lambda_i)$ and $\mu_1(1;\lambda_i)$, we have the following result.

THEOREM 2.

$$\mu_1(f;\lambda_i)\|f\|_n^2\leqq\mu_1(1;\lambda_i)\|1\|_n^2=\lambda_i\|1\|_n^2$$

and the equality holds if and only if f is constant, where

$$\|f\|_n=\left\{\int_B f^n dV_g\right\}^{1/n}.$$

It is interesting to compare the least positive eigenvalue of the warped product $B\times_f F$ with that of the ordinary Riemannian product $B\times F$. Using Theorem 2, we obtain the following result.

THEOREM 3. *Assume that the least positive eigenvalue of B is not less than that of F. Then,*

$$\lambda_1(f)\operatorname{Vol}(f)^{2/n}\leqq\lambda_1(1)\operatorname{Vol}(1)^{2/n}$$

and the equality holds only if f is constant, where $n=\dim F$ and $\lambda_1(f)$ and $\operatorname{Vol}(f)$ denote the least positive eigenvalue and the volume of $B\times_f F$, respectively.

REMARK. Theorem 3 implies that the ordinary Riemannian product is distinguished by its least positive eigenvalue in the class of warped products. In fact, if the assumption in Theorem 3 is satisfied and $\operatorname{Vol}(f)=\operatorname{Vol}(1)$, then $\lambda_1(f)\leqq\lambda_1(1)$ and the equality holds if and only if $f\equiv 1$.

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§1. Spectrum of a warped product—Review of the Ejiri's Theorem.

We review some results about the spectrum of a warped product which N. Ejiri showed in [3].

Let (B, g) (resp. (F, h)) be an m(resp. n)-dimensional compact connected Riemannian manifold and f be a positive differentiable function on B. By $C^\infty(B)$ we denote the space of differentiable functions on

B with the scalar product $\langle\varphi, \psi\rangle=\int_B f^n\varphi\psi dV_g$. For a non-negative parameter λ, we define a differential operator $L_{(f;\lambda)}$ acting on $C^\infty(B)$ by $L_{(f;\lambda)}=\Delta^B-(n/f)\,\mathrm{grad}\, f+\lambda/f^2$, where Δ^B is the Laplacian of (B, g) and $\mathrm{grad}\, f$ is the gradient of f introduced by the metric tensor g. Then $L_{(f;\lambda)}$ becomes a self-adjoint differential operator of $L^2(B)$ $(\supset C^\infty(B))$. So $L_{(f;\lambda)}$ has a discrete spectrum with finite multiplicities, which we denote by $\mathrm{Spec}\,(L_{(f;\lambda)})$.

About the spectrum of the warped product, N. Ejiri showed that $\mathrm{Spec}\,(B\times_f F)=\sum_{i=0}^\infty \mathrm{Spec}\,(L_{(f;\lambda_i)})$, where $\lambda_i\in \mathrm{Spec}\,(F, h)$.

§2. Proof of Theorem 1.

We use the same notations as in section 1. We denote the least eigenvalue of the operator $L_{(f;\lambda)}$ introduced by the warped product $B\times_f F$ by $\mu_1(f;\lambda)$.

LEMMA 1. *If* $0\leqq\lambda\leqq\lambda'$, *then* $\mu_1(f;\lambda)\leqq\mu_1(f;\lambda')$. *The equality holds if and only if* $\lambda=\lambda'$.

PROOF. We apply the minimum principle to the self-adjoint differential operator $L_{(f;\lambda)}$.

For any non-zero differentiable function $\varphi\in C^\infty(B)$,

$$\mu_1(f;\lambda)\leqq\frac{\langle L_{(f;\lambda)}\varphi, \varphi\rangle}{\langle\varphi, \varphi\rangle},$$

and the equality holds if and only if $L_{(f;\lambda)}\varphi=\mu_1(f;\lambda)\varphi$. Using the formula

$$\int_B \Delta^B\varphi\cdot\psi dV_g=\int_B (d\varphi\,|\,d\psi)dV_g,$$

we have

$$\begin{aligned}\int_B f^n\Delta^B\varphi\cdot\varphi dV_g&=\int_B (d\varphi\,|\,d(f^n\varphi))dV_g\\ &=\int_B f^n(d\varphi\,|\,d\varphi)dV_g+\int_B nf^{n-1}\varphi(df\,|\,d\varphi)dV_g\\ &=\int_B f^n(d\varphi\,|\,d\varphi)dV_g+\int_B nf^{n-1}\varphi(\mathrm{grad}\, f)\varphi dV_g,\end{aligned}$$

where $(d\varphi\,|\,d\psi)=g^{ji}\varphi_j\psi_i$.

Therefore

$$\begin{aligned}\langle L_{(f;\lambda)}\varphi, \varphi\rangle&=\int_B f^n(\Delta^B\varphi-(n/f)(\mathrm{grad}\, f)\varphi+(\lambda/f^2)\varphi)\varphi dV_g\\ &=\int_B f^n\Delta^B\varphi\cdot\varphi dV_g-\int_B nf^{n-1}\varphi(\mathrm{grad}\, f)\varphi dV_g\end{aligned}$$

$$+\lambda\int_B f^{n-2}\varphi^2 dV_g$$
$$=\int_B f^n(d\varphi\,|\,d\varphi)dV_g+\lambda\int_B f^{n-2}\varphi^2 dV_g\ .$$

Let $\varphi\in C^\infty(B)$ be an eigenfunction of the operator $L_{(f;\lambda')}$ with the least eigenvalue, that is, $L_{(f;\lambda')}\varphi=\mu_1(f;\lambda')\varphi$. Then

$$\begin{aligned}\mu_1(f;\lambda)&\leqq\langle L_{(f;\lambda)}\varphi,\varphi\rangle/\langle\varphi,\varphi\rangle\\&=\left\{\int_B f^n(d\varphi\,|\,d\varphi)dV_g+\lambda\int_B f^{n-2}\varphi^2 dV_g\right\}\Big/\langle\varphi,\varphi\rangle\\&\leqq\left\{\int_B f^n(d\varphi\,|\,d\varphi)dV_g+\lambda'\int_B f^{n-2}\varphi^2 dV_g\right\}\Big/\langle\varphi,\varphi\rangle\\&=\langle L_{(f;\lambda')}\varphi,\varphi\rangle/\langle\varphi,\varphi\rangle\\&=\mu_1(f;\lambda')\ .\end{aligned}$$

The equality implies that $\lambda=\lambda'$.

REMARK. Let λ_i be the i-th eigenvalue of the Laplacian of (F,h). The least eigenvalue of $L_{(f;\lambda_0)}$ is 0 and its eigenfunction is constant. The least eigenvalue of $L_{(f;\lambda_1)}$ is strictly positive by Lemma 1. So the least positive eigenvalue of the warped product $B\times_f F$ is the minimum of the least positive eigenvalue of $L_{(f;\lambda_0)}$ and the least eigenvalue of $L_{(f;\lambda_1)}$.

PROOF OF THEOREM 1. We denote the i-th eigenvalue of (M,h) (resp. (M',h')) by λ_i (resp. λ'_i). Since $\mathrm{Spec}(B\times_f M)=\mathrm{Spec}(B\times_f M')$, $\dim(B\times M)=\dim(B\times M')$ and hence $\dim M=\dim M'$. We apply mathematical induction: First of all, $\lambda_0=\lambda'_0=0$. We assume next that, for any non-negative integer k, $\lambda_i=\lambda'_i$ for $i=0,1,\cdots,k$. Since $\dim M=\dim M'$, $L_{(f;\lambda_i)}=L_{(f;\lambda'_i)}$ and hence $\mathrm{Spec}(L_{(f;\lambda_i)})=\mathrm{Spec}(L_{(f;\lambda'_i)})$ for $i=0,1,\cdots,k$. Thus $\mathrm{Spec}(B\times_f M)-\sum_{i=0}^k\mathrm{Spec}(L_{(f;\lambda_i)})=\mathrm{Spec}(B\times_f M')-\sum_{i=0}^k\mathrm{Spec}(L_{(f;\lambda'_i)})$. By Lemma 1, the minimum of $\mathrm{Spec}(B\times_f M)-\sum_{i=0}^k\mathrm{Spec}(L_{(f;\lambda_i)})$ is the least eigenvalue of $L_{(f;\lambda_{k+1})}$. By the same reason, the minimum of $\mathrm{Spec}(B\times_f M')-\sum_{i=0}^k\mathrm{Spec}(L_{(f;\lambda'_i)})$ is the least eigenvalue of $L_{(f;\lambda'_{k+1})}$. So $\mu_1(f;\lambda_{k+1})=\mu_1(f;\lambda'_{k+1})$ and Lemma 1 implies that $\lambda_{k+1}=\lambda'_{k+1}$. Therefore, for arbitrary non-negative integer i, we have $\lambda_i=\lambda'_i$.

§3. Proof of Theorem 2.

We use the same notations as in section 1. We denote the i-th eigenvalue of F by λ_i. Applying the minimum principle to the operator $L_{(f;\lambda_i)}$, we obtain

$$\mu_1(f;\lambda_i)\leqq\langle L_{(f;\lambda_i)}1,1\rangle/\langle 1,1\rangle$$
$$=\lambda_i\int_B f^{n-2}dV_g\Big/\int_B f^n dV_g\,.$$

By Hölder's inequality, we have

$$\int_B f^{n-2}dV_g\leqq\left\{\int_B (f^{n-2})^{n/(n-2)}dV_g\right\}^{(n-2)/n}\left\{\int_B dV_g\right\}^{2/n}$$
$$=\left\{\int_B f^n dV_g\right\}^{(n-2)/n}\left\{\int_B dV_g\right\}^{2/n},$$

and hence

$$\mu_1(f;\lambda_i)\leqq\frac{\lambda_i\left\{\int_B f^n dV_g\right\}^{(n-2)/n}\left\{\int_B dV_g\right\}^{2/n}}{\int_B f^n dV_g}=\frac{\lambda_i\left\{\int_B dV_g\right\}^{2/n}}{\left\{\int_B f^n dV_g\right\}^{2/n}}\,.$$

Since $\mu_1(1;\lambda_i)=\lambda_i$, we have

$$\mu_1(f;\lambda_i)\|f\|_n^2\leqq\lambda_i\|1\|_n^2=\mu_1(1;\lambda_i)\|1\|_n^2\,.$$

If the equality holds, f is constant as the equality holds in Hölder's inequality. Conversely, if f is constant, then $\mu_1(f;\lambda_i)=\lambda_i/f^2$ and $\|f\|_n=f\|1\|_n$. Therefore the equality holds.

§4. Proof of Theorem 3.

The assumption of Theorem 3 implies that the least positive eigenvalue of F is the least positive eigenvalue of Riemannian product $B\times F$. Thus we have $\lambda_1(1)=\lambda_1$. On the other hand, by the remark in section 2, we obtain $\lambda_1(f)\leqq\mu_1(f;\lambda_1)$. The proof of Theorem 2 implies that

$$\mu_1(f;\lambda_1)\leqq\frac{\lambda_1\left\{\int_B dV_g\right\}^{2/n}}{\left\{\int_B f^n dV_g\right\}^{2/n}}=\frac{\lambda_1\left\{\int_B dV_g\right\}^{2/n}\left\{\int_F dV_h\right\}^{2/n}}{\left\{\int_B f^n dV_g\right\}^{2/n}\left\{\int_F dV_h\right\}^{2/n}}$$
$$=\lambda_1\,\mathrm{Vol}\,(1)^{2/n}/\mathrm{Vol}\,(f)^{2/n}\,,$$

that is, $\mu_1(f;\lambda_1)\,\mathrm{Vol}\,(f)^{2/n}\leqq\lambda_1\,\mathrm{Vol}\,(1)^{2/n}$. Therefore, we have $\lambda_1(f)\,\mathrm{Vol}\,(f)^{2/n}\leqq\lambda_1(1)\,\mathrm{Vol}\,(1)^{2/n}$. If the equality holds, Theorem 2 implies f is constant.

REMARK. Even if f is constant, the equality does not necessarily hold. In fact, consider the case where the least positive eigenvalue of $L_{(f;\lambda_0)}$ is less than $\mu_1(f;\lambda_1)$.

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