TOKYO J. MATH.
VOL. 4, NO. 2, 1981

Note on Differentiable Maps and Liapunov Index

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ABSTRACT. We prove that if a differentiable map f has a negative Lirpunov index at a point x, then there exists an attractive neighborhood of x, and prove in addition that, if f is a unimodal map on an interval which has negative Schwarzian derivative, and satisfies the same condition as above, then there exists an attractive periodic point which absorbs x.

Introduction

Ruelle [1] and Pesin [2] investigated the properties of dynamical systems by means of Liapunov index. The most remarkable result among these is Pesin's work, in which he constructed a Markov partition for the dynamical systems with smooth invariant measure having non-zero Liapunov index. On the other hand, without assuming the existence of smooth invariant measure, Ruelle succeeded in the construction of the stable and the unstable manifold. In our paper, we will construct attractive domains for differentiable maps. Unfortunately, it will be difficult to calculate the Liapunov indices exactly. But we hope that the study of Liapunov indices will enable us to classify ω-limit sets of dynamical systems, and will help us to obtain characterizations of orbits moving densely in the space, characterizations of orbits getting absorbed in a strage attractor, or characterizations of attractive yeriodic points. This paper is intended for the first step toward the realization of this hope.

The reason why we treat differentiable maps rather than diffeomorphisms is that we want to treat the logistic equation in our analysis. Although this equation is one-dimensional, its orbits exhibit the most complicated behavior. One of the most important results on this map was given by Guckenheimer [3].

Received May 8, 1980
Revised July 14, 1980

He proved that if it has an attractive periodic point, then almost all points are absorbed in this attractive periodic point. Our second result (cf. Abstract) gives a sufficient condition for the existence of an attractive periodic point. It is desirable to prove the existence of an attractive periodic point for an arbitrary one-dimensional map, especially for a unimodal map.

§ 1. Attractive domain of an n-dimensional differentiable map.

Let m be a compact domain in R^n with smooth boundary, let f be a $C^{1,\theta}$ (differentiable and the differential $Df(x)$ is θ-Hölder continuous). We denote the euclidean norm in M by $|\cdot|$ and put $B(x, \alpha)=\{y \in M; |x-y|<\alpha\}$. Let us put $N(f)=\{x; \det(Df(x))=0\}$ and $O(f, x)=\{f^n(x); n \geqq 0\}$. If $N(f) \cap O(f, x)=\phi$, we can define the Liapunov index $L(f, x)$ of f at x as follows;

$$L(f, x)=\lim_{n\to\infty} \frac{1}{n} \log \|\{Df(f^n(x))\}^{-1}\|^{1/\theta} \cdot \|Df^n(x)\| ,$$

where $\|Df(x)\|$ is the operator norm of $Df(x)$. We assume that there exists such an x in M that $L(f, x)<0$. Define the number $\alpha_{x,\mu}$ by

$$\alpha_{x,\mu}=\inf_{n\geqq 0} e^{n\mu}\{\|\{Df(f^n(x))\}^{-1}\|^{1/\theta} \cdot \|Df^n(x)\|\}^{-1} \exp\left(-K\frac{1}{1-e^{\theta\mu}}\right) ,$$

where K is the Hölder constant which is defined as $\|Df(x)-Df(y)\| \leqq K\|x-y\|^\theta$, and μ is a constant satisfying $L(f, x)<\mu<0$. Then we can prove that there exists an attractive domain for x.

THEOREM 1. *If $L(f, x)<\mu<0$, then there exists a positive constant C such that $|f^n(x+u)-f^n(x)|<Ce^{n\mu}$ for $x+u$ in $B(x, \alpha_{x,\mu})$.*

PROOF. At first, let us estimate $f^n(x+u)-f^n(x)$ for $n\geqq 0$, where $x+u$ and x in M.

$$\begin{aligned} f^n(x+u)-f^n(x) &= \int_0^1 \frac{d}{dt} f(t(f^{n-1}(x+u)-f^{n-1}(x))+f^{n-1}(x))dt \\ &= \int_0^1 Df(t(f^{n-1}(x+u)-f^{n-1}(x))dt \circ (f^{n-1}(x+u)-f^{n-1}(x)) \\ &= \prod_{i=0}^{n-1} \int_0^1 \{Df(f^i(x))+Df(t(f^i(x+u)-f^i(x))+f^i(x))-Df(f^i(x))\}dt \circ u . \end{aligned}$$

It follows that

$$|f^n(x+u)-f^n(x)|\leqq\|Df^n(x)\|\prod_{i=0}^{n-1}\int_0^1\{1+\|\{Df(f^i(x))\}^{-1}\|$$
$$\times Kt|f^i(x+u)-f^i(x)|^\theta\}dt\cdot|u|\ ,$$

that is,

$$\|\{Df(f^n(x))\}^{-1}\|^{1/\theta}|f^n(x+u)-f^n(x)|\leqq\|\{Df(f^n(x))\}^{-1}\|^{1/\theta}$$
$$\times\|Df^n(x)\|\prod_{i=1}^{n-1}\{1+K\|\{Df(f^i(x))\}^{-1}\|\,|f^i(x+u)-f^i(x)|^\theta|u|$$
$$\leqq\|Df(f^n(x))\}^{-1}\|^{1/\theta}\cdot\|Df^n(x)\|\exp\Big(K\sum_{i=0}^{n-1}\|\{Df(f^i(x))\}^{-1}\|\,|f^i(x+u)$$
$$-f^i(x)|^\theta\Big)|u|\ .$$

Now we prove the following inequality inductively for u in $B(x,\alpha_{x,\mu})$.

$$\|\{Df(f^n(x))\}^{-1}\|^{1/\theta}|f^n(x+u)-f^n(x)|\leqq e^{n\mu}\ . \tag{1.1}$$

For the case of $n=0$, it is clear. Suppose that

$$\|\{Df(f^i(x))\}^{-1}\|^{1/\theta}|f^i(x+u)-f^i(x)|\leqq e^{i\mu}\qquad\text{for}\quad i<n\ .$$

Then we get

$$\|\{Df(f^n(x))\}^{-1}\|^{1/\theta}\cdot\|Df^n(x)\|\exp\Big(\frac{K}{1-e^{\mu\theta}}\Big)|u|<\alpha_{x,\mu}^{-1}e^{n\mu}|u|<e^{n\mu}\ . \tag{1.2}$$

Thus (1.1) was proved. If we put $C=\sup_{y\in M}\|Df(y)\|^{1/\theta}$, then the theorem is proved using (1.2) again.

§2. The application for one dimensional maps.

Let I be a closed bounded interval. Let f be a C^3 map from I into I and let $f'(x)$ be the differential of f at x. We assume the following condition (2.1).

(2.1)
(i) $N(f)=\{x\in I;\ f'(x)=0\}$ consists of only one point c.
(ii) $\left(\dfrac{f''(x)}{2}\right)^2-f'(x)\cdot\dfrac{f'''(x)}{3}>0$ for $x\in I-\{c\}$.

REMARK. It is easy to see that important functions $f_\lambda(x)=\lambda x(1-x)$, $g_\lambda(x)=\lambda\sin x$ and $h_\lambda(x)=\lambda xe^{-x}$ satisfy the condition (2.1).

The condition (2.1) (ii) means that Schwarzian derivative of f is negative. Singer [4] showed that f^n has at most one inflection point on interval where f^n is monotonic for $n>0$. We give the proof of this result in Lemma 2 for the sake of completeness. We proved this result independently. It will be proved more direct than Singer's one.

Let $I=[a, b]$ and $C_0=(c, b)$. Let $C_{\varepsilon_0\varepsilon_1\cdots\varepsilon_{n-1}}=\bigcap_{i=0}^{n-1} f^{-i}(C_{\varepsilon_i})$, $\varepsilon_i=0$ or 1. Clearly, $C_{\varepsilon_0\varepsilon_1\cdots\varepsilon_{n-1}}$ is the maximal open interval where f^n is monotonic.

LEMMA 2. *If f satisfies (2.1), then*

$$\min\{|(f^n)'(\alpha)|, |(f^n)'(\beta)|\}=\inf_{\alpha<t<\beta}|(f^n)'(t)| \qquad \text{for} \quad \alpha, \beta$$

in $C_{\varepsilon_0\varepsilon_1\cdots\varepsilon_{n-1}}$.

SUBLEMMA. *Under the same condition, for any $x_0\neq c$, there exists a $\gamma>0$ such that*

$$|f'(y)f'(z)|\leqq\frac{1}{|y-z|}|f(y)-f(z)|$$
$$\text{for} \quad x_0-\gamma<y<x_0<z<x_0+\gamma\ .$$

PROOF OF SUBLEMMA. We consider the Taylor expansion of $f(x)$ at $x=x_0$,

$$f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+a_3(x-x_0)^3+O((x-x_0)^4)\ .$$

Let $y=x_0+h$, $z=x_0+k$ and $t=|h|\vee|k|$, then $f'(y)f'(z)=a_1^2+2a_1a_2(h+k)+3a_1a_3(h^2+k^2)+4a_2^2hk+O(t^3)$. On the other hand,

$$\frac{|f(y)-f(z)|^2}{|y-z|^2}a_1^2+2a_1a_2(h+k)+2a_1a_3(h^2+hk+k^2)+a_2^2(h+k)^2+O(t^3)\ .$$

So we have

$$\frac{|f(y)-f(z)|^2}{|y-z|^2}-f'(y)f'(z)=(a_2^2-a_1a_3)(h-k)^2+O(t^3)\ .$$

By the condition (2.1) (ii), we obtain

$$\frac{|f(y)-f(z)|^2}{|y-z|^2}\geqq f'(y)f'(z)$$

for sufficiently small t.

PROOF OF LEMMA 2. For any x in $C_{\varepsilon_0\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}}$, $f^i(x)$ is monotonic and $(f^i)'(x)=0$ for $0\leqq i\leqq n-1$. By sublemma there exists an $\gamma>0$ such that

$$|z-y|\leqq\frac{1}{\sqrt{f'(z)f'(y)}}\cdot\frac{1}{\sqrt{f'(f(z))f'(f(y))}}\cdots\frac{1}{\sqrt{f'(f^{n-1}(z))f'(f^{n-1}(y))}}$$
$$\times|f^n(z)-f^n(y)| \qquad \text{for} \quad x-\gamma\leqq x\leqq z\leqq x+\gamma\ ,$$

hence,

$$|(f^n)'(z)(f^n)'(y)|\leqq\frac{|f^n(z)-f^n(y)|^2}{|z-y|^2}=\frac{1}{|z-y|^2}\left(\int_y^z|(f^n)'(t)|dt\right)^2 .$$

Using Schwarz's inequality, we have

$$|(f^n)'(z)\cdot(f^n)'(y)|\leqq\frac{1}{|z-y|}\int_y^z|(f^n)'(t)|^2dt \tag{2.2}$$

for $x-\gamma\leqq y\leqq x\leqq z\leqq x+\gamma$. Suppose that x_0, t_0, y_0 in $C_{\varepsilon_0\varepsilon_1\cdots\varepsilon_n}$ satisfy $x_0\leqq t_0\leqq y_0$ and

$$|(f^n)'(t_0)|=\min_{x_0\leqq t\leqq y_0}|(f^n)'(t)| .$$

Then there exist y, z sufficiently near t_0 which satisfy

$$|(f^n)'(x)|\leqq|(f^n)'(y)|=|(f^n)'(z)|$$

for $y<x<z$. Hence we have

$$|(f^n)'(z)(f^n)'(y)|\geqq\frac{1}{|z-y|}\int_y^z|(f^n)'(t)|^2dt ,$$

which contradicts the inequality (2.2).

LEMMA 3. *If there exists an x in (a, b) such that* (i) $L(f, x)<0$ *and* (ii) $\overline{O(f, x)}$=*closure of* $O(f, x)=\overline{\{f^n(x);\, n\geqq 0\}}\not\ni c$ *(c is the point in* (2.1)), *then there exists an attractive periodic point x_p which absorbs x.*

PROOF. Let $\mu<0$ and $\alpha_{x,\mu}$ be the one given in Theorem 1. At first, we prove

$$\sum_{i=0}^{\infty}\alpha_{f^i(x),\mu}=\infty . \tag{2.3}$$

It is enough to show that

$$\inf_{n\geqq 0}\frac{e^{n\mu}}{(f^n)'(f^i(x))}\geqq 1$$

for infinitely many i. Assume that there exists an i_0 such that for any $i\geqq i_0$,

$$\inf_{n\geqq 0}\frac{e^{n\mu}}{(f^n)'(f^i(x))}<1 .$$

Then, for each $i \geqq i_0$, there exists an $n_i > 0$ such that $e^{n_i\mu} < |(f^{n_i})'(f^i(x))|$. Then we can construct the sequence $\{k_i\}$, $k_1 = i_0$, $k_2 = n_{k_1}$, $k_3 = n_{k_1+k_2}, \cdots$. We put $n = k_1 + k_2 + \cdots + k_r$, then $e^{n\mu} = \prod_{i=1}^{r} e^{k_i\mu} < \prod_{i=1}^{r} |f^{k_i}(f^{k_1+\cdots+k_{i-1}}(x))| = |(f^n)'(x)|$. On the other hand, we have

$$L(f, x) = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \frac{|(f^n)'(x)|}{|f'(f^n(x))|^{1/\theta}}$$
$$= \overline{\lim_{n\to\infty}} \frac{1}{n} \log |(f^n)'(x)| \quad \text{by condition (ii)} .$$

Hence we have $L(f, x) \geqq \mu$, which is a contradiction; thus we proved (2.3). By (2.3), there exist $q, p > 0$ such that

$$B(f^q(x), \alpha_{f^q(x),\mu}) \cap B(f^{q+p}(x), \alpha_{f^{q+p}(x),\mu}) \neq \phi .$$

Then $L = \bigcup_{n=0}^{\infty} f^{np}(B(f^q(x), \alpha_{f^q(x),\mu}))$ is an f^p-invariant interval. Hence, $\bar{L}$ contains a fixed point x_p of f^p. By the attractivity of f on $B(y, \alpha_{y,\mu})$, we can prove that $\omega_x = \{y \in I; n_1 < n_2 < \cdots$ such that $\lim_{k\to\infty} f^{n_k}(x) = y\} = O(f, x)$.

THEOREM 2. *Assume that f satisfies* (2.1) *and that there exists an x in (a, b) satisfying $L(f, x) < 0$. Then there exists an attractive periodic point x_p such that $\omega_x = O(f, x_p)$.*

PROOF. In Lemma 3, we proved the theorem for the case of $c \notin \overline{O(f, x)}$, hence we can assume

$$c \in \overline{O(f, x)} . \tag{2.4}$$

Let us take a $C_{\epsilon_0\epsilon_1\cdots\epsilon_{n-1}} = (a_n, b_n)$ which contains x for each n. We will show that for sufficiently large n, $f^n(a_n)$ or $f^n(b_n)$ is contained in $B(f^n(x), \alpha(f^n(x)))$. By Lemma 2, we have $|(f^n)'(t)| \leqq |(f^n)'(x)|$ for any t with $a_n < t < x$ or t with $x < t < b_n$. We consider the first case. Then, $|f^n(x) - f^n(a_n)| \leqq |(f^n)'(x)||x - a_n| \leqq |(f^n)'(x)||b - a|$. By the assumption of the theorem, we can take n_0 so large that

$$e^{-n_0\mu} > |b - a| \alpha_{x,\mu}^{-1}$$

for $L(f, x) < \mu < 0$.

Then we have,

$$\alpha_{f^{n_0}(x),\mu} = \inf_{n\geqq 0} e^{n\mu} \frac{|f'(f^{n+n_0}(x))|^{1/\theta}}{|(f^n)'(f^{n_0}(x))|}$$
$$= \inf_{n\geqq 0} e^{(n+n_0)\mu} \frac{|f'(f^{n+n_0}(x))|^{1/\theta}}{|(f^{n+n_0})'(x)|} |(f^{n_0})'(x)| e^{-n_0\mu}$$

$$\geqq \alpha_{x,\mu} |(f^{n_0})'(x)| e^{-n_0 \mu} .$$

Hence, we have $\alpha_{f^{n_0}(x),\mu} \geqq |b-a||(f^{n_0})'(x)|$. It proves that $f^{n_0}(a_{n_0})$ is contained in $B(f^{n_0}(x), \alpha_{f^{n_0}(x)})$.

On the other hand, it is easily shown from $f^{n_0}(a_{n_0})=0$, that there exists $0 \leqq q < n_0$ such that $f^q(a_{n_0})=c$. Therefore, $f^{n_0}(a_{n_0,\mu})=f^{n_0-q}(c)$. By (2.4), it follows that $B(f^{n_0}(x), \alpha_{f^{n_0}(x),\mu})$ contains $f^r(x)$ for some $r>n_0$. Hence we have $B(f^{n_0}(x), \alpha_{f^{n_0}(x),\mu}) \cap B(f^r(x), \alpha_{f^r(x)}) \neq \phi$. We can prove by the same argument as in Lemma 2 the existence of an attractive periodic point x_p which absorbs x.

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