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Vanishing Theorems of Cohomology Groups with Values in the Sheaves $\mathcal{O}_{\mathrm{inc},\varphi}$ and $\mathcal{O}_{\mathrm{dec}}$

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Introduction

In this paper we study vanishing theorems of cohomology groups with values in the sheaves of holomorphic functions with exponential bounds. We treat the sheaf $\mathcal{O}_{\mathrm{inc},\varphi}$ of holomorphic functions with some exponential growth condition and the sheaf $\mathcal{O}_{\mathrm{dec}}$ of holomorphic functions with some exponential decay condition. The sheaves $\mathcal{O}_{\mathrm{inc},\varphi}$ and $\mathcal{O}_{\mathrm{dec}}$ are proposed by Professors M. Sato and T. Kawai to define modified Fourier hyperfunctions. Those are modifications of the sheaf $\tilde{\mathcal{O}}$ of holomorphic functions with the infra-exponential growth condition and the sheaf $\underset{\sim}{\mathcal{O}}$ of holomorphic functions with some exponential decay condition in Kawai [12]. We have two motivations:

The first is to give a foundation for our forthcoming paper (Saburi [26]) on the theory of modified Fourier hyperfunctions.

The second is to improve Kawai's proof of vanishing theorems of cohomology groups with the value in the sheaf $\tilde{\mathcal{O}}$. Our methods of proof are valid for the $\tilde{\mathcal{O}}$ without difficulties.

Kawai proved the Cartan Theorem B and the Malgrange theorem for the sheaf $\tilde{\mathcal{O}}$ (Theorems 2.1.4 and 3.1.8 in Kawai [12] respectively). His proof of the Cartan Theorem B for the sheaf $\tilde{\mathcal{O}}$ is somewhat complicated. Moreover it seems to the author that his proof of the Malgrange theorem for the sheaf $\tilde{\mathcal{O}}$ is not complete.

We give a direct method of the calculation of the cohomology groups with the value in the sheaf $\mathcal{O}_{\mathrm{inc},\varphi}$, and prove the Cartan Theorem B for that sheaf (Theorem I in §1.2). We also prove in details the Malgrange theorem for the sheaf $\mathcal{O}_{\mathrm{inc},\varphi}$ (Theorem III in §1.2).

There are some works relevant to Kawai [12]. Those are Ito-Nagamachi

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[8], Junker [9], [10], Nagamachi-Muguibayashi [17], [18], [19], Nagamachi [20], and Saburi [23]. Except Saburi [23], they proved vanishing theorems of cohomology groups with values in the sheaves ${}^{E}\tilde{\mathcal{O}}$ and ${}^{E}\underset{\sim}{\mathcal{O}}$ of (Hilbert or Fréchet) vector valued holomorphic functions with the infra-exponential growth condition and some exponential decay condition respectively. To prove those theorems, they used the same method as Kawai's [12].

In this paper, following to Kawai [12], we mainly rely on the theory of L^2 estimates for the $\bar{\partial}$ operator in Hörmander [6], [7] and that of Fréchet-Kômura spaces and dual Fréchet-Kômura spaces in Komatsu [13]. Fréchet-Kômura spaces and dual Fréchet-Kômura spaces are the two classes of topological vector spaces proposed by Komatsu [13]. (In the terminology of Komatsu [13], Fréchet-Kômura spaces are referred as FS^* spaces and dual Fréchet-Kômura spaces are referred as DFS^* spaces. But following to Komatsu [15], we use the terminologies Fréchet-Kômura spaces and dual Fréchet-Kômura spaces for those classes of topological vector spaces.) We use those theories mainly in § 2.

The plan of this paper is as follows:

In § 1 we give the definitions of the sheaves $\mathcal{O}_{\text{inc},\varphi}$ and $\mathcal{O}_{\text{dec},\varphi}$ and formulate the main results. Those are the Cartan Theorem B for the sheaves $\mathcal{O}_{\text{inc},\varphi}$ and $\mathcal{O}_{\text{dec}}$ and Malgrange theorem for the sheaf $\mathcal{O}_{\text{inc},\varphi}$.

In § 2 we introduce the sheaf $\mathscr{X}_{\varphi}$ of locally square summable functions with some exponential growth condition, which will be defined in § 2.1. Moreover we study the Dolbeault complex $(\mathscr{X}_{\varphi}^{(p,\cdot)}; \bar{\partial})$ and its dual complex $(\mathscr{Y}_{\varphi,comp}^{(p,\cdot)}; \vartheta)$, where $\mathscr{Y}_{\varphi}$ is the sheaf of locally square summable functions with some exponential decay condition defined also in § 2.1. Here we give a sufficient condition so that the Dolbeault complex constitutes an exact sequence. The result of this section gives a direct method of calculation of cohomology groups with values in the sheaves $\mathcal{O}_{\text{inc},\varphi}$ and $\mathcal{O}_{\text{dec}}$, which is a somewhat different way from that of Kawai [12]. (See Remark 3 in § 2.2, and compare § 2 in Kawai [12] with §§ 2 and 3 of ours.)

In § 3, using the result in § 2, we construct a soft resolution of the sheaves $\mathcal{O}_{\text{inc},\varphi}$ and $\mathcal{O}_{\text{dec}}$. Moreover we prove the Cartan Theorem B for those sheaves (Theorems 3.1.2 and 3.1.4). We also prove the Malgrange theorem for the sheaf $\mathcal{O}_{\text{inc},\varphi}$ (Theorem 3.2.1).

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§ 1. Definitions of the sheaves $\mathcal{O}_{\mathrm{inc},\varphi}$ and $\mathcal{O}_{\mathrm{dec},\varphi}$ and main results.

1.1. Definitions.

DEFINITION 1.1.1. We denote by $\boldsymbol{D}^k$ the radial compactification $\boldsymbol{R}^k \sqcup S_\infty^{k-1}$ of $\boldsymbol{R}^k$, where S_∞^{k-1} is a $(k-1)$-dimensional sphere put at infinity. Here we identify S_∞^{k-1} with $(\boldsymbol{R}^k\backslash\{0\})/\boldsymbol{R}_+$. Let us consider the natural projection $\varpi: \boldsymbol{R}^k\backslash\{0\} \to S_\infty^{k-1} = (\boldsymbol{R}^k\backslash\{0\})/\boldsymbol{R}_+$. We denote $x_\infty = \varpi(x)$ for $x \in \boldsymbol{R}^k\backslash\{0\}$. Then the topology of $\boldsymbol{D}^k$ is defined as follows. Let B^k be the closed unit ball in $\boldsymbol{R}^k$ centered at the origin. We define a bijection κ of $\boldsymbol{D}^k$ onto B^k by the following:

$$\kappa(y) = \begin{cases} x/|x| & \text{if } \ y = x_\infty \in S_\infty^{k-1} \\ x/(\|x\|+1) & \text{if } \ y = x \in \boldsymbol{R}^k \,, \end{cases}$$

where $\|x\|$ is a slight modification of $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ near the origin so as to be C^∞ as a function of x and monotone increasing as a function of $|x|$. We equip $\boldsymbol{D}^k$ with the weakest topology so that κ gives a homeomorphism of $\boldsymbol{D}^k$ onto B^k. We can also equip $\boldsymbol{D}^k$ with a differential structure so that κ gives a diffeomorphism of $\boldsymbol{D}^k$ onto B^k.

With this topology, a fundamental system of neighborhoods of a point $x_\infty \in S_\infty^{k-1}$ is given by a family $\{(C+a) \sqcup C_\infty\}$ $(C_\infty = \{y_\infty \in S_\infty^{k-1};\, y \in C\})$, where a runs all vectors in $\boldsymbol{R}^k$, and C runs all open cones in $\boldsymbol{R}^k$ with the vertex at the origin containing x. Thus we have $x_\infty = \lim_{\lambda\to+\infty}(\lambda x + a)$ for any $x \in \boldsymbol{R}^k\backslash\{0\}$ and any $a \in \boldsymbol{R}^k$. On $\boldsymbol{R}^k$ this topology coincide with the usual topology.

We identify $\boldsymbol{C}^n$ with $\boldsymbol{R}^{2n}$ and denote by $\boldsymbol{Q}^n$ the radial compactification of $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$.

For a real valued function φ on $\boldsymbol{C}^n$ which is bounded on any compact set in $\boldsymbol{C}^n$, we define two sheaves $\mathcal{O}_{\mathrm{inc},\varphi}$ and $\mathcal{O}_{\mathrm{dec},\varphi}$.

DEFINITION 1.1.2. We denote by $\mathcal{O}_{\mathrm{inc},\varphi}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{O}_{\mathrm{inc},\varphi}(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{O}_{\mathrm{inc},\varphi}(W) = \{f \in \mathcal{O}(W \cap \boldsymbol{C}^n);\ \sup_{z \in K \cap \boldsymbol{C}^n} |f(z)| \exp(-\varphi(z) - \varepsilon|z|) < \infty$$
$$\text{for any } K \Subset W \text{ and any } \varepsilon > 0\} \,,$$

where the notation $K \Subset W$ means that K is a relatively compact subset of W.

DEFINITION 1.1.3. We denote by $\mathcal{O}_{\mathrm{dec},\varphi}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{O}_{\mathrm{dec},\varphi}(W)$ over an open set W in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{O}_{\mathrm{dec},\varphi}(W)=\{f\in\mathcal{O}(W\cap C^n);\ \text{for any } K\subset W \text{ there exists an } \varepsilon>0 \text{ such that } \sup_{z\in K\cap C^n}|f(z)|\exp(\varphi(z)+\varepsilon|z|)<\infty\}.$$

When the function φ is identically 0, we denote $\mathcal{O}_{\mathrm{inc},\varphi}$ and $\mathcal{O}_{\mathrm{dec},\varphi}$ by $\mathcal{O}_{\mathrm{inc}}$ and $\mathcal{O}_{\mathrm{dec}}$ respectively.

REMARK. The restrictions of the sheaves $\mathcal{O}_{\mathrm{inc},\varphi}$ and $\mathcal{O}_{\mathrm{dec},\varphi}$ to C^n coincide with the sheaf $\mathcal{O}$ of holomorphic functions on C^n.

DEFINITION 1.1.4. We call an open set W in Q^n to be *acute*, if it satisfies the following condition:

$$\sup_{z\in W\cap C^n}|\mathrm{Im}\, z|/(|\mathrm{Re}\, z|+A)<1 \quad \text{for some} \quad A>0.$$

DEFINITION 1.1.5. We call an open set V in Q^n to be $\mathcal{O}_{\mathrm{inc}}$-*pseudoconvex* if it is acute and if there exists a strictly plurisubharmonic C^∞ function p on $V\cap C^n$ satisfying the following condition (P):

$$(P)\qquad \begin{cases} \text{i)} & \{z\in V\cap C^n;\ p(z)<c\}\subset V \quad \text{for any} \quad c\in R, \\ \text{ii)} & \sup_{z\in K\cap C^n} p(z)<\infty \quad \text{for any} \quad K\subset V. \end{cases}$$

DEFINITION 1.1.6. We say a function φ on C^n to be of *linear variation* if there exists a constant $A>0$ such that

$$|\varphi(z')-\varphi(z)|<A \quad \text{for every} \quad z, z'\in C^n \text{ with } |z'-z|<1.$$

EXAMPLE (a). For $e^1_\infty=(1, 0, \cdots, 0)_\infty\in S^{2n-1}_\infty$, there exists a fundamental system of neighborhoods consisting of $\mathcal{O}_{\mathrm{inc}}$-pseudoconvex open sets in Q^n. Put

$$U_\delta=\left\{z\in C^n;\ |\mathrm{Im}\, z_1|^2+\sum_{j=2}^n|z_j|^2<\delta^2|\mathrm{Re}\, z_1-(1/\delta)|^2,\ \mathrm{Re}\, z_1>1/\delta\right\},$$
$$V_\delta=\mathring{\bar{U}}_\delta,$$

then $\{V_\delta\}_{0<\delta<1}$ gives a fundamental system of neighborhoods of e^1_∞ consisting of $\mathcal{O}_{\mathrm{inc}}$-pseudoconvex open sets. It is easy to see that $\{V_\delta\}_{0<\delta<1}$ gives a fundamental system of neighborhoods of e^1_∞ and each V_δ $(0<\delta<1)$ is acute. To show the $\mathcal{O}_{\mathrm{inc}}$-pseudoconvexity of V_δ $(0<\delta<1)$, we define the following functions:

$$q_\delta(z)=|\mathrm{Im}\, z_1|^2+\sum_{j=2}^n|z_j|^2-\delta^2|\mathrm{Re}\, z_1-(1/\delta)|^2$$
$$p_\delta(z)=1/(-q_\delta(z)) \qquad (z\in V_\delta\cap C^n).$$

Then p_δ gives a strictly plurisubharmonic C^∞ function on $V_\delta \cap C^n$ $(0<\delta<1)$ which satisfies the condition (P) in Definition 1.1.5. This shows the $\mathscr{O}_{\mathrm{inc}}$-pseudoconvexity of V_δ $(0<\delta<1)$.

REMARK 1. Considering the function $p(z)+|z|^2$, we fined that $V\cap C^n$ is pseudoconvex in the usual sense, if V is an $\mathscr{O}_{\mathrm{inc}}$-pseudoconvex open set in Q^n.

1.2. Main results.

Using the above definitions we can descrive the main results of this paper.

For a sheaf $\mathscr{F}$ on a topological space X, we denote by $H^q(E; \mathscr{F})$ the q-th cohomology group of an open (or closed) set E with the value in the sheaf $\mathscr{F}$.

THEOREM I (Theorem 3.1.2). *For any $\mathscr{O}_{\mathrm{inc}}$-pseudoconvex open set V in Q^n and any plurisubharmonic function φ on C^n of linear variation, we have*

$$H^q(V; \mathscr{O}_{\mathrm{inc},\varphi})=0 \quad (q\geqq 1)\,.$$

THEOREM II (Theorem 3.1.4). *Let K be a compact set in Q^n. Suppose that there exists a fundamental system of neighborhoods of K consisting of $\mathscr{O}_{\mathrm{inc}}$-pseudoconvex open sets in Q^n. Then we have*

$$H^q(K; \mathscr{O}_{\mathrm{dec}})=0 \quad (q\geqq 1)\,.$$

THEOREM III (Theorem 3.2.1). *For any acute open set W in Q^n and any plurisubharmonic function φ on C^n of linear variation, we have*

$$H^n(W; \mathscr{O}_{\mathrm{inc},\varphi})=0\,.$$

§ 2. The Dolbeault complex for the sheaves $\mathscr{H}_\varphi^{(p,\cdot)}$.

2.1. Preparation.

In this section we define the sheaf on Q^n, $\mathscr{H}_\varphi$ of $\mathscr{L}^2_{\mathrm{loc}}$ functions with some exponential growth condition and $\mathscr{Y}_\varphi$ of $\mathscr{L}^2_{\mathrm{loc}}$ functions with some exponential decay condition. We define a topology of $\mathscr{H}_\varphi(W)$ for an open set W in Q^n. We consider the following dual Dolbeault complexes:

$$\begin{array}{ccccccccc}
\mathscr{H}_\varphi^{(p,0)}(W) & \xrightarrow{\bar\partial} & \mathscr{H}_\varphi^{(p,1)}(W) & \xrightarrow{\bar\partial} & \cdots & \xrightarrow{\bar\partial} & \mathscr{H}_\varphi^{(p,n)}(W) & \longrightarrow & 0 \\
\updownarrow & & \updownarrow & & & & \updownarrow & & \\
\mathscr{Y}_{\varphi,comp}^{(p,0)}(W) & \xleftarrow[\vartheta]{} & \mathscr{Y}_{\varphi,comp}^{(p,1)}(W) & \xleftarrow[\vartheta]{} & \cdots & \xleftarrow[\vartheta]{} & \mathscr{Y}_{\varphi,comp}^{(p,n)}(W) & \longleftarrow & 0\,,
\end{array}$$

and make some remarks for the dual operator ϑ of the Cauchy-Riemann operator $\bar{\partial}$.

a) For a real valued measurable function φ on C^n which is bounded on any compact set in C^n, we define two sheaves $\mathscr{X}_\varphi$ and $\mathscr{Y}_\varphi$ on Q^n:

DEFINITION 2.1.1. We denote by $\mathscr{X}_\varphi$ the sheaf on Q^n whose section module $\mathscr{X}_\varphi(W)$ over an open set W in Q^n is given by the following:

$$\mathscr{X}_\varphi(W)=\left\{f \in \mathscr{L}^2_{\mathrm{loc}}(W\cap C^n);\ \int_{K\cap C^n} |f(z)|^2 \exp(-\varphi(z)-\varepsilon|z|)d\lambda<\infty \right.$$
$$\left. \text{for any } K\subset W \text{ and any } \varepsilon>0\right\},$$

where $d\lambda$ is the Lebesgue measure on $C^n\cong R^{2n}$.

DEFINITION 2.1.2. We denote by $\mathscr{Y}_\varphi$ the sheaf on Q^n whose section module $\mathscr{Y}_\varphi(W)$ over an open set W in Q^n is given by the following:

$$\mathscr{Y}_\varphi(W)=\left\{f \in \mathscr{L}^2_{\mathrm{loc}}(W\cap C^n);\ \text{for any } K\subset W \text{ there exists an } \varepsilon>0\right.$$
$$\left.\text{such that } \int_{K\cap C^n} |f(z)|^2 \exp(\varphi(z)+\varepsilon|z|)d\lambda<\infty\right\}.$$

REMARK. The restrictions of the sheaves $\mathscr{X}_\varphi$ and $\mathscr{Y}_\varphi$ to C^n coincide with the sheaf $\mathscr{L}^2_{\mathrm{loc}}$ of locally square summable functions on $C^n\cong R^{2n}$.

Let W be an open set in Q^n. We are going to equip the function space $\mathscr{X}_\varphi(W)$ with a topology.

Let $\{K_j\}$ be an increasing sequence of compact subsets of W which exhausts W. Then we define Hilbert spaces $X_j(\varphi)$ $(j=1, 2, \cdots)$, using the notation in Hörmander [6], [7]:

$$\begin{aligned} X_j(\varphi)&=L^2(\mathring{K}_j\cap C^n;\ \varphi(z)+(1/j)|z|)\\ &=\left\{f \in \mathscr{L}^2_{\mathrm{loc}}(\mathring{K}_j\cap C^n);\ \int_{\mathring{K}_j\cup C^n} |f(z)|^2 \exp(-\varphi(z)-(1/j)|z|)\,d\lambda<\infty\right\}. \end{aligned}$$

Consider the projective system:

$$X_1(\varphi) \xleftarrow{\rho_1^2} X_2(\varphi) \xleftarrow{\rho_2^3} \cdots \xleftarrow{\rho_{k-1}^k} X_k(\varphi) \xleftarrow{\rho_k^{k+1}} X_{k+1}(\varphi) \xleftarrow{\rho_{k+1}^{k+2}} \cdots,$$

where ρ_j^{j+1} are natural restriction mappings. We note that ρ_j^{j+1} are weakly compact operators with dense ranges. As to the weak compactness of ρ_j^{j+1}, we remark that a continuous operator of a Hilbert space into another is always weakly compact. ρ_j^{j+1} have dense ranges. Because every $f\in X_j(\varphi)$ can be approximated by the sequence $\{f_\nu\}\subset X_{j+1}(\varphi)$ defined as follows:

$$f_\nu(z)=\begin{cases} f(z)\exp(-(1/\nu)|z|^2) & \text{if} \quad z\in \mathring{K}_j\cap C^n \\ 0 & \text{if} \quad z\in(\mathring{K}_{j+1}\backslash\mathring{K}_j)\cap C^n\,. \end{cases}$$

Now we equip $\mathscr{H}_\varphi(W)$ with a following projective limit topology of Hilbert spaces:

$$\mathscr{H}_\varphi(W)=\lim_j \operatorname{proj} X_j(\varphi)\,.$$

Since ρ_j^{j+1} are weakly compact, $\mathscr{H}_\varphi(W)$ is a Fréchet-Kômura space. We remark that the topology of $\mathscr{H}_\varphi(W)$ just defined above does not depend on the choice of exhaustion $\{K_j\}$ of W.

Next we study the dual space of $\mathscr{H}_\varphi(W)$.

We represent the dual space of $X_j(\varphi)$ by

$$\begin{aligned} Y_j(\varphi)&=L^2(\mathring{K}_j\cap C^n;\ -\varphi(z)-(1/j)|z|) \\ &=\left\{f\in\mathscr{L}^2_{\mathrm{loc}}(\mathring{K}_j\cap C^n);\ \int_{K_j\cap C^n}|f(z)|^2\exp(\varphi(z)+(1/j)|z|)d\lambda<\infty\right\}. \end{aligned}$$

The pairing between $X_j(\varphi)$ and $Y_j(\varphi)$ is given by the following:

$$\langle f, g\rangle=\int_{K_j\cap C^n} f\bar{g}\,d\lambda \quad \text{for} \quad (f, g)\in X_j(\varphi)\times Y_j(\varphi)\,.$$

Since ρ_j^{j+1} have dense ranges, the strong dual space of the Fréchet-Kômura space $\mathscr{H}_\varphi(W)$ is represented by the following dual Fréchet-Kômura space:

$$\mathscr{Y}_{\varphi,comp}(W)=\lim_j \operatorname{ind} Y_j(\varphi)\,,$$

thanks to Theorem 11 in Komatsu [13] (p. 376), where the injection ρ'_j: $Y_j(\varphi)\to Y_{\varphi,\mathrm{comp}}(W)$ is given by the following:

$$\rho'_j g(z)=\begin{cases} g(z) & \text{if} \quad z\in\mathring{K}_j\cap C^n \\ 0 & \text{if} \quad z\in(W\backslash\mathring{K}_j)\cap C^n\,. \end{cases}$$

The pairing between $\mathscr{H}_\varphi(W)$ and $\mathscr{Y}_{\varphi,comp}(W)$ is given by the following:

$$\langle f, g\rangle=\int_{W\cap C^n} f\bar{g}\,d\lambda \quad \text{for} \quad (f, g)\in\mathscr{H}_\varphi(W)\times\mathscr{Y}_{\varphi,comp}(W)\,.$$

REMARK. We note that $\mathscr{H}_\varphi(W)$ coincide with $\mathscr{H}_{\varphi_A}(W)$ ($\varphi_A(z)=\varphi(z)+A\log(|z|^2+1)$) for any $A>0$. To see this fact, it is sufficient to consider the following projeceive system:

$$\cdots \longleftarrow X_j(\varphi_A) \longleftarrow X_j(\varphi) \longleftarrow X_{j+1}(\varphi_A) \longleftarrow X_{j+1}(\varphi) \longleftarrow \cdots .$$

b) Now we go on to the definition of the Dolbeault complex $\mathscr{X}_\varphi^{(p,\cdot)}$, and make an elementary study for its dual complex.

DEFINITION 2.1.3. We denote by $\mathscr{X}_\varphi^{(p,q)}$ (resp. $\mathscr{Y}_\varphi^{(p,q)}$) the sheaf on Q^n whose section module $\mathscr{X}_\varphi^{(p,q)}(W)$ (resp. $\mathscr{Y}_\varphi^{(p,q)}(W)$) over an open set W in Q^n is given by the following:

$$\mathscr{X}_\varphi^{(p,q)}(W)=\{\sum_{|I|=p,|J|=q} f_{I,J}dz^I\wedge d\bar{z}^J;\ f_{I,J}\in\mathscr{X}_\varphi(W)\}$$
$$(\text{resp. } \mathscr{Y}_\varphi^{(p,q)}(W)=\{\sum_{|I|=|p,|J|=q} g_{I,J}dz^I\wedge d\bar{z}^J;\ g_{I,J}\in\mathscr{Y}_\varphi(W)\}) .$$

Let W be an open set in Q^n. Then $\mathscr{X}_\varphi^{(p,q)}(W)$ is an Fréchet-Kômura space:

$$\mathscr{X}_\varphi^{(p,q)}(W)=\lim_j \operatorname{proj} X_j^{(p,q)}(\varphi)$$
$$=\lim_j \operatorname{proj} L^2_{(p,q)}(\mathring{K}_j\cap C^n;\ \varphi(z)+(1/j)|z|) ,$$

where we followed the notation in Hormander [6], [7]:

$$L^2_{(p,q)}(\mathring{K}_j\cap C^n;\ \varphi(z)+(1/j)|z|)$$
$$=\{\sum_{|I|=p,|J|=q} f_{I,J}dz^I\wedge dz^J;\ f_{I,J}\in L^2(\mathring{K}_j\cap C^n;\ -\varphi(z)-(1/j)|z|)\} .$$

The strong dual space of the Fréchet-Kômura space $\mathscr{X}_\varphi^{(p,q)}(W)$ is represented by the following dual Fréchet-Kômura space:

$$\mathscr{Y}_{\varphi,comp}^{(p,q)}(W)=\lim_j \operatorname{ind} Y_j^{(p,q)}(\varphi)$$
$$=\lim_j \operatorname{ind} L^2_{(p,q)}(\mathring{K}_j\cap C^n;\ -\varphi(z)-(1/j)|z|) .$$

The pairing between $\mathscr{X}_\varphi^{(p,q)}(W)$ and $\mathscr{Y}_{\varphi,comp}^{(p,q)}(W)$ is given by the following:

$$\langle f, g\rangle=\sum_{|I|=p,|J|=q}\int_{W\cap C^n} f_{I,J}\bar{g}_{I,J}d\lambda \quad \text{for} \quad (f, g)\in\mathscr{X}_\varphi^{(p,q)}(W)\times\mathscr{Y}_{\varphi,comp}^{(p,q)}(W) ,$$

where we put

$$f=\sum_{|I|=p,|J|=q} f_{I,J}dz^I\wedge d\bar{z}^J \quad (f_{I,J}\in\mathscr{X}_\varphi(W)) ,$$
$$g=\sum_{|I|=p,|J|=q} g_{I,J}dz^I\wedge d\bar{z}^J \quad (g_{I,J}\in\mathscr{Y}_{\varphi,comp}(W)) .$$

Let us consider the Cauchy-Riemann operators:

$$\bar{\partial}_j\colon X_j^{(p,q-1)}(\varphi_1)\longrightarrow X_j^{(p,q)}(\varphi_2) \quad (j=1, 2, \cdots) ,$$
$$\bar{\partial}\colon \mathscr{X}_{\varphi_1}^{(p,q-1)}(W)\longrightarrow \mathscr{X}_{\varphi_2}^{(p,q)}(W) .$$

The domains $D_{\bar\partial_j}$ and $D_{\bar\partial}$ of $\bar\partial_j$ and $\bar\partial$ respectively are defined as follows:

$$D_{\bar\partial_j}=\{f\in X_j^{(p,q-1)}(\varphi_1);\ \bar\partial f\in X_j^{(p,q)}(\varphi_2)\}\ ,$$
$$D_{\bar\partial}=\{f\in \mathscr{H}_{\varphi_1}^{(p,q-1)}(W);\ \bar\partial f\in \mathscr{H}_{\varphi_2}^{(p,q)}(W)\}\ ,$$

where $\bar\partial f$ is defined in the sense of distributions.

We note that the following diagram is commutative:

$$\begin{array}{ccccc} & \mathscr{H}_{\varphi_1}^{(p,q-1)}(W) & \xrightarrow{\ \bar\partial\ } & \mathscr{H}_{\varphi_2}^{(p,q)}(W) & \\ \rho_j & \downarrow & & \downarrow & \rho_j\ . \\ & X_j^{(p,q-1)}(\varphi_1) & \xrightarrow[\bar\partial_j]{} & X_j^{(p,q)}(\varphi_2) & \end{array} \tag{2.1.1}$$

The operators $\bar\partial_j$ and $\bar\partial$ are densely defined and closed. Hence the operators $\bar\partial_j$ and $\bar\partial$ have their dual operators ϑ_j and ϑ respectively. The operators ϑ_j and ϑ are also closed. We note that, if $g=\vartheta v$ (or $\vartheta_j v$), then the following equation holds:

$$g=\sum_{|I|=p,|K|=q-1}\ \sum_{1\le k\le n,\{k\}\cup K=J}\frac{\partial}{\partial z_k}v_{I,J}dz^I\wedge d\bar z^K$$

as distributions on C^n.

c) Next we go on to study the relation of the operators ϑ_j and ϑ.

We note that, if $g\in D_{\vartheta_j}$, then by the commutative diagram (2.1.1), we have $\rho_j'g\in D_\vartheta$ and $\vartheta\rho_j'g=\rho_j'\vartheta_j g$. However we have to check whether $g\in D_\vartheta\cap Y_j^{(p,q)}(\varphi_2)$ implies $g\in D_{\vartheta_j}$ or not. Therefore we make a little argument about this problem. In what follows we identify $\rho_j'g$ and $\rho_j'^{j+1}g$ with g.

First we need

LEMMA 2.1.1. *Let W be an open set in Q^n. Then we can choose an exhaustion $\{K_j\}$ of W consisting of compact subsets of W such that each $K_j\cap C^n$ has a C^∞ boundary.*

PROOF. Using a partition of unity by C^∞ functions, we can construct a C^∞ function q on $W\cap C^n$ which satisfies the following conditions:

$$W_c=\{z\in W\cap C^n;\ q(z)<c\}\subset W \quad \text{for any} \quad c\in R\ ,$$
$$\sup_{z\in K\cap C^n} q(z)<\infty \quad \text{for any} \quad K\subset W\ .$$

Then by the Sard theorem (see, for example, Guillemin, Pollack [1], p. 205) we can choose an increasing sequence $\{c_j\}\subset R$ ($c_j\uparrow\infty$) so that each W_{c_j} has a C^∞ boundary. We put $K_j=\bar W_{c_j}$ ($j=1, 2, \cdots$), then $\{K_j\}$ gives an exhaustion of W with required properties. q.e.d.

REMARK. Recall that the topology of $\mathscr{H}_\varphi(W)$ does not depend on the choice of an exhaustion $\{K_j\}$ of W. Therefore in what follows, by Lemma 2.1.1, we may assume that each $K_j \cap C^n$ has a C^∞ boundary.

PROPOSITION 2.1.2. *Suppose* $g \in D_\vartheta \cap Y_j^{(p,q)}(\varphi_2)$ *and* $\vartheta g \in Y_k^{(p,q-1)}(\varphi_1)$. *Then we have* $g \in D_{\vartheta_l}$ *and* $\vartheta_l g = \vartheta g$, *where* $l = \max\{j, k\}$.

PROOF. We prove the proposition only in the case $j \geqq k$. Since in the case $j<k$, the proof goes similarly as in the case $j \geqq k$. By the previous remark, we may assume that each $K_j \cap C^n$ has a C^∞ boundary. Hence by Proposition 2.1.1 in Hörmander ([6], p. 100), for any $f \in D_{\bar\partial_j}$ there exists a sequence $\{f_\nu\} \subset \dot C^1_{(p,q-1)}(\mathring K_j \cap C^n)$ such that

$$\|f_\nu - f\|_{\varphi_1, j} + \|\bar\partial f_\nu - \bar\partial_j f\|_{\varphi_2, j} \longrightarrow 0 \quad (\nu \longrightarrow \infty)\,, \tag{2.1.2}$$

where the notations $\|\cdot\|_{\varphi_1,j}$ and $\|\cdot\|_{\varphi_2,j}$ are the norms of the Hilbert spaces $X_j^{(p,q-1)}(\varphi_1)$ and $X_j^{(p,q)}(\varphi_2)$ respectively, and $\dot C^1_{(p,q-1)}(\mathring K_j \cap C^n)$ is defined as follows:

$$\dot C^1_{(p,q-1)}(\mathring K_j \cap C^n) = \{\sum_{|I|=p, |J|=q-1} h_{I,J} dz^I \wedge d\bar z^J;\ h_{I,J} = H_{I,J}|_{\mathring K_j \cup C^n},\ H_{I,J} \in C_0^1(C^n)\}\,.$$

Then we have from (2.1.2)

$$\left.\begin{matrix} \langle \bar\partial f_\nu, g\rangle \longrightarrow \langle \bar\partial_j f, g\rangle_j \\ \langle f_\nu, \vartheta g\rangle \longrightarrow \langle f, \vartheta g\rangle_j \end{matrix}\right. \quad (\nu \longrightarrow \infty)\,.$$

Thus we have $\langle \bar\partial_j f, g\rangle_j = \langle f, \vartheta g\rangle_j$ for each $f \in D_{\bar\partial_j}$. This shows $g \in D_{\vartheta_j}$ and $\vartheta_j g = \vartheta g$. q.e.d.

2.2. The Dolbeault complex for the sheaves $\mathscr{H}_\varphi^{(p,q)}$.

Under the above preparations, we now discuss a sufficient condition so that the Dolbeault complex $\mathscr{H}_\varphi^{(p,\cdot)}$ constitutes an exact sequence. That is

THEOREM 2.2.1. *Let* V *be an* $\mathscr{O}_{\mathrm{inc}}$*-pseudoconvex open set in* Q^n, φ *a plurisubharmonic function on* $V \cap C^n$. *Then the following sequence*

$$\mathscr{H}_\varphi^{(p,0)}(V) \xrightarrow{\bar\partial^{(1)}} \mathscr{H}_\varphi^{(p,1)}(V) \xrightarrow{\bar\partial^{(2)}} \cdots \xrightarrow{\bar\partial^{(n)}} \mathscr{H}_\varphi^{(p,n)}(V) \longrightarrow 0 \tag{2.2.1}$$

is exact.

PROOF. a) From the definition of $\mathscr{O}_{\mathrm{inc}}$-pseudoconvexity, there exists a strictly plurisubharmonic C^∞ function p on $V \cap C^n$ satisfying the following condition (P):

$$(P)\quad \begin{cases} U_c=\{z\in V\cap C^n;\, p(z)<c\}\subset V & \text{for any}\quad c\in \boldsymbol{R}\,, \\ \sup_{z\in K\cap C^n} p(z)<\infty & \text{for any}\quad K\subset V\,. \end{cases}$$

Hence as in the proof of Lemma 2.1.1, we can choose a sequence $\{c_j\}\subset \boldsymbol{R}$ ($c_j\uparrow\infty$) so that each U_{c_j} has a C^∞ boundary. Here we note that each U_{c_j} is pseudoconvex in the usual sense. Put $K_j=\bar{U}_{c_j}$. Then $\{K_j\}$ gives an exhaustion of V, and we have the following representation for $\mathscr{X}_\varphi^{(p,q)}(V)$:

$$\mathscr{X}_\varphi^{(p,q)}(V)=\lim_j \operatorname{proj} X_j^{(p,q)}(\varphi)\,.$$

We put $\varphi_q=\varphi+2(n-q)\log(1+|z|^2)$, and replace the above representation by

$$\mathscr{X}_\varphi^{(p,q)}(V)=\lim_j \operatorname{proj} X_j^{(p,q)}(\varphi_q)\,.$$

(See Remark at the end of a) in § 2.1.)

b) By Theorem 4.4.2 in Hörmander ([7], p. 94), the following sequence

$$(2.2.2)\qquad X_j^{(p,0)}(\varphi_0)\xrightarrow{\bar\partial_j^{(1)}} X_j^{(p,1)}(\varphi_1)\xrightarrow{\bar\partial_j^{(2)}}\cdots\xrightarrow{\bar\partial_j^{(n)}} X_j^{(p,n)}(\varphi_n)\longrightarrow 0$$

is exact for each j. Since the kernel of a closed operator is closed, $\bar\partial_j^{(q)}$ ($q=1,\cdots,n$) has a closed range for each j. Hence from the closed range theorem for Banach spaces, we conclude that the dual sequence of (2.2.2):

$$(2.2.3)\qquad Y_j^{(p,0)}(\varphi_0)\xleftarrow[\vartheta_j^{(1)}]{} Y_j^{(p,1)}(\varphi_1)\xleftarrow[\vartheta_j^{(2)}]{}\cdots\xleftarrow[\vartheta_j^{(n)}]{} Y_j^{(p,n)}(\varphi_n)\longleftarrow 0$$

is exact, and that $\vartheta_j^{(q)}$ ($q=1,\cdots,n$) has a closed range for each j.

c) Consider the following dual complexes

$$\begin{array}{llllll}(2.2.1) & \mathscr{X}_\varphi^{(p,0)}(V) \xrightarrow{\overline{\partial^{(1)}}} & \mathscr{X}_\varphi^{(p,1)}(V)\xrightarrow{\overline{\partial^{(2)}}}\cdots & \xrightarrow{\overline{\partial^{(n)}}} \mathscr{X}_\varphi^{(p,n)}(V) & \longrightarrow 0 \\ & \updownarrow & \updownarrow & \updownarrow & \\ (2.2.4) & \mathscr{Y}_{\varphi,comp}^{(p,0)}(V)\xleftarrow[\vartheta^{(1)}]{} & \mathscr{Y}_{\varphi,comp}^{(p,1)}(V)\xleftarrow[\vartheta^{(2)}]{}\cdots & \xleftarrow[\vartheta^{(n)}]{} \mathscr{Y}_{\varphi,comp}^{(p,n)}(V) & \longleftarrow 0\,. \end{array}$$

In order to prove the exactness of the sequence (2.2.1), we will show that the sequence (2.2.4) is exact and that $\vartheta^{(q)}$ ($q=1,\cdots,n$) has a closed range. Then by the Serre-Komatsu duality theorem (see Theorem 19 in Komatsu [13], p. 381), we have the exactness of (2.2.1).

d) (The exactness of the sequence (2.2.4)) Let $\vartheta^{(q)}g=0$ and $g\in Y_j^{(p,q)}(\varphi)$. Then by Proposition 2.1.2, we have $g\in D_{\vartheta_j^{(q)}}$ and $\vartheta_j^{(q)}g=\vartheta^{(q)}g=0$.

Hence by the exactness of the sequence (2.2.3), there exists a $v \in Y_j^{(p,q+1)}(\varphi_{q+1})$ such that $\vartheta_j^{(q+1)}v=g$. We note that $v \in D_{\vartheta_j^{(q+1)}}$ implies $v \in D_{\vartheta^{(q+1)}}$ and $\vartheta^{(q+1)}v=\vartheta_j^{(q+1)}v$. Thus we have $\vartheta^{(q+1)}v=g$. This shows the exactness of the sequence (2.2.4).

e-1) (Closedness of $\operatorname{Im}\vartheta^{(q)}$ in the case $q\geqq 2$) If $q\geqq 2$, then we have $\operatorname{Im}\vartheta^{(q)}=\operatorname{Ker}\vartheta^{(q-1)}$ by the exactness of the sequence (2.2.4). Since $\vartheta^{(q-1)}$ is closed, $\operatorname{Ker}\vartheta^{(q-1)}$ is closed. Thus we have the closedness of $\operatorname{Im}\vartheta^{(q)}$ for $q\geqq 2$.

e-2) (Closedness of $\operatorname{Im}\vartheta^{(q)}$ in the case $q=1$) First we will find a sufficient condition to obtain the closedness of $\operatorname{Im}\vartheta^{(1)}$ by a functional analytic consideration.

The Fréchet-Kômura space $\mathscr{X}_\varphi^{(p,0)}(V)$ is reflexive by Theorem 1 in Komatsu ([13], p. 369). Hence in the strong dual space $\mathscr{Y}_{\varphi,comp}^{(p,0)}(V)$ of $\mathscr{X}_\varphi^{(p,0)}(V)$, the closedness, the weak closedness and the weak* closedness are equivalent for a convex set, especially for a subspace. On the other hand, a Fréchet space is fully complete by the Banach theorem (see, for example, Bourbaki ([1], p. 75). Therefore to obtain the closdness of $\operatorname{Im}\vartheta^{(1)}$, it is sufficient to show the weak closedness $\operatorname{Im}\vartheta^{(1)}\cap N^\circ$ in N° for any neighborhood N of $0\in\mathscr{X}_\varphi^{(p,0)}(V)$. Here N° denotes the polar set of N:

$$N^\circ=\{g\in\mathscr{Y}_{\varphi,comp}^{(p,0)}(V);\ \operatorname{Re}\langle f, g\rangle\geqq -1 \text{ for all } f\in N\}\,.$$

(As for the above argument, see Komatsu [2], p. 168 for example.) Since N° is bounded, there exists a bounded set B_j in some $Y_j^{(p,0)}(\varphi_0)$ such that $N^\circ=\rho_j'(B_j)$, by Theorem 6 in Komatsu ([13], p. 372). Here we note that ρ_j' gives a weak homeomorphism and that we have

$$\begin{aligned}\rho_j'^{-1}(\operatorname{Im}\vartheta^{(1)}\cap N^\circ)&=\rho_j'^{-1}(\operatorname{Im}\vartheta^{(1)})\cap\rho_j'^{-1}(N^\circ)\\&=\rho_j'^{-1}(\operatorname{Im}\vartheta^{(1)})\cap B_j\,.\end{aligned}$$

Therefore to obtain the weak closedness of $\operatorname{Im}\vartheta^{(1)}\cap N^\circ$ in N°, it is sufficient to show the weak closedness of $\rho_j'^{-1}(\operatorname{Im}\vartheta^{(1)})$ in $Y_j^{(p,0)}(\varphi_0)$. On the other hand, we found the closedness of $\operatorname{Im}\vartheta_j^{(1)}$ in b) of the proof. Since $\operatorname{Im}\vartheta_j^{(1)}$ is a subspace, $\operatorname{Im}\vartheta_j^{(1)}$ is also weakly closed.

Thus we have found that the condition $\rho_j'^{-1}(\operatorname{Im}\vartheta^{(1)})=\operatorname{Im}\vartheta_j^{(1)}$ $(j=1, 2, \cdots)$ is a sufficient condition for the closedness of $\operatorname{Im}\vartheta^{(1)}$. We will show this fact as

LEMMA 2.2.2. *Under the assumptions in Theorem* 2.2.1, *consider the operators*

$$\vartheta^{(q)}: \mathscr{Y}^{(p,q-1)}_{\varphi,comp}(V) \longleftarrow \mathscr{Y}^{(p,q)}_{\varphi,comp}(V)$$
$$\vartheta^{(q)}_j: Y^{(p,q-1)}_j(\varphi_{q-1}) \longleftarrow Y^{(p,q)}_j(\varphi_q) \quad (j=1, 2, \cdots).$$

Then we have $\rho_j'^{-1}(\mathrm{Im}\, \vartheta^{(q)}) = \mathrm{Im}\, \vartheta^{(q)}_j$ $(j=1, 2, \cdots)$.

PROOF. We found $\mathrm{Im}\, \vartheta^{(q)}_j \subset \rho_j'^{-1}(\mathrm{Im}\, \vartheta^{(q)})$ in Proposition 2.1.2. Hence we have only to show $\mathrm{Im}\, \vartheta^{(q)}_j \supset \rho_j'^{-1}(\mathrm{Im}\, \vartheta^{(q)})$.

We will show that the conditions $g \in D_{\vartheta^{(q)}} \cap Y^{(p,q)}_k(\varphi_q)$ and $\vartheta^{(q)} g \in Y^{(p,q-1)}_j(\varphi_{q-1})$ imply $\vartheta^{(q)} g \in \mathrm{Im}\, \vartheta^{(q)}_j$. If $k \leqq j$, then we have $\vartheta^{(q)} g \in \mathrm{Im}\, \vartheta^{(q)}_j$ by Proposition 2.1.2. Therefore we have only to show in the case $k > j$.

a) We have $\vartheta^{(q)} g = \vartheta^{(q)}_k g$ by Proposition 2.1.2.

b) We recall $K_j = \bar{U}_{c_j}$ $(U_{c_j} = \{z \in V \cap C^n; p(z) < c_j\})$ and $\sup_{z \in K \cap C^n} p(z) < \infty$ for any $K \subset V$. Since $\mathrm{supp}\, \vartheta^{(q)}_k g = \mathrm{supp}\, \vartheta^{(q)} g \subset K_j \cap C^n$, there exists a $\tilde{g} \in D_{\vartheta_k^{(q)}}$ such that $\vartheta^{(q)}_k \tilde{g} = \vartheta^{(q)} g$ and $\mathrm{supp}\, \tilde{g} \subset K_j \cap C^n$ by Proposition 2.3.2 in Hörmander ([6], p. 109).

c) Consider the following dual diagram:

$$\begin{array}{ccc} L^2_{(p,q-1)}(\mathring{K}_j \cap C^n; \varphi_{q-1} + (1/k)|z|) & \xrightarrow{\tilde{\bar{\partial}}^{(q)}_k} & L^2_{(p,q)}(\mathring{K}_j \cap C^n; \varphi_q + (1/k)|z|) \\ \updownarrow & & \updownarrow \\ L^2_{(p,q-1)}(\mathring{K}_j \cap C^n; -\varphi_{q-1} - (1/k)|z|) & \xleftarrow[\tilde{\vartheta}^{(q)}_k]{} & L^2_{(p,q)}(\mathring{K}_j \cap C^n; -\varphi_q - (1/k)|z|) . \end{array}$$

Then we have $\tilde{\vartheta}^{(q)}_k \tilde{g} = \vartheta^{(q)} g$ in a similar way to the proof of Proposition 2.1.2.

d) We approximate $\tilde{\vartheta}^{(q)} g = \vartheta^{(q)}_k \tilde{g} \in Y^{(p,q-1)}_j(\varphi_{q-1})$ by elements of $\mathrm{Im}\, \vartheta^{(q)}_j$ in the topology of $Y^{(p,q-1)}_j(\varphi_{q-1})$:

Put $b_\nu(z) = \exp(-(1/\nu) z^2)$ $(z^2 = z_1^2 + \cdots + z_n^2)$. Since V is acute, we have $\bar{b}_\nu \tilde{g} \in Y^{(p,q)}_j(\varphi_q)$ $(\nu = 1, 2, \cdots)$, where $\bar{b}_\nu$ denotes the complex conjugate of b_ν. Similarly we have, for each $f \in D_{\bar{\partial}^{(q)}_j}$, $b_\nu f \in D_{\tilde{\bar{\partial}}^{(q)}_k}$ and $\tilde{\bar{\partial}}^{(q)}_k (b_\nu f) = b_\nu \bar{\partial}^{(q)}_j f$. Then for any $f \in D_{\bar{\partial}^{(q)}_j}$, we have

$$\begin{aligned} \langle \bar{\partial}^{(q)}_j f, \bar{b}_\nu \tilde{g} \rangle_j &= \langle b_\nu \bar{\partial}^{(q)}_j f, \tilde{g} \rangle_k^{\sim} = \langle \tilde{\bar{\partial}}^{(q)}_k (b_\nu f), \tilde{g} \rangle_k^{\sim} = \langle b_\nu f, \tilde{\vartheta}^{(q)}_k \tilde{g} \rangle_k^{\sim} \\ &= \langle f, \bar{b}_\nu \tilde{\vartheta}^{(q)}_k \tilde{g} \rangle_j = \langle f, b_\nu \vartheta^{(q)} g \rangle_j , \end{aligned}$$

where $\langle \cdot, \cdot \rangle_k^{\sim}$ denotes the pairing between $L^2_{(p,\cdot)}(K_j \cap C^n; \varphi_\cdot + (1/k)|z|)$ and $L^2_{(p,\cdot)}(K_j \cap C^n; -\varphi_\cdot - (1/k)|z|)$. Hence we have $\bar{b}_\nu \tilde{g} \in D_{\vartheta^{(q)}_j}$ and $\vartheta^{(q)}_j (\bar{b}_\nu \tilde{g}) = \bar{b}_\nu \vartheta^{(q)} g$. Since $\vartheta^{(q)} g \in Y^{(p,q-1)}_j(\varphi_{q-1})$, we have

$$\vartheta^{(q)}_j (\bar{b}_\nu g) = \bar{b}_\nu \vartheta^{(q)} g \longrightarrow \vartheta^{(q)} g \quad \text{in} \quad Y^{(p,q-1)}_j(\varphi_{q-1}) \quad (\nu \longrightarrow \infty) .$$

This shows that $\vartheta^{(q)} g \in \overline{\mathrm{Im}\, \vartheta^{(q)}_j}$.

e) Since $\operatorname{Im}\vartheta_j$ is closed, we have $\vartheta^{(q)}g \in \operatorname{Im}\vartheta_j^{(q)}$.
Thus we have had $\rho_j'^{-1}(\operatorname{Im}\vartheta^{(q)}) = \operatorname{Im}\vartheta_j^{(q)}$. q.e.d.

At the same time we complete the proof of Theorem 2.2.1.

REMARK 1. In the course of the above proof of Theorem 2.2.1, we have proved the following:

Without the assumption of acuteness of an open set V in $\boldsymbol{Q}^n$, if there exists a strictly plurisubharmonic C^∞ function p on $V \cap \boldsymbol{C}^n$ satisfying the condition (P) in Definition 1.1.5, for any plurisubharmonic function φ, we have the following exact sequence:

$$\mathscr{H}_\varphi^{(p,1)}(V) \xrightarrow{\bar\partial} \mathscr{H}_\varphi^{(p,2)}(V) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{H}_\varphi^{(p,n)}(V) \longrightarrow 0 .$$

REMARK 2. The difficulty of the proof of Theorem 2.2.1 has been to show the closedness of $\operatorname{Im}\vartheta^{(1)}$. If we can find holomorphic functions which play the same role as b_ν (appeared in d) of the proof of Lemma 2.2.2) for wider classes of open sets than that of acute open sets, then we are able to show Theorem 2.2.1 for wider classes of pseudoconvex open sets.

REMARK 3. Kawai proved the exactness of the following sequence (Lemma 2.1.1 in Kawai [12]):

$$X \xrightarrow{\bar\partial} Y \xrightarrow{\bar\partial} Z ,$$

where

$$X = \lim_{\substack{\longleftarrow \\ j}} \operatorname{proj} L^2_{(p,q-1)}(\Omega; (1/j)\|z\| + 4\log(1+|z|^2) + \varphi(z))$$

$$Y = \lim_{\substack{\longleftarrow \\ j}} \operatorname{proj} L^2_{(p,q)}(\Omega; (1/j)\|z\| + 2\log(1+|z|^2) + \varphi(z))$$

$$Z = \lim_{\substack{\longleftarrow \\ j}} \operatorname{proj} L^2_{(p,q+1)}(\Omega; (1/j)\|z\| + \varphi(z)) ,$$

Ω is a pseudoconvex open set in $\boldsymbol{C}^n$, φ is a plurisubharmonic function on Ω and $\|z\|$ is a slight modification of $|z|$ near $\{z_j=0$ for some $j\}$ so as to be C^∞ and convex. Using this lemma, he proved the Cartan Theorem B for the sheaf $\tilde{\mathscr{O}}$ in the form of the Čech cohomology groups and constructed a soft (L^2-) resolution of that sheaf.

On the other hand our method gives a direct construction of a soft (L^2-) resolution of the sheaf $\mathscr{O}_{\mathrm{inc},\varphi}$ and a direct proof of the Cartan Theorem B for this sheaf at the same time. (See § 3.1.)

The reader should notice the difference of growth conditions between the spaces X, Y, Z and the spaces $\mathscr{H}_\varphi^{(p,q)}(W)$: The spaces X, Y and Z are the intersections of the spaces of globally square summable differential forms on Ω with respect to the weighted measures $(\varphi(z)+(1/j)|z|)d\lambda(z)$. On the other hand the spaces $\mathscr{X}_\varphi^{(p,q)}(W)$ are the intersections of the spaces of (in a certain sense) locally square summable functions on $W\cap C^n$ with respect to the weighted measures $(\varphi(z)+(1/j)|z|)d\lambda(z)$.

§ 3. Vanishing theorems of cohomology groups with values in the sheaves $\mathscr{O}_{\mathrm{inc},\varphi}$ and $\mathscr{O}_{\mathrm{dec}}$.

3.1. The Cartan Theorem B for the sheaves $\mathscr{O}_{\mathrm{inc},\varphi}$ and $\mathscr{O}_{\mathrm{dec}}$.

DEFINITION 3.1.1. For a continuous function φ on C^n, we denote by $\mathscr{X}_\varphi^{1,(p,q)}$ the sheaf on Q^n whose section module $\mathscr{X}_\varphi^{1,(p,q)}(W)$ over an open set W in Q^n is given by the following:

$$\mathscr{X}_\varphi^{1,(p,q)}(W)=\{f\in\mathscr{X}_\varphi^{(p,q)}(W);\ \bar\partial f\in\mathscr{X}_\varphi^{(p,q+1)}(W)\}\ ,$$

where $\bar\partial f$ is defined in the sense of distributions.

DEFINITION 3.1.2. We denote by $\mathscr{Y}_\varphi^{1,(p,q)}$ the sheaf on Q^n whose section module $\mathscr{Y}_\varphi^{1,(p,q)}(W)$ over an open set W in Q^n is given by the following:

$$\mathscr{Y}_\varphi^{1,(p,q)}(W)=\{g\in\mathscr{Y}_\varphi^{(p,q)}(W);\ \bar\partial g\in\mathscr{Y}_\varphi^{(p,q+1)}(W)\}\ ,$$

where $\bar\partial g$ is defined in the sense of distributions.

REMARK 1. We note that $\mathscr{X}_\varphi^{1,(p,q)}$ and $\mathscr{Y}_\varphi^{1,(p,q)}$ are soft sheaves on Q^n.

REMARK 2. The restrictions of the sheaves $\mathscr{X}_\varphi^{1,(p,q)}$ and $\mathscr{Y}_\varphi^{1,(p,q)}$ to C^n coinside with the sheaf $\mathscr{H}^{(p,q)}$ on C^n, where $\mathscr{H}^{(p,q)}$ denotes the sheaf on C^n whose section module $\mathscr{H}^{(p,q)}(W)$ over an open set W in C^n is given by the following:

$$\mathscr{H}^{(p,q)}(W)=\{f\in\mathscr{L}_{\mathrm{loc}}^{2,(p,q)}(W);\ \bar\partial f\in\mathscr{L}_{\mathrm{loc}}^{2,(p,q+1)}(W)\}\ .$$

PROPOSITION 3.1.1. *Let φ be a plurisubharmonic function on C^n of linear variation. Then we have the following soft resolution of the sheaf $\mathscr{O}_{\mathrm{inc},\varphi}$ on Q^n:*

$$(3.1.1)\qquad 0\longrightarrow\mathscr{O}_{\mathrm{inc},\varphi}\longrightarrow\mathscr{X}_{2\varphi}^{1,(0,0)}\xrightarrow{\bar\partial}\mathscr{X}_{2\varphi}^{1,(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{X}_{2\varphi}^{1,(0,n)}\longrightarrow 0\ .$$

PROOF. a) (Exactness of (3.1.1) on C^n) The restriction of (3.1.1) to C^n is the following sequence:

$$(3.1.1)'\qquad 0\longrightarrow\mathscr{O}\longrightarrow\mathscr{H}^{(0,0)}\xrightarrow{\bar\partial}\mathscr{H}^{(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{H}^{(0,n)}\longrightarrow 0\ .$$

This is the ordinary L^2-resolution of the sheaf $\mathscr{O}$ of holomorphic functions on C^n. Therefore the sequence (3.1.1) is exact on C^n.

b) (Exactness of (3.1.1) on S_∞^{2n-1}) We have only to show the exactness of (3.1.1) at $e_\infty^1=(1, 0, \cdots, 0)_\infty \in S_\infty^{2n-1}$. Because from the exactness at e_∞^1, we can show the exactness of (3.1.1) at each point in S_∞^{2n-1} by using of linear unitary transforms of C^n.

b-1) The exactness at the term $\mathscr{L}_{2\varphi}^{1,(0,0)}$ at e_∞^1 follows from the ellipticity of $\bar{\partial}^{(1)}$, the assumption of linear variationality of φ and the estimation of sup-norms by L^2-norms for holomorphic functions:

Let $f \in \mathscr{L}_{2\varphi,e_\infty^1}^{1,(0,0)}$ and $\bar{\partial} f=0$. Then there exists an open neighborhood W of e_∞^1 in Q^n such that $f \in \mathscr{O}(W \cap C^n)$ and

$$\int_{K\cap C^n} |f(w)|^2 \exp(-2\varphi(w)-\varepsilon|w|)d\lambda<\infty$$

holds for any $K \subset W$ and any $\varepsilon>0$. On the other hand since φ is of linear variation, there exists a constant $A>0$ such that

$$\varphi(z)-A \leqq \varphi(w) \leqq \varphi(z)+A \quad \text{for any} \quad z, w \in C^n \quad \text{with} \quad |w-z| \leqq 1$$

holds. Hence if we choose compact neighborhoods L and L' of e_∞^1 in Q^n so that $L \subset L' \subset W$ and $\mathrm{dist}(L\cap C^n, \partial L' \cap C^n) \geqq 2$, we have the following estimation for all $z \in L\cap C^n$ and all $\varepsilon>0$:

$$\begin{aligned}
&\sup_{w\in B_z} |f(w)| \exp(-\varphi(w)-\varepsilon|w|)\\
&\qquad \leqq \exp(-\varphi(z)-\varepsilon|z|+A+\varepsilon) \sup_{w\in B_z} |f(w)|\\
&\qquad \leqq B\exp(-\varphi(z)-\varepsilon|z|+A+\varepsilon)\left[\int_{B_z'} |f(w)|^2 d\lambda(w)\right]^{1/2}\\
&\qquad = Be^{A+\varepsilon}\left[\int_{B_z'} |f(w)|^2 \exp(-2\varphi(z)-2\varepsilon|z|)d\lambda(w)\right]^{1/2}\\
&\qquad \leqq Be^{A+\varepsilon}\left[\int_{B_z'} |f(w)|^2 \exp(-2\varphi(w)-2\varepsilon|w|+4(A+\varepsilon))d\lambda(w)\right]^{1/2}\\
&\qquad \leqq B\exp(3(A+\varepsilon))\left[\int_{B_z'} |f(w)|^2 \exp(-2\varphi(w)-\varepsilon|w|)d\lambda(w)\right]^{1/2}\\
&\qquad <\infty ,
\end{aligned}$$

where we put $B_z=\{w \in C^n; |w-z| \leqq 1\}$ and $B_z'=\{w \in C^n; |w-z| \leqq 2\}$, and B is a positive constant depend only on $\mathrm{dist}(B_z, \partial B_z')$. Moreover, from the arbitrariness of $z \in L\cap C^n$ in the above estimation, we have

$$\sup_{z\in L\cap C^n} |f(z)| \exp(-\varphi(z)-\varepsilon|z|)<\infty$$

for all $\varepsilon>0$. Thus we have $f \in \mathscr{O}_{\mathrm{inc},\varphi,e_\infty^1}$, and the exactness at the term

$\mathscr{X}_{2\varphi}^{1,(0,0)}$ at e_∞^1 was shown.

b-2) The exactness at the term $\mathscr{X}_{2\varphi}^{1,(0,q)}$ $(q\geqq 1)$ at e_∞^1 follows from Theorem 2.2.1. Because e_∞^1 has a fundamental system of neighborhoods consisting of $\mathcal{O}_{\rm inc}$-pseudoconvex open sets. (See Example (a) in § 1.1.) q.e.d.

THEOREM 3.1.2. *Let V be an $\mathcal{O}_{\rm inc}$-pseudoconvex open set in Q^n, and φ plurisubharmonic function on C^n of linear variation. Then we have*

$$H^q(V; \mathcal{O}_{{\rm inc},\varphi})=0 \quad (q\geqq 1)\,.$$

PROOF. This is an immediate consequence of Proposition 3.1.1 and Theorem 2.2.1. q.e.d.

Now we go on to the case of the sheaf $\mathcal{O}_{\rm dec}$.

LEMMA 3.1.3. *For $\delta>0$ and $A>0$, we put*

$$U_{\delta,A}=\{z\in C^n;\, |{\rm Im}\, z|^2<\delta^2|{\rm Re}\, z|^2+A^2\}\,.$$

Assume $0<\delta<1$, then for any $0<\varepsilon<\sqrt{1-\delta^2}/\sqrt{2}\,A$, we have the following estimation from below:

$$|\cosh(\varepsilon\sqrt{z^2})|\geqq C_{\delta,A,\varepsilon}\exp(\varepsilon\sqrt{1-\delta^2}|z|/\sqrt{1+\delta^2}) \quad (z\in U_{\delta,A}) \tag{3.1.2}$$

for some constant $C_{\delta,A,\varepsilon}>0$. Here we put $z^2=z_1^2+\cdots+z_n^2$. Also we have the following estimation from above:

$$|\cosh(\varepsilon\sqrt{z^2})|\leqq\exp(\varepsilon|z|) \quad (z\in C^n)\,. \tag{3.1.3}$$

REMARK. Since $\cosh w$ is an even function, $\cosh(\varepsilon\sqrt{z^2})$ defines an entire function on C^n.

PROOF OF LEMMA 3.1.3. This is a consequence of an explicit calculation. q.e.d.

PROPOSITION 3.1.4. *For the sheaf $\mathcal{O}_{\rm dec}$, we have the following soft resolution:*

$$0\longrightarrow\mathcal{O}_{\rm dec}\longrightarrow\mathscr{Y}^{1,(0,0)}\xrightarrow{\bar\partial}\mathscr{Y}^{1,(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{Y}^{1,(0,n)}\longrightarrow 0\,. \tag{3.1.4}$$

PROOF. The restriction of the sequence (3.1.4) to C^n is the L^2-resolution (3.1.1)′ of the sheaf $\mathcal{O}$ of holomorphic functions on C^n. Therefore it is sufficient to show the exactness of (3.1.4) on S_∞^{2n-1}. To show this we have only to prove the exactness of (3.1.4) at e_∞^1. Because from the exactness at e_∞^1, we can show the exactness of (3.1.4) at each point in S_∞^{2n-1} by using of linear unitary transforms of C^n.

Exactness at the term $\mathscr{Y}^{1,(0,0)}$ at e^1_∞ follows from the ellipticity of the operator $\bar\partial^{(1)}$ and the estimation sup-norms by L^2-norms for holomorphic functions as in the proof of Proposition 3.1.1. For simplicity, we denote e^1_∞ by e_∞.

Let $g \in \mathscr{Y}^{1,(0,q)}_{e_\infty}$ ($q \geqq 1$) and $\bar\partial g = 0$. Then we can find a $v \in Y^{1,(0,q-1)}_{e_\infty}$ such that $\bar\partial v = g$ as follows. Put $g_\varepsilon(z) = \cosh(\varepsilon\sqrt{z^2})$. Then there exists an $\varepsilon > 0$ so that $g_\varepsilon g \in \mathscr{X}^{1,(0,q)}_{e_\infty}$ by Lemma 3.1.3. We note $\bar\partial(g_\varepsilon g) = g_\varepsilon \bar\partial g = 0$. Therefore there exists a $w \in \mathscr{X}^{1,(0,q-1)}_{e_\infty}$ such that $\bar\partial w = g_\varepsilon g$ by Proposition 3.1.1. Hence we have $\bar\partial(w/g_\varepsilon) = g$ and $w/g_\varepsilon \in \mathscr{Y}^{1,(0,q-1)}_{e_\infty}$ again by Lemma 3.1.3. Thus we have proved the exactness of the sequence (3.1.4) on S^{2n-1}_∞. q.e.d.

THEOREM 3.1.5. *Let K be a compact set in Q^n which has a fundamental system of neighborhoods consisting of $\mathscr{O}_{\mathrm{inc}}$-pseudoconvex open sets in Q^n. Then we have*

$$H^q(K; \mathscr{O}_{\mathrm{dec}}) = 0 \quad (q \geqq 1) .$$

PROOF. The proof goes similarly to that of Proposition 3.1.4 by using Theorem 3.1.2 instead of Proposition 3.1.1. q.e.d.

Here the author thanks Professor M. Morimoto, who suggest the author to use the function $\cosh(\varepsilon\sqrt{z^2})$ as a damping function.

REMARK 1. By Remark 1 in § 2.2 and Proposition 3.1.1, the following proposition is valid;

Without the assumption of acuteness for an open set V in Q^n, under the assumption of existence of a strictly plurisubharmonic C^∞ function p on $V \cap C^n$ satisfying the conditions (P) in Definition 1.1.5, we have

$$H^q(V; \mathscr{O}_{\mathrm{inc},\varphi}) = 0 \quad (q \geqq 2)$$

for any pulrisubharmonic function φ on C^n of linear variation.

REMARK 2. We do not know whether following statements (*), (**) and (***) are valid or not:

(*) $H^q(V; \mathscr{O}_{\mathrm{dec}}) = 0$ ($q \geqq 1$) for any $\mathscr{O}_{\mathrm{inc}}$-pseudoconvex open set V in Q^n.

(**) $H^q(K; \mathscr{O}_{\mathrm{dec},\varphi}) = 0$ ($q \geqq 1$) for any compact set K in Q^n which has a fundamental system of neighborhoods consisting of $\mathscr{O}_{\mathrm{inc}}$-pseudoconvex open sets and any plurisubharmonic function φ on C^n of linear variation.

(***) $H^q(V; \mathscr{O}_{\mathrm{dec},\varphi}) = 0$ ($q \geqq 1$) for any $\mathscr{O}_{\mathrm{inc}}$-pseudoconvex open set V in Q^n and any plurisubharmonic function φ on C^n of linear variation.

As for the first statements, since $\mathscr{O}_{\mathrm{dec}}(V)=\lim\mathrm{proj}_{K\subset V}\mathscr{O}_{\mathrm{dec}}(K)$ is a space of projective limit of dual Fréchet-Kômura spaces. We can not use the usual argument of Mittag-Lefler type by using of an approximation theorem.

As for the second and third statements, we do not know whether the following sequence constitute a soft resolution or not for the sheaf $\mathscr{O}_{\mathrm{dec},\varphi}$ for any φ such as above:

$$0\longrightarrow\mathscr{O}_{\mathrm{dec},\varphi}\longrightarrow\mathscr{Y}_{\varphi}^{(0,0)}\xrightarrow{\bar\partial}\mathscr{Y}_{\varphi}^{(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{Y}_{\varphi}^{(0,n)}\longrightarrow 0\,.$$

3.2. The Malgrange theorem for the sheaf $\mathscr{O}_{\mathrm{inc},\varphi}$.

Next we go on to the Malgrange theorem for the sheaf $\mathscr{O}_{\mathrm{inc},\varphi}$:

THEOREM 3.2.1. *Let W be an acute open set in $\boldsymbol{Q}^n$, and φ a plurisubharmonic function on $\boldsymbol{C}^n$ of linear variation. Then we have*

$$H^n(W;\mathscr{O}_{\mathrm{inc},\varphi})=0\,.$$

PROOF. Since we have the soft resolution (3.1.1) for the sheaf $\mathscr{O}_{\mathrm{inc},\varphi}$, it is sufficient to show the sequence

$$\mathscr{X}_{\psi}^{1,(0,n-1)}(W)\xrightarrow{\bar\partial^{(n)}}\mathscr{X}_{\psi}^{1,(0,n)}(W)\longrightarrow 0$$

is exact, where we put $\psi=2\varphi$. In particular it is sufficient to prove

$$\mathscr{X}_{\psi}^{(0,n-1)}(W)\xrightarrow{\bar\partial^{(n)}}\mathscr{X}_{\psi}^{(0,n)}(W)\longrightarrow 0 \tag{3.2.1}$$

is exact. To show this, as in the proof of Theorem 2.2.1, we consider the following dual complex:

$$\begin{array}{ccccc}\mathscr{X}_{\psi}^{(0,n-1)}(W) & \xrightarrow{\bar\partial^{(n)}} & \mathscr{X}_{\psi}^{(0,n)}(W) & \longrightarrow & 0\\ \updownarrow & & \updownarrow & & \\ \mathscr{Y}_{\psi,comp}^{(0,n-1)}(W) & \xleftarrow[\vartheta^{(n)}]{} & \mathscr{Y}_{\psi,comp}^{(0,n)}(W) & \longleftarrow & 0\end{array} \tag{3.2.2}$$

and prove the exactness of (3.2.2) and the closedness of $\mathrm{Im}\,\vartheta^{(n)}$. Then by the Serre-Komatsu duality theorem (Theorem 19 in Komatsu [13], p. 381), we are able to obtain the exatness of (3.2.1).

a) (Injectivity of $\vartheta^{(n)}$) The injectivity of $\vartheta^{(n)}$ follows from following two facts: If $\vartheta^{(n)}g=0$, then g is anti-holomorphic (i.e., $\bar g$ is holomorphic). If $g\in\mathscr{Y}_{\psi,comp}^{(0,n)}(W)$ and $\vartheta^{(n)}g=0$, then we have $g=0$ by the uniquess of analytic continuation.

b) (Closedness of $\mathrm{Im}\,\vartheta^{(n)}$) To show the closedness of $\mathrm{Im}\,\vartheta^{(n)}$, using the same argument as in e) of the proof of Theorem 2.2.1, we will prove the closedness of $\operatorname{Im}\vartheta_j^{(n)}$ and $\rho_j'^{-1}(\operatorname{Im}\vartheta^{(n)})=\operatorname{Im}\vartheta_j^{(n)}$. We will prove these facts as Lemma 3.2.2 and Lemma 3.2.3.

LEMMA 3.2.2. *Let U be an acute open set in Q^n, φ a plurisubharmonic function on C^n. We define the following spaces of differential forms:*

$$\begin{aligned} X^{(n-1)} &= L^2_{(p,n-1)}(U\cap C^n;\ \varphi(z)+2\log(1+|z|^2))\ , \\ X^{(n)} &= L^2_{(p,n)}(U\cap C^n;\ \varphi(z))\ , \\ Y^{(n-1)} &= L^2_{(p,n-1)}(U\cap C^n;\ -\varphi(z)-2\log(1+|z|^2))\ , \\ Y^{(n)} &= L^2_{(p,n)}(U\cap C^n;\ -\varphi(z))\ . \end{aligned}$$

Consider the following dual diagram:

$$\begin{array}{ccc} X^{(n-1)} & \xrightarrow{\bar\partial^{(n)}} & X^{(n)} \\ \updownarrow & & \updownarrow \\ Y^{(n-1)} & \xleftarrow[\vartheta^{(n)}]{} & Y^{(n)}\ . \end{array}$$

If $U\cap C^n$ has a C^∞ boundary, then $\vartheta^{(n)}$ has a closed range.

PROOF. For $\delta>0$ and $a>0$, we put

$$\begin{aligned} U_{\delta,a} &= \{z\in C^n;\ |\operatorname{Im} z|^2<\delta^2|\operatorname{Re} z|^2+a^2\}\ , \\ V_{\delta,a} &= \mathring{\bar U}_{\delta,a}\ . \end{aligned}$$

By the acuteness of U there exist a δ $(0<\delta<1)$ and a $a>0$ such that $U_{\delta,a}\subset U$ holds. We put

$$\begin{aligned} q_{\delta,a}(z) &= 1/(a^2+\delta^2|\operatorname{Re} z|^2-|\operatorname{Im} z|^2)\ , \\ p(z) &= q_{\delta,a}(z)+|z|^2\ . \end{aligned}$$

We note that functions $q_{\delta,a}$ and p are plurisubharmonic on $U_{\delta,a}$, that $U_{\delta,a}$ is pseudoconvex and that there exists a constant $c>0$ so that $K_c=\overline{\{z\in C^n;\ q_{\delta,a}(z)\leqq c\}}\supset U$.

Next we define the spaces $\tilde X^{(n-1)}$, $\tilde X^{(n)}$, $\tilde Y^{(n-1)}$ and $\tilde Y^{(n)}$ modifying $X^{(n-1)}$, $X^{(n)}$, $Y^{(n-1)}$ and $Y^{(n)}$ respectively with the replacement of U by $V_{\delta,a}$. Consider the following dual diagram:

$$\begin{array}{ccc} \tilde X^{(n-1)} & \xrightarrow{\tilde{\bar\partial}^{(n)}} & \tilde X^{(n)} \\ \updownarrow & & \updownarrow \\ \tilde Y^{(n-1)} & \xleftarrow[\tilde\vartheta^{(n)}]{} & \tilde Y^{(n)}\ . \end{array}$$

Since $U\cap C^n$ has a C^∞ boundary, we have $D_{\bar\partial}\subset D_{\tilde{\bar\partial}}$, $D_\vartheta\subset D_{\tilde\vartheta}$ and $\tilde\vartheta g=\vartheta g$ for $g\in D_\vartheta$.

Let $\{\vartheta g_\nu\}$ be a sequence in $\mathrm{Im}\,\vartheta$ such that ϑg_ν converges to some h_0 in $Y^{(n-1)}$. Then for any $f\in\mathrm{Ker}\, D_{\tilde{\bar\partial}}$, we have

$$0=\langle\tilde{\bar\partial} f,\, g_\nu\rangle^\sim=\langle f,\, \tilde\vartheta g_\nu\rangle^\sim \longrightarrow \langle f,\, h_0\rangle^\sim \quad (\nu\longrightarrow\infty)\,.$$

Since $\mathrm{supp}\, h_0\subset K_c\cap C^n$, there exists a $g_0\in D_{\tilde\vartheta}$ such that $\tilde\vartheta g_0=h_0$ and $\mathrm{supp}\, g_0\subset K_c\cap C^n$ by Proposition 2.3.2 in Hörmander ([6], p. 109).

If we have $\mathrm{supp}\, g_0\subset \bar U\cap C^n$, then we can prove $g_0\in D_\vartheta$ and $\vartheta g_0=\tilde\vartheta g_0$ in a similar way to the proof of Proposition 2.1.2, because $U\cap C^n$ has a C^∞ boundary. Therefore we have only to prove $\mathrm{supp}\, g_0\subset U\cap C^n$.

Let $z\in(V_{\delta,a}\cap C^n)\backslash\bar U$ and B_z a small open ball in C^n with the center at z so that $B_z\cap\bar U=\phi$. It is sufficient to show

$$\langle\omega,\, g_0\rangle=0 \quad \text{for all} \quad \omega\in\mathscr{D}^{(q,n)}(B_z)\,.$$

We fix a differential form $\omega\in\mathscr{D}^{(q,n)}(B_z)$. Since $\mathscr{D}^{(p,n)}(B_z)\subset\mathrm{Ker}\,\tilde{\bar\partial}^{(n)}$, we can find a differential form $\eta\in\tilde X^{(p,n-1)}$ such that $\tilde{\bar\partial}\eta=\omega$ by Theorem 4.4.2 in Hörmander ([7], p. 94). Since $(\mathrm{supp}\, g_\nu)\cap(\mathrm{supp}\,\omega)=\phi$, we have

$$\langle\eta,\, \tilde\vartheta g_\nu\rangle^\sim=\langle\tilde{\bar\partial}\eta,\, g_\nu\rangle^\sim=\langle\omega,\, g_\nu\rangle^\sim=0$$

for all ν. On the other hand, since ϑg_ν converges to h_0 in $Y^{(n-1)}$ (and also in $\tilde Y^{(n-1)}$), we have

$$0=\langle\eta,\, \tilde\vartheta g_\nu\rangle^\sim=\langle\eta,\, \vartheta g_\nu\rangle^\sim\longrightarrow\langle\eta,\, h_0\rangle^\sim \quad (\nu\longrightarrow\infty)\,.$$

Thus we have

$$\langle\omega,\, g_0\rangle^\sim=\langle\tilde{\bar\partial}\eta,\, g_0\rangle^\sim=\langle\eta,\, \tilde\vartheta g_0\rangle^\sim=\langle\eta,\, h_0\rangle^\sim=0\,. \qquad \text{q.e.d.}$$

From Lemma 3.2.2, we immediately have the closedness of $\mathrm{Im}\,\vartheta_j^{(n)}$ which was in the question in b) of the proof of Theorem 3.2.1.

LEMMA 3.2.3. *Let U be an acute open set in Q^n, φ a plurisubharmonic fuuction on C^n. Then in the following dual diagram:*

$$\begin{array}{ccc}
\mathscr{H}_\varphi^{(p,n-1)}(U) & \xrightarrow{\bar\partial^{(n)}} & \mathscr{H}_\varphi^{(p,n)}(U) \\
\updownarrow & & \updownarrow \\
\mathscr{Y}_{\varphi,comp}^{(p,n-1)}(U) & \xrightarrow[\vartheta^{(n)}]{} & \mathscr{Y}_{\varphi,comp}^{(p,n)}(U)\,,
\end{array}$$

we have $\rho_j'^{-1}(\mathrm{Im}\,\vartheta^{(n)})=\mathrm{Im}\,\vartheta_j^{(n)} \quad (j=1, 2, \cdots)\,.$

PROOF. First, if it is necessary we replace K_j by $\hat{K}_j$ which was used in the definition of the topology of the space $\mathscr{X}_\varphi^{(p,q)}(U)$. Here we meaned by $\hat{K}_j$ the union of K_j and all of relatively compact sets in the connected components of $U\backslash K_j$. The reason why we replace K_j by $\hat{K}_j$ is that $\operatorname{supp}\vartheta^{(n)}g\subset\hat{K}_jC^n$ implies $\operatorname{supp}g\subset\hat{K}_j\cap C^n$ by the uniqueness of analytic continuation. We note that $\hat{K}_j$ has also C^∞ boundary and $\hat{K}_j\subset\hat{K}_{j+1}$.

Thus we have the following representation:

$$\begin{aligned}\mathscr{X}_\varphi^{(p,q)}(U)&=\lim_j\operatorname{proj} X_j^{(p,q)}(\varphi_q)\\&=\lim_j\operatorname{proj} L^2_{(p,q)}(\dot{\hat{K}}_j\cap C^n;\ \varphi_q(z)+(1/j)|z|)\ ,\\ \mathscr{Y}_{\varphi,comp}^{(p,q)}(U)&=\lim_j\operatorname{ind} Y_j^{(p,q)}(\varphi_q)\\&=\lim_j\operatorname{ind} L^2_{(p,q)}(\check{\hat{K}}_j\cap C^n;\ -\varphi_q(z)-(1/j)|z|)\ .\end{aligned}$$

Since $\operatorname{Im}\vartheta_j\subset\rho_j'^{-1}(\operatorname{Im}\vartheta)$ always holds, we have only to prove $\rho_j'^{-1}(\operatorname{Im}\vartheta^{(n)})\subset\operatorname{Im}\vartheta_j^{(n)}$.

Let $g\in D_\vartheta\cap Y_k^{(p,n)}(\varphi_n)$ and $\vartheta g\in Y_j^{(p,n-1)}(\varphi_{n-1})$. If $j\geq k$, then we have $\vartheta g=\vartheta_j g\in\operatorname{Im}\vartheta_j$ by Proposition 2.1.2. We are going to prove $\vartheta^{(n)}g\in\operatorname{Im}\vartheta_j^{(n)}$ for $j<k$ in five steps:

a) By Proposition 2.1.2, we have $\vartheta g=\vartheta_k g$.

b) Since $\operatorname{supp}\vartheta^{(n)}g\subset\hat{K}_j\cap C^n$, we have $\operatorname{supp}g\subset\hat{K}_j\cap C^n$ by the uniqueness of analytic continuation.

c) As in the proof of Lemma 3.2.2, we choose $0<\delta<1$ and $a>0$ so that $V_{\delta,a}\supset K_j$. Put

$$\begin{aligned}\tilde{X}_l^{(p,q)}&=L^2_{(p,q)}(V_{\delta,a}\cap C^n;\ \varphi_q(z)+(1/l)|z|)\\ \tilde{Y}_l^{(p,q)}&=L^2_{(p,q)}(V_{\delta,a}\cap C^n;\ -\varphi_q(z)-(1/l)|z|)\quad(l=1,2,\cdots)\ ,\end{aligned}$$

and consider the following dual diagram:

$$\begin{array}{ccc}\tilde{X}_l^{(p,n-1)}(\varphi_{n-1}) & \xrightarrow{\tilde{\partial}_l^{(n)}} & \tilde{X}_l^{(p,n)}(\varphi_n)\\ \updownarrow & & \updownarrow\\ \tilde{Y}_l^{(p,n-1)}(\varphi_{n-1}) & \xleftarrow[\tilde{\vartheta}_l^{(n)}]{} & \tilde{Y}_l^{(p,n)}(\varphi_n)\end{array}\quad(l=1,2,\cdots)\ .$$

We may regard as $D_{\tilde{\vartheta}_l}\supset D_{\vartheta_l}$. We note that $g\in D_{\vartheta_l}$ implies $g\in D_{\tilde{\vartheta}_l}$ and $\tilde{\vartheta}_l g=\vartheta_l g$.

Now our g belongs to D_{ϑ_k}. Put $b_\nu(z)=\exp(-(1/\nu)z^2)$ ($\nu=1,2,\cdots$, $z^2=z_1^2+\cdots+z_n^2$). Then we have $\bar{b}_\nu g\in Y_j^{(p,n)}(\varphi_n)$ ($\nu=1,2,\cdots$), since $V_{\delta,a}$ is acute. For all $f \in \operatorname{Ker} \tilde{\bar{\partial}}_j$, we have

$$0=\langle \tilde{\bar{\partial}}_j f, \bar{b}_\nu g\rangle_{\tilde{j}} = \langle b_\nu \tilde{\bar{\partial}}_j f, g\rangle_{\tilde{k}} = \langle \tilde{\bar{\partial}}_k (b_\nu f), g\rangle_{\tilde{k}}$$
$$= \langle b_\nu f, \tilde{\vartheta}_k g\rangle_{\tilde{k}} = \langle f, \bar{b}_\nu \vartheta_k g\rangle_{\tilde{j}} = \langle f, \bar{b}_\nu \vartheta g\rangle_j .$$

Since $\vartheta g \in Y_j^{(p,n-1)}(\varphi_{n-1}) \subset \tilde{Y}_j^{(p,n-1)}(\varphi_{n-1})$, we have by the Lebesgue dominated convergence theorem

$$\langle f, \bar{b}_\nu \vartheta g\rangle_{\tilde{j}} \longrightarrow \langle f, \vartheta g\rangle_{\tilde{j}} \quad (\nu \longrightarrow \infty) ,$$

for all $f \in D_{\tilde{\bar{\partial}}_j}$. Hence we have

$$\langle f, \vartheta g\rangle_{\tilde{j}} = 0$$

for all $f \in \operatorname{Ker} \tilde{\bar{\partial}}_j$. Here we note again $\operatorname{supp} \vartheta g \subset \hat{K}_j \subset V_{\delta,a}$. Then by Proposition 2.3.2 in Hörmander ([6], p. 109), there exists a $h \in \tilde{Y}_j^{(p,n)}(\varphi_n)$ such that

$$\tilde{\vartheta}_j h = \vartheta g , \qquad \operatorname{supp} h \subset V_{\delta,a} .$$

d) We note that

$$\vartheta(h-g) = \sum_{k=1}^{n} \frac{\partial}{\partial z_k}(h-g) dz_{\hat{k}} = 0$$

holds on C^n, and $\operatorname{supp}(h-g) \subset V_{\delta,a}$. Hence we have $h=g$ on $V_{\delta,a} \cap C^n$ by the uniqueness of analytic continuation. This and b) implies $g = h \in Y_j^{(p,n)}(\varphi_n)$.

e) Since $K_j \cap C^n$ has a C^∞ boundary, we have $g \in D_{\vartheta_j}$ and $\vartheta g = \vartheta_j g \in \operatorname{Im} \vartheta_j$ by Proposition 2.1.2. q.e.d.

Thus we have completed the proof of Theorem 3.2.1.

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