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Ambiguous Numbers over $P(\zeta_3)$ of Absolutely Abelian Extensions of Degree 6

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Let K be an abelian number field of degree 6 over the rational number field P and suppose K contains a primitive 3rd root ζ_3 of unity. Then the ambiguous number of $K/P(\zeta_3)$ is 3^{2t-2} when 3 unramifies in $K/P(\zeta_3)$ and it is 3^{2t-1} when 3 ramifies in $K/P(\zeta_3)$ where $t+1$ is the number of prime numbers which ramify in K/P.

Let Γ be the genus field of K/P, then Γ/K is unramified and the number of these ideal classes of K which are principal in Γ is a multiple of $(\Gamma: K)$ and it is larger than $(\Gamma: K)$ if $t \geqq 2$.

§1. Preliminaries.

Throughout this paper we shall use the following notations.

P The rational number field.

ζ_n A primitive n-th root of unity.

In this paper, the conductor of K is the minimal number f such that $K \subset P(\zeta_f)$ when K is abelian over P.

I_K	The group of ideals in K.
P_K	The group of principal ideals in K.
$h_K = [I_K: P_K]$	The class number of K.
$\mathfrak{A} \sim 1$	An ideal $\mathfrak{A}$ is principal in the field.
$\mathfrak{A} \sim \mathfrak{B}$	Ideals $\mathfrak{A}$ and $\mathfrak{B}$ are contained in a same ideal class in the field.

We call $\mathfrak{A} \in I_K$ an ambiguous ideal if $\mathfrak{A}^\sigma = \mathfrak{A}$ for all $\sigma \in \mathrm{Gal}(K/k)$ and we call $\mathfrak{A} \in I_K$ an ambiguous class ideal if $\mathfrak{A}^{1-\sigma} \in P_K$ for all $\sigma \in \mathrm{Gal}(K/k)$.

$A_{0,K/k}$	The subgroup of I_K/P_K consisting of classes each of which contains an ambiguous ideal for K/k.
$a_{0,K/k}$	The order of $A_{0,K/k}$.
$A_{K/k}$	The subgroup of I_K/P_K consisting of classes each of which

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contains an ambiguous class ideal for K/k.

$a_{K/k}$ The order of $A_{K/k}$.

We call $a_{0,K/k}$ the ambiguous number and we call $a_{K/k}$ the ambiguous class number.

E_K The group of all units in K.

Let K be an abelian number field over k. If a number field Γ satisfies the conditions

(i) Γ/k is abelian

(ii) no prime divisor in K ramifies in Γ/K

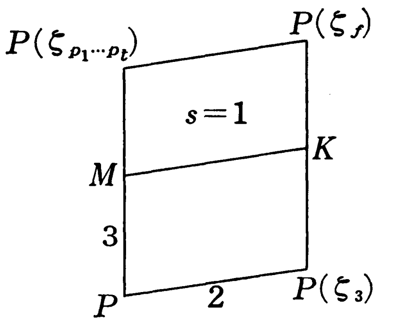
(iii) Γ is maximal under the conditions (i) and (ii),

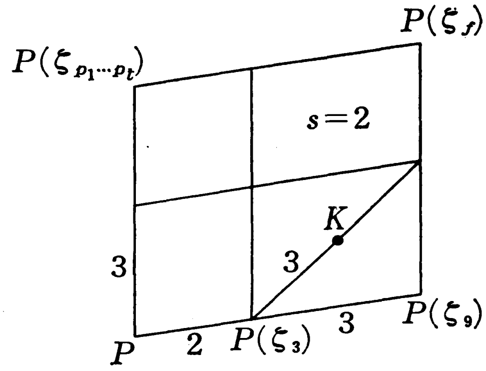
then we call Γ *the genus field of* K/k (in the wide sense).

§2. Ambiguous numbers.

THEOREM. *Let K be an abelian number field of degree 6 over P and suppose K contains ζ_3. Then the factorizaton of the conductor f of K into prime numbers is $f=3^s p_1 p_2\cdots p_t$, $s=1$ or 2. When $t\geqq 1$, we have $N_{K/P(\zeta_3)}E_K=\{\pm 1\}$ and $[E_{P(\zeta_3)}: N_{K/P(\zeta_3)}E_K]=3$. When $s=1$, we have $a_{0,K/P(\zeta_3)}=3^{2t-2}$. When $s=2$ and $t\geqq 1$, we have $a_{0,K/P(\zeta_3)}=3^{2t-1}$. When $s=2$ and $t=0$, we have $a_{0,K/P(\zeta_3)}=1$* and *$N_{K/P(\zeta_3)}E_K=\{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}=E_{P(\zeta_3)}$.*

PROOF. $p_i\equiv 1 \pmod 3$ for $i=1, 2, \cdots, t$.

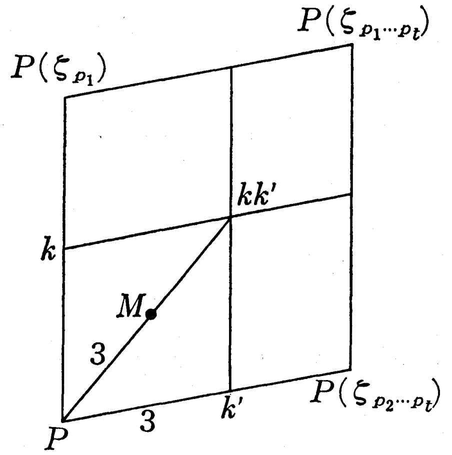


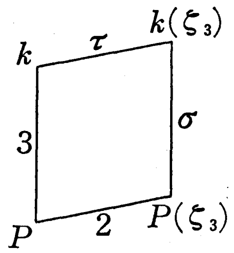


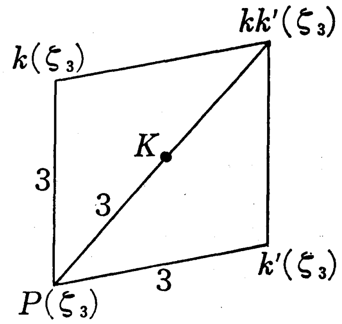
First we consider the case $s=1$. We put $M=P(\zeta_{p_1 p_2\cdots p_t})\cap K$. These factorizations of p_i into prime ideals are $p_i=\mathfrak{P}_i^3$ in M, $p_i=\mathfrak{p}_{i,1}\mathfrak{p}_{i,2}$ in $P(\zeta_3)$ since $p_i\equiv 1 \pmod 3$ and $\mathfrak{P}_i=\mathfrak{P}_{i,1}\mathfrak{P}_{i,2}$ in K. And $\mathfrak{p}_{i,j}=\mathfrak{P}_{i,j}^3$ $(j=1, 2)$ *in* K. Since the class number of $P(\zeta_3)$ is 1,

$$(1) \qquad \mathfrak{P}_i^3\sim 1 \quad \text{in} \quad M, \ \mathfrak{P}_{i,j}^3\sim 1 \quad \text{in} \quad K \ (i=1, 2, \cdots, t;\ j=1, 2)\,.$$

We put $k=P(\zeta_{p_1})\cap M(\zeta_{p_2\cdots p_t})$ and $k'=P(\zeta_{p_2\cdots p_t})\cap M(\zeta_{p_1})$. Then we have the following diagrams:







We have $k(\zeta_3)=P(\zeta_3, \sqrt[3]{b})$ for $b\in P(\zeta_3)$ and $b=\mathfrak{p}_{1,1}^{s_1}\mathfrak{p}_{1,2}^{s_2}\mathfrak{a}^3$ for $\mathfrak{a}\in I_{P(\zeta_3)}$, $(\mathfrak{a}, p_1)=1$, $3\nmid s_1$, $3\nmid s_2$, since $k(\zeta_3)/P(\zeta_3)$ is a cubic Kummer extension, $\mathfrak{p}_{1,1}\nmid 3$, $\mathfrak{p}_{1,2}\nmid 3$ and only $\mathfrak{p}_{1,1}$, $\mathfrak{p}_{1,2}$ ramify in $k(\zeta_3)/P(\zeta_3)$. Since $h_{P(\zeta_3)}=1$, we can put $b=\mathfrak{p}_{1,1}^{s_1}\mathfrak{p}_{1,2}^{s_2}$, $s_1, s_2=1$ or 2. We can put $\mathrm{Gal}\,(k(\zeta_3)/P(\zeta_3))=(\sigma)$ and $\sqrt[3]{b}^{\,1-\sigma}=\zeta_3$. We put $\mathrm{Gal}\,(k(\zeta_3)/k)=(\tau)$. Then $\sqrt[3]{b}^{\,(1+\tau)(1-\sigma)}=\zeta_3^{1+\tau}=1$. Therefore $\sqrt[3]{b}^{\,1+\tau}\in P(\zeta_3)\cap k=P$. Since $\sqrt[3]{b}^{\,1+\tau}=(\mathfrak{P}_{1,1}^{s_1}\mathfrak{P}_{1,2}^{s_2})^{1+\tau}=\mathfrak{P}_1^{s_1+s_2}$, we have $3|s_1+s_2$. So we can put $s_1=1$ and $s_2=2$. Therefore $b=\mathfrak{p}_{1,1}\mathfrak{p}_{1,2}^2$ and $\sqrt[3]{b}=\mathfrak{P}_{1,1}\mathfrak{P}_{1,2}^2=\mathfrak{P}_1\mathfrak{P}_{1,2}$. Hence we have $\mathfrak{P}_{1,2}\sim 1$ in $k(\zeta_3)$ from $\mathfrak{P}_1\sim 1$ in k and $\mathfrak{P}_1\mathfrak{P}_{1,2}\sim 1$ in $k(\zeta_3)$. Therefore $\mathfrak{P}_{1,1}\sim 1$ in $k(\zeta_3)$. Accordingly $\mathfrak{P}_{1,1}\sim 1$ in $kk'(\zeta_3)$. Since $kk'(\zeta_3)/K$ is unramified, $kk'(\zeta_3)$ is contained in the genus field Γ of K/P. Hence

(2) $$\mathfrak{P}_{i,j}\sim 1 \quad \text{in } \Gamma\,.$$

Ambiguous ideal group $A_{0,M/P}$ is generated $\mathfrak{P}_i$ and $A_{0,K/P(\zeta_3)}$ is generated by $\mathfrak{P}_{i,j}$. Since $\mathfrak{P}_i^3\sim 1$ in M, $A_{0,M/P}$ is an abelian group of the type $(3, 3, \cdots, 3)$. Since $a_{0,M/P}=3^{t-1}$, there exists

(3) $(r_1, r_2, \cdots, r_t)\neq(0, 0, \cdots, 0)$ such that $r_i=0$ or 1 or 2,
$i=1, 2, \cdots, t$ $\mathfrak{P}_1^{r_1}\mathfrak{P}_2^{r_2}\cdots\mathfrak{P}_t^{r_t}\sim 1$ in M.

Also we have:

$K=P(\zeta_3, \sqrt[3]{c})$, $c=\mathfrak{p}_{1,1}\mathfrak{p}_{1,2}^2\mathfrak{p}_{2,1}\mathfrak{p}_{2,2}^2\cdots\mathfrak{p}_{t,1}\mathfrak{p}_{t,2}^2$, $c\in P(\zeta_3)$ since $K/P(\zeta_3)$ is a cubic Kummer extension, $\mathfrak{p}_{i,j}\nmid 3$, only $\mathfrak{p}_{i,j}$ ramify in $K/P(\zeta_3)$, $h_{P(\zeta_3)}=1$ and $N_{K/M}\sqrt[3]{c}\in P$. Therefore we have:

(4) $$\sqrt[3]{c}=\mathfrak{P}_{1,1}\mathfrak{P}_{1,2}^2\mathfrak{P}_{2,1}\mathfrak{P}_{2,2}^2\cdots\mathfrak{P}_{t,1}\mathfrak{P}_{t,2}^2$$

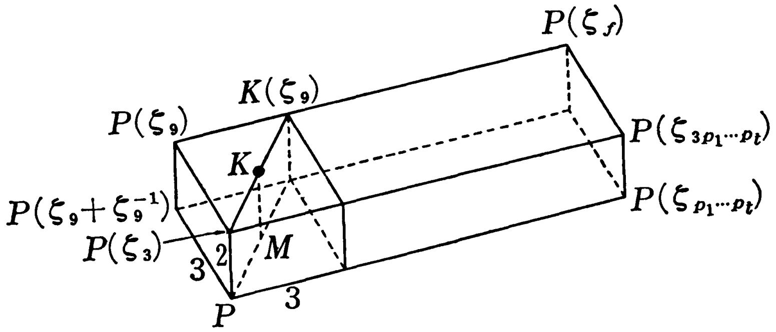
From (1), (3), (4) and $\mathfrak{P}_i=\mathfrak{P}_{i,1}\mathfrak{P}_{i,2}$, $a_{0,K/P(\zeta_3)}\,|\,3^{2t-2}$. Therefore, by the formula of the ambiguous number

$$a_{0,K/P(\zeta_3)}=\frac{3^{2t}}{(K:P(\zeta_3))[E_{P(\zeta_3)}:N_{K/P(\zeta_3)}E_K]}\leqq 3^{2t-2} .$$

Consequently $3\leqq[E_{P(\zeta_3)}:N_{K/P(\zeta_3)}E_K]=$ a power of 3. As $E_{P(\zeta_3)}=\{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$, $[E_{P(\zeta_3)}:N_{K/P(\zeta_3)}E_K]|6$. Therefore $[E_{P(\zeta_3)}:N_{K/P(\zeta_3)}E_K]=3$. Hence we have $a_{0,K/P(\zeta_3)}=3^{2t-2}$ and $N_{K/P(\zeta_3)}E_K=\{\pm 1\}$.

Secondly we consider the case of $s=2$ and $t\geqq 1$. Let M be the maximal real subfield of K. The factorization of 3 is:

$3=\mathfrak{P}^6$ in K, $3=\mathfrak{p}^2$ in $P(\zeta_3)$, $\mathfrak{P}=(1-\zeta_9)$ in $P(\zeta_9)$, $\mathfrak{p}=(1-\zeta_3)$ in $P(\zeta_3)$, $\mathfrak{p}=\mathfrak{P}^3$ in K, $3=\mathfrak{Q}$ in M, $\mathfrak{Q}=\mathfrak{P}^2$ in K.



These factorizations of p_i are $p_i=\mathfrak{p}_{i,1}\mathfrak{p}_{i,2}$ in $P(\zeta_3)$, $p_i=\mathfrak{P}_i^3$ in M and $p_{i,j}=\mathfrak{P}_{i,j}^3$ in K. Hence $\mathfrak{P}_i=\mathfrak{P}_{i,1}\mathfrak{P}_{i,2}$ in K. We have $K=P(\zeta_3, \sqrt[3]{c})$, $c\in P(\zeta_3)$, $c=\mathfrak{p}^s\mathfrak{p}_{1,1}\mathfrak{p}_{1,2}^2\cdots\mathfrak{p}_{t,1}\mathfrak{p}_{t,2}^2$ in the same way as (4). Since $N_{K/M}\sqrt[3]{c}\in P$, $\sqrt[3]{c}=\mathfrak{P}^s\mathfrak{P}_{1,1}\mathfrak{P}_{1,2}^2\cdots\mathfrak{P}_{t,1}\mathfrak{P}_{t,2}^2$ and $N_{K/M}\sqrt[3]{c}=\mathfrak{Q}^s p_1p_2\cdots p_t$, we have $3|s$. Hence we can put $s=0$ by $h_{P(\zeta_3)}=1$. Hence

(5) $$\sqrt[3]{c}=\mathfrak{P}_{1,1}\mathfrak{P}_{1,2}^2\cdots\mathfrak{P}_{t,1}\mathfrak{P}_{t,2}^2$$

Evidently

(6) $$\mathfrak{P}^3\sim 1 \quad\text{in}\quad K .$$

Also, in the same way as in (1),

(7) $$\mathfrak{P}_{i,j}^3\sim 1 \quad\text{in}\quad K .$$

Moreover,

(8) $\mathfrak{P}\sim 1$ and $\mathfrak{P}_{i,j}\sim 1$ in the genus field Γ of K/P in the same way as (2).

As $\mathfrak{Q}^3\sim 1$, $\mathfrak{P}_i^3\sim 1$ in M, $A_{0,M/\Gamma}$ is an abelian group of the type $(3, 3, \cdots, 3)$ and is generated by $\mathfrak{Q}$, $\mathfrak{P}_i$. Since $a_{0,M/P}=3^t$, we have:

$$(9)\quad \begin{gathered}\mathfrak{Q}^{s}\mathfrak{P}_1^{s_1}\mathfrak{P}_2^{s_2}\cdots\mathfrak{P}_t^{s_t}\sim 1 \quad \text{in} \quad M \\ (s, s_1, s_2, \cdots, s_t)\neq(0, 0, 0, \cdots, 0)\,, \qquad s, s_i=0 \quad \text{or} \quad 1 \text{ or } 2\,.\end{gathered}$$

$A_{0,K/P(\zeta_3)}$ is generated by $\mathfrak{P}$, $\mathfrak{P}_{i,j}$. From (5), (6), (7), (9), $\mathfrak{Q}=\mathfrak{P}^2$ and $\mathfrak{P}_i=\mathfrak{P}_{i,1}\mathfrak{P}_{i,2}$, we have $a_{0,K/P(\zeta_3)}\,|\,3^{2t-1}$. Hence we have

$$a_{0,K/P(\zeta_3)}=\frac{3^{2t+1}}{(K\colon P(\zeta_3))[E_{P(\zeta_3)}\colon N_{K/P(\zeta_3)}E_K]}\leqq 3^{2t-1}\,.$$

Consequently we have $a_{0,K/P(\zeta_3)}=3^{2t-1}$, $N_{K/P(\zeta_3)}E_K=\{\pm 1\}$. When $t=0$, it is easy to show $a_{0,K/P(\zeta_3)}=1$, $N_{K/P(\zeta_3)}E_K=E_{P(\zeta_3)}=\{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$.

In relation to principal ideal problem in unramified abelian extensions, we have the following Corollary by (2) and (8).

COROLLARY. *Let Γ be the genus field of K/P in Theorem. Γ/K is unramified and the number of these ideal classes of K which are principal in Γ is:*
a multiple of $3^{2t-2}=(\Gamma\colon K)^2$ *when* $s=1$
a multiple of $3^{2t-1}=3^{-1}(\Gamma\colon K)^2$ *when* $s=2$ *and* $t\geqq 1$.
$(\Gamma\colon K)=1$ *when* $s=2$ *and* $t=0$.

REMARK. In Theorem, we have

$$a_{K/P(\zeta_3)}=\frac{3^r}{(K\colon P(\zeta_3))[E_{P(\zeta_3)}\colon E_{P(\zeta_3)}\cap N_{K/P(\zeta_3)}K]}$$

where

$$r=\begin{cases}2t & (\text{if } s=1)\\ 2t+1 & (\text{if } s=2)\,.\end{cases}$$

Therefore, we have

$$a_{K/P(\zeta_3)}=\begin{cases}3^{2t-2}=a_{0,K/P(\zeta_3)} & \text{if} \quad s=1 \quad \text{and} \quad \zeta_3\notin N_{K/P(\zeta_3)}K\\ 3^{2t-1}=3a_{0,K/P(\zeta_3)} & \text{if} \quad s=1 \quad \text{and} \quad \zeta_3\in N_{K/P(\zeta_3)}K\\ 3^{2t-1}=a_{0,K/P(\zeta_3)} & \text{if} \quad s=2\,, \quad t\geqq 1 \quad \text{and} \quad \zeta_3\notin N_{K/P(\zeta_3)}K\\ 3^{2t}\;=3a_{0,K/P(\zeta_3)} & \text{if} \quad s=2\,, \quad t\geqq 1 \quad \text{and} \quad \zeta_3\in N_{K/P(\zeta_3)}K\\ 1=a_{0,K/P(\zeta_3)} & \text{if} \quad s=2 \quad \text{and} \quad t=0\,.\end{cases}$$

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