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On Homogeneous Convex Cones of Non-Positive Curvature

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Introduction

It is well known that homogeneous convex cones play an important role in the theory of homogeneous bounded domains (cf. e.g., [3], [7]). On a homogeneous convex cone V, a canonical Riemannian metric is defined which is closely related to the Bergman metric of the tube domain over V. With respect to this Riemannian metric, every homogeneous self-dual cone is a Riemannian symmetric space of non-positive sectional curvature (cf. [2], [8]). But it is little known about the Riemannian geometric properties of homogeneous non-self-dual cones. The main purposes of the present paper are to give a necessary condition for a homogeneous convex cone to be of non-positive sectional curvature with respect to the canonical metric and to determine such cones of rank 3 or of low dimensions.

The relation between homogeneous convex cones and homogeneous affine hyperspheres has been studied by Calabi [1] and Sasaki [9]. In § 1, we will recall some of definitions and the fundamental results on homogeneous convex cones and homogeneous affine hyperspheres from [13], [1] and [9]. In § 2, by using results of Sasaki [9] and Meschiari [5], we will see that every homogeneous convex cone with dimension $\geqq 2$ is homothetically equivalent to a product Riemannian manifold of a homogeneous hyperbolic affine hypersphere and the half line of positive real numbers (Proposition 2.1). As an application of this and a result in [10] or [12], a characterization for a homogeneous hyperbolic affine hypersphere to be Riemannian symmetric with respect to the affine metric will be given (Theorem 2.2). By making use of a result in Calabi [1] we will see that the Ricci curvature of a homogeneous convex cone is always non-positive (Theorem 2.3). In § 3, by using the results obtained in [12] we will calculate explicitly the curvature tensor of the canonical metric (Lemmas 3.1 and 3.2) and give a sufficient condition

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for a homogeneous convex cone to have a plane section with positive curvature (Theorem 3.4). By applying this to certain homogeneous convex cones, we will show that the sectional curvature of a homogeneous convex cone with respect to the canonical metric is not necessarily non-positive (Corollary 3.5). In § 4, we will consider homogeneous convex cones with non-positive sectional curvature and determine such cones of rank 3 (Theorem 4.6) or of low dimensions (Theorem 4.7). Finally, some corrections to the misstatements contained in [11] will be given.

Some of our results have been announced in the note [11].

Throughout this paper, the same notation and terminologies as in [12] will be employed.

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§ 1. Preliminaries.

In this section, following Vinberg [13], Calabi [1] and Sasaki [9] we recall some of definitions and the fundamental results on homogeneous convex cones and homogeneous affine hyperspheres.

1.1. Let V be a convex cone in the n-dimensional real number space $\boldsymbol{R}^n$. We denote by $G(V)$ the group of all linear automorphisms of $\boldsymbol{R}^n$ leaving V invariant. If the group $G(V)$ acts transitively on V, then the cone V is said to be *homogeneous*. Let $\langle\ ,\ \rangle$ be an inner product on $\boldsymbol{R}^n$ and V^* the dual cone of V with respect to this inner product. Then the *characteristic function* ϕ_V of a convex cone V is defined on V by

$$\phi_V(x)=\int_{V^*}\exp(-\langle x, y\rangle)dy\ , \tag{1.1}$$

where dy is the canonical Euclidean measure on $\boldsymbol{R}^n$. The characteristic function has the following property:

$$\phi_V(Ax)=\phi_V(x)/|\det A| \tag{1.2}$$

for every $A\in G(V)$.

Let us take a system of linear coordinates $(x_1, x_2, \cdots, x_n)$ of $\boldsymbol{R}^n$. Then from the characteristic function we can define a $G(V)$-invariant Riemannian metric g_V on V by

$$g_V=\sum_{i,j}\frac{\partial^2\log\phi_V}{\partial x_i\partial x_j}dx_i dx_j\ .$$

This Riemannian metric g_V is called the *canonical metric* of V.

1.2. Next, we recall briefly the relation between homogeneous convex cones and T-algebras due to Vinberg. The detailed proofs may be found in [13] or [14].

Let $\mathfrak{A}=\sum_{1\leq i,j\leq r}\mathfrak{A}_{ij}$ be a T-algebra of rank r with an involution $*$. We put

$$T=T(\mathfrak{A})=\{t=(t_{ij})\in\mathfrak{A};\ t_{ii}>0 \text{ and } t_{ij}=0 \text{ for } j<i\}$$

and

$$V=V(\mathfrak{A})=\{tt^*;\ t\in T(\mathfrak{A})\}\ .$$

Then the set V is a homogeneous convex cone in the real vector space $X(\mathfrak{A})=\{x\in\mathfrak{A};\ x^*=x\}$. Conversely every homogeneous convex cone is realized in this form (up to linear equivalence). The number r is called the *rank* of V. The set T is a connected $\boldsymbol{R}$-triangular solvable Lie subgroup of $G(V)$ which acts simply transitively on V. The tangent space of V at the point $e\in V$ may be naturally identified with the ambient space $X(\mathfrak{A})$ or with the Lie algebra $\mathfrak{t}$ of the Lie group T, where $e=(e_{ij})$, $e_{ii}=1$ and $e_{ij}=0$ for $1\leqq i\neq j\leqq r$. Under this identification, the canonical metric g_V at the point e may be considered as an inner product $\langle\ ,\ \rangle$ on $\mathfrak{t}$. The Lie algebra $\mathfrak{t}$ is bigraded with subspaces $\mathfrak{A}_{ij}$ $(1\leqq i\leqq j\leqq r)$ such that

$$\mathfrak{t}=\sum_{1\leqq i\leqq j\leqq r}\mathfrak{A}_{ij}\ \text{(orthogonal direct sum)}\ ,\qquad \mathfrak{A}_{ii}=\boldsymbol{R}\quad(1\leqq i\leqq r)\ . \tag{1.3}$$

The natural identification of $\mathfrak{t}$ with $X(\mathfrak{A})$ is given by $a\in\mathfrak{t}\rightarrow a+a^*\in X(\mathfrak{A})$. For every i, j with $1\leqq i\leqq j\leqq r$, we put

$$n_{ij}=\dim\mathfrak{A}_{ij}\ ,\qquad n_i=1+1/2\sum_{k<i}n_{ki}+1/2\sum_{i<k}n_{ik}\ . \tag{1.4}$$

Then the condition

$$\max\{n_{ij},\ n_{jk}\}\leqq n_{ik} \tag{1.5}$$

is satisfied for $1\leqq i<j<k\leqq r$ with $n_{ij}n_{jk}\neq 0$.

For every i with $1\leqq i\leqq r$, we put

$$e_i=(1/2\sqrt{n_i})e_{ii}\in\mathfrak{A}_{ii}=\boldsymbol{R}\ .$$

Then we have $\langle e_i, e_i\rangle=1$. For every i, j with $n_{ij}\neq 0$ $(1\leqq i<j\leqq r)$, we take an orthonormal basis $\{e_{ij}^\lambda\}_{1\leq\lambda\leq n_{ij}}$ of $\mathfrak{A}_{ij}$. If $n_{ij}n_{jk}\neq 0$ with $1\leqq i<j<k\leqq r$, then there exists a system $\{T_\lambda\}$ of linear operators

$$T_\lambda: \mathfrak{A}_{jk} \longrightarrow \mathfrak{A}_{ik} \quad (1 \leqq \lambda \leqq n_{ij})$$

satisfying the following conditions:

$$[e_{ij}^\lambda, a] = T_\lambda a$$

for every $a \in \mathfrak{A}_{jk}$ and

$${}^tT_\lambda T_\mu + {}^tT_\mu T_\lambda = (1/n_j)\delta_{\lambda\mu} I_{jk} \quad (1 \leqq \lambda, \mu \leqq n_{ij}) ,$$

where I_{jk} is the identity operator on $\mathfrak{A}_{jk}$. An irreducible homogeneous self-dual cone is characterized by the condition that $n_{ij}=c$ for every i, j with $1 \leqq i < j \leqq r$, where c is a positive constant.

1.3. Let S be a hypersurface in $\boldsymbol{R}^n$ $(n \geqq 2)$ and $f: S \to \boldsymbol{R}^n$ its imbedding. For each point p in S, we take a local coordinate neighborhood $(U, (u_1, u_2, \cdots, u_{n-1}))$ around p. For every i, j with $1 \leqq i, j \leqq n-1$, we put a function Λ_{ij} on U by

$$\Lambda_{ij} = \det (f_{ij}, f_1, \cdots, f_{n-1}) \tag{1.6}$$

where $f_{ij} = \partial^2 f / \partial u_i \partial u_j$, $f_i = \partial f / \partial u_i$ and f is considered as a column vector valued mapping. From now on, we assume that the symmetric matrix (Λ_{ij}) defined by (1.6) is positive definite on a suitable local coordinate neighborhood U around every point p in S. Then for $1 \leqq i, j \leqq n-1$, by putting

$$g_{ij} = \Lambda^{-1/(n+1)} \Lambda_{ij} , \qquad \Lambda = \det (\Lambda_{ij}) \tag{1.7}$$

we have a Riemannian metric $g_S = \sum_{i,j} g_{ij}\, du_i du_j$ on S. The Riemannian metric g_S defined by (1.7) is called the *affine metric* of S. A hypersurface S is called a *hyperbolic affine hypersphere* (with its center at the origin), if the imbedding f satisfies the following equation:

$$\Delta f = -(n-1) H f ,$$

where H is a negative constant and Δ is the Laplace-Beltrami operator with respect to the affine metric g_S (cf. Calabi [1]). A closed hyperbolic affine hypersphere S is said to be *homogeneous* if there exists a unimodular subgroup $G(S)$ of $GL(n, \boldsymbol{R})$ which acts transitively on S. In this case, S is a homogeneous Riemannian manifold, that is, the affine metric g_S is $G(S)$-invariant.

For a convex cone V in $\boldsymbol{R}^n$ and a positive real number c, we denote by S_c the level surface of the characteristic function ϕ_V with level c. Then the following result is known in Sasaki [9]: For a homogeneous convex cone V in $\boldsymbol{R}^n$, the level surface S_c is a homogeneous hyperbolic affine hypersphere, and on S_c the canonical metric g_V coincides with the affine metric g_{S_c} up to a constant factor. Conversely every homogeneous hyperbolic affine hypersphere is realized in this form.

§ 2. Homogeneous convex cones and homogeneous affine hyperspheres.

In this section, we investigate the Riemannian geometric properties of homogeneous convex cones by making use of the results due to Calabi [1] and Sasaki [9] stated in § 1.

2.1. For a homogeneous convex cone V in $\boldsymbol{R}^n$, we denote by $S=S_1$ the level surface of the characteristic function ϕ_V with level one and $\iota\colon S \to \boldsymbol{R}^n$ the inclusion mapping of S into $\boldsymbol{R}^n$. We define a mapping ξ by

$$\xi\colon S\times \boldsymbol{R}^+ \to V\,, \qquad \xi(p,t)=tp\,, \tag{2.1}$$

where $\boldsymbol{R}^+=\{t\in \boldsymbol{R};\, t>0\}$.

The following proposition is essentially due to Sasaki [9] and Meschiari [5].

PROPOSITION 2.1. *Let V be a homogeneous convex cone in $\boldsymbol{R}^n$ with $n\geqq 2$ and c a positive real number. Then the Riemannian manifold (V, g_V) is homothetically equivalent to the Riemannian product of the homogeneous hyperbolic affine hypersphere (S_c, g_{S_c}) and the half line of positive real numbers.*

PROOF. For every positive real number λ, the homothety $\lambda\,\mathrm{id.}$ is contained in $G(V)$. Therefore, by Sasaki's result stated in the subsection 1.3 and the property (1.2), the affine hyperspheres S_c and $S=S_1$ are homothetically equivalent. Hence, in order to prove the above assertion it is sufficient to show that the Riemannian manifold $(S, \iota^* g_V)\times(\boldsymbol{R}^+, nt^{-2}dt^2)$ is isometric to (V, g_V). In fact, it is easy to see that the mapping ξ defined by (2.1) is an isometry (cf. Meschiari [5]). q.e.d.

2.2. It is known in [2] or [8] that a homogeneous self-dual cone is Riemannian symmetric with respect to the canonical metric. Conversely, it was proved in the recent papers (cf. [10] or [12]) that every Riemannian symmetric homogeneous convex cone is self-dual. Therefore, by Proposition 2.1 we have the following

THEOREM 2.2. *For a homogeneous convex cone V in $\boldsymbol{R}^n$ with $n\geqq 2$,* *the homogeneous hyperbolic affine hypersphere S_e is Riemannian symmetric with respect to the affine metric if and only if V is self-dual.*

We remark that the sufficient condition in the above statement has been proved by Sasaki [9].

It is known in Calabi [1] that the Ricci curvature of a complete hyperbolic affine hypersphere with respect to the affine metric is non-positive. Combining this with Proposition 2.1 we have the following

THEOREM 2.3. *The Ricci curvature of a homogeneous convex cone with respect to the canonical metric is non-positive.*

§ 3. Sectional curvature of the canonical metric.

In this section, after calculating the curvature tensor by making use of the results obtained in [12], we give a sufficient condition for a homogeneous convex cone to have a plane section with positive curvature. Furthermore, we show that the sectional curvature of a homogeneous convex cone with respect to the canonical metric (or equivalently, a homogeneous hyperbolic affine hypersphere with respect to the affine metric) is not necessarily non-positive.

From now on, we will consider exclusively the canonical metric. So, for the sake of brevity the terminology *with respect to the canonical metric* may be omitted.

3.1. Let V be a homogeneous convex cone of rank r and $\mathfrak{t}=\sum_{1\leq i\leq j\leq r}\mathfrak{A}_{ij}$ the corresponding simply transitive subalgebra of $\mathfrak{g}(V)$ given by (1.3), where $\mathfrak{g}(V)$ denotes the Lie algebra of $G(V)$. The bracket operation [,] in the Lie algebra $\mathfrak{t}$ and the connection function α of the canonical metric have been obtained explicitly in [12]. The curvature tensor R of the canonical metric is given by the following formula (cf. Nomizu [6]):

$$(3.1)\qquad R\colon \mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t}\,,$$
$$R(a, b, c)=R(a, b)c=\alpha(a, \alpha(b, c))-\alpha(b, \alpha(a, c))-\alpha([a, b], c)$$

for every $a, b, c\in\mathfrak{t}$. Therefore, the following two lemmas can be proved in a straightforward manner from the above formula (3.1), the lists of the bracket operations in $\mathfrak{t}$ and the connection function of the canonical metric given by Lemmas 1.1 and 2.2 of [12]. So, we may omit their proofs. Here we will denote by a_{ij} the $\mathfrak{A}_{ij}$-component of an element $a=(a_{ij})\in\mathfrak{t}$.

LEMMA 3.1. *If $n_{ij}n_{jk}\neq 0$ with $1\leqq i<j<k\leqq r$, then the curvature tensor R of the canonical metric satisfies the following conditions*:

$$(1)\qquad \begin{aligned} R(e_\beta, a_{\alpha\beta}, b_{\alpha\beta}) &= \langle a_{\alpha\beta}, b_{\alpha\beta}\rangle((1/4\sqrt{n_\alpha n_\beta})e_\alpha-(1/4n_\beta)e_\beta)\ ,\\ R(e_\alpha, a_{\alpha\beta}, b_{\alpha\beta}) &= \langle a_{\alpha\beta}, b_{\alpha\beta}\rangle((1/4\sqrt{n_\alpha n_\beta})e_\beta-(1/4n_\alpha)e_\alpha) \end{aligned}$$

for $(\alpha, \beta)=(i, j)$, (i, k) *or* (j, k).

$$(2)\qquad \begin{aligned} R(e_k, a_{jk}, a_{ik}) &= R(e_k, a_{ik}, a_{jk}) = (1/4\sqrt{n_k})\sum_{1\leqq\lambda\leqq n_{ij}}\langle T_\lambda a_{jk}, a_{ik}\rangle e_{ij}^\lambda\ ,\\ R(e_i, a_{ij}, a_{ik}) &= R(e_i, a_{ik}, a_{ij}) = (1/4\sqrt{n_i})\sum_{1\leqq\lambda\leqq n_{ij}}\langle a_{ij}, e_{ij}^\lambda\rangle {}^tT_\lambda a_{ik}\ . \end{aligned}$$

$$(3)\qquad \begin{aligned} R(a_{ij}, a_{jk}, b_{jk}) &= (1/4n_j)\langle a_{jk}, b_{jk}\rangle a_{ij} - 1/4\sum_{1\leqq\lambda,\mu\leqq n_{ij}}\langle a_{ij}, e_{ij}^\lambda\rangle\\ &\qquad \times(\langle T_\lambda b_{jk}, T_\mu a_{jk}\rangle + 2\langle T_\lambda a_{jk}, T_\mu b_{jk}\rangle)e_{ij}^\mu\ ,\\ R(a_{ij}, a_{jk}, b_{ij}) &= -(1/4n_j)\langle a_{ij}, b_{ij}\rangle a_{jk} + 1/4\sum_{1\leqq\lambda,\mu\leqq n_{ij}}\langle a_{ij}, e_{ij}^\lambda\rangle\\ &\qquad \times\langle b_{ij}, e_{ij}^\mu\rangle(2\,{}^tT_\mu T_\lambda + {}^tT_\lambda T_\mu)a_{jk}\ . \end{aligned}$$

$$(4)\qquad \begin{aligned} R(a_{ij}, a_{jk}, a_{ik}) &= \sum_{1\leqq\lambda\leqq n_{ij}}\langle a_{ij}, e_{ij}^\lambda\rangle\langle T_\lambda a_{jk}, a_{ik}\rangle((1/4\sqrt{n_k})e_k-(1/4\sqrt{n_i})e_i)\ ,\\ R(a_{ij}, a_{ik}, a_{jk}) &= \sum_{1\leqq\lambda\leqq n_{ij}}\langle a_{ij}, e_{ij}^\lambda\rangle\langle T_\lambda a_{jk}, a_{ik}\rangle((1/4\sqrt{n_k})e_k-(1/4\sqrt{n_j})e_j)\ . \end{aligned}$$

$$(5)\qquad \begin{aligned} R(a_{ij}, a_{ik}, b_{ik}) &= -(1/4n_i)\langle a_{ik}, b_{ik}\rangle a_{ij}\\ &\qquad + 1/4\sum_{1\leqq\lambda,\mu\leqq n_{ij}}\langle a_{ij}, e_{ij}^\lambda\rangle\langle {}^tT_\lambda b_{ik}, {}^tT_\mu a_{ik}\rangle e_{ij}^\mu\ ,\\ R(a_{jk}, a_{ik}, b_{ik}) &= -(1/4n_k)\langle a_{ik}, b_{ik}\rangle a_{jk} + 1/4\sum_{1\leqq\lambda\leqq n_{ij}}\langle T_\lambda a_{jk}, b_{ik}\rangle\,{}^tT_\lambda a_{ik}\ . \end{aligned}$$

Similarly we have

LEMMA 3.2. *If $n_{ij}n_{ik}\neq 0$, $n_{jk}=0$ or $n_{jk}n_{ik}\neq 0$, $n_{ij}=0$ with $1\leqq i<j<k\leqq r$, then the curvature tensor R of the canonical metric satisfies the following conditions*: *For* $(\alpha, \beta)=(i, j)$, (i, k) *or* (j, k),

$$(1)\qquad \begin{aligned} R(e_\beta, a_{\alpha\beta}, b_{\alpha\beta}) &= \langle a_{\alpha\beta}, b_{\alpha\beta}\rangle((1/4\sqrt{n_\alpha n_\beta})e_\alpha-(1/4n_\beta)e_\beta)\ ,\\ R(e_\alpha, a_{\alpha\beta}, b_{\alpha\beta}) &= \langle a_{\alpha\beta}, b_{\alpha\beta}\rangle((1/4\sqrt{n_\alpha n_\beta})e_\beta-(1/4n_\alpha)e_\alpha)\ , \end{aligned}$$

$$(2)\qquad \begin{aligned} &R(e_\alpha, a_{\alpha\beta}, e_\alpha) = (1/4n_\alpha)a_{\alpha\beta}\ ,\qquad R(e_\alpha, a_{\alpha\beta}, e_\beta) = -(1/4\sqrt{n_\alpha n_\beta})a_{\alpha\beta}\ ,\\ &R(e_\beta, a_{\alpha\beta}, e_\beta) = (1/4n_\beta)a_{\alpha\beta}\ , \end{aligned}$$

$$(3)\qquad R(a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}) = (1/4n_\alpha + 1/4n_\beta)(\langle a_{\alpha\beta}, c_{\alpha\beta}\rangle b_{\alpha\beta} - \langle b_{\alpha\beta}, c_{\alpha\beta}\rangle a_{\alpha\beta})\ .$$

$$(4)\qquad \begin{aligned} R(a_{ij}, a_{ik}, b_{ij}) &= (1/4n_i)\langle a_{ij}, b_{ij}\rangle a_{ik}\ ,\\ R(a_{ij}, a_{ik}, b_{ik}) &= -(1/4n_i)\langle a_{ik}, b_{ik}\rangle a_{ij}\ . \end{aligned}$$

(5)
$$R(a_{jk}, a_{ik}, b_{jk}) = (1/4n_k)\langle a_{jk}, b_{jk}\rangle a_{ik}\,,$$
$$R(a_{jk}, a_{ik}, b_{ik}) = -(1/4n_k)\langle a_{ik}, b_{ik}\rangle a_{jk}\,.$$

(6) *Every one of the following values is zero*:

$R(e_\alpha, e_\beta)$ *for* $\alpha, \beta \in \{i, j, k\}$,
$R(e_i, a_{ij}, e_k)$, $R(e_i, a_{ij}, a_{jk})$, $R(e_i, a_{ij}, a_{ik})$, $R(e_i, a_{jk})$,
$R(e_i, a_{ik}, e_j)$, $R(e_i, a_{ik}, a_{ij})$, $R(e_i, a_{ik}, a_{jk})$, $R(e_j, a_{ij}, e_k)$,
$R(e_j, a_{ij}, a_{jk})$, $R(e_j, a_{ij}, a_{ik})$, $R(e_j, a_{jk}, a_{ij})$, $R(e_j, a_{jk}, a_{ik})$,
$R(e_j, a_{ik})$, $R(e_k, a_{ij})$, $R(e_k, a_{jk}, a_{ij})$, $R(e_k, a_{jk}, a_{ik})$,
$R(e_k, a_{ik}, a_{ij})$, $R(e_k, a_{ik}, a_{jk})$, $R(a_{ij}, b_{ij}, a_{jk})$, $R(a_{ij}, b_{ij}, a_{ik})$,
$R(a_{ij}, a_{jk})$, $R(a_{ij}, a_{ik}, a_{jk})$, $R(a_{jk}, b_{jk}, a_{ik})$.

3.2. If $n_{ij} \neq 0$ for some pair (i, j) with $1 \leqq i < j \leqq r$, then from the formulas (1) in Lemmas 3.1 or 3.2 it follows that there are always plane sections with negative curvature (cf. also Theorem 2.3). Now we give a sufficient condition for a homogeneous convex cone to have a plane section with positive curvature and show that the sectional curvature of a homogeneous convex cone is not necessarily non-positive.

LEMMA 3.3. *If* $n_{ij}n_{jk} \neq 0$ *with* $1 \leqq i < j < k \leqq r$, *then the following equalities* (1) *and* (2) *are valid*:
(1) $\langle R(X, Y)Y, X\rangle = -\alpha^2/2n_k + \alpha/\sqrt{2n_jn_k} - 1/4n_i$,
(2) $\langle R(U, V)V, U\rangle = -\beta^2/2n_i + \beta/\sqrt{2n_in_j} - 1/4n_k$
for $X = \alpha e_k + e_{ij}$, $Y = e_{ik} + e_{jk}$, $U = \beta e_i + e_{jk}$ *and* $V = e_{ij} + e_{ik}$, *where* $\alpha, \beta \in \boldsymbol{R}$, $e_{ij} = e^1_{ij}$, $e_{jk} = e^1_{jk}$, $e_{ik} = e^1_{ik}$ *and* $e_{ik} = \sqrt{2n_j}T_1e_{jk}$.

PROOF. Substituting the formulas (1)-(5) in Lemma 3.1 to $R(X, Y)Y = \alpha R(e_k, e_{ik}, e_{ik}) + \alpha R(e_k, e_{jk}, e_{jk}) + \alpha R(e_k, e_{ik}, e_{jk}) + \alpha R(e_k, e_{jk}, e_{ik}) + R(e_{ij}, e_{jk}, e_{jk}) + R(e_{ij}, e_{ik}, e_{ik}) + R(e_{ij}, e_{jk}, e_{ik}) + R(e_{ij}, e_{ik}, e_{jk})$, we have the first equality. The second equality follows similarly from the formulas in Lemma 3.1 and the Bianchi's identity. q.e.d.

From the above lemma we have the following

THEOREM 3.4. *If a triple* (i, j, k) *with* $n_{ij}n_{jk} \neq 0$ $(1 \leqq i < j < k \leqq r)$ *satisfies the conditions* $n_i > n_j$ *or* $n_k > n_j$, *then there exists a plane section in* $\mathfrak{t}$ *with positive curvature.*

PROOF. If $n_i > n_j$, then by the formula (1) in Lemma 3.3 $\langle R(X, Y)Y, X\rangle$ attains a positive value for a suitable real number α. By making use of the formula (2) in Lemma 3.3 we may similarly show the statement for the case of $n_k > n_j$. q.e.d.

Now we give two concrete examples of homogeneous convex cones which satisfy the conditions in the above theorem. Let l, m, n be positive integers which satisfy the condition $n \leqq m$ with $(m, n) \neq (1, 1)$ and consider the irreducible homogeneous non-self-dual cones defined by the following (3.2) or (3.3):

$$(3.2) \qquad V = \{x = (x_1, x_2, \cdots, x_{7+l}) \in \boldsymbol{R}^{7+l};\ x_4 > 0,\ A(x) \text{ is positive definite}\},$$

where $A(x) = (a_{ij}(x))$ is a real symmetric matrix of degree 3 such that $a_{11}(x) = x_1 x_4 - \sum_{8 \leqq k \leqq 7+l} x_k^2$, $a_{12}(x) = x_4 x_5$, $a_{13}(x) = x_4 x_6$, $a_{22}(x) = x_2 x_4$, $a_{23}(x) = x_4 x_7$ and $a_{33}(x) = x_3 x_4$.

$$(3.3) \qquad V = \{x = (x_1, x_2 \cdots, x_{m+n+4}) \in \boldsymbol{R}^{m+n+4};\ x_3 > 0,\ P_1(x) > 0,\ P_2(x) > 0\},$$

where $P_1(x) = x_1 x_3 - \sum_{5 \leqq k \leqq m+4} x_k^2$ and $P_2(x) = P_1(x)(x_2 x_3 - \sum_{m+5 \leqq k \leqq m+n+4} x_k^2) - (x_3 x_4 - \sum_{5 \leqq k \leqq n+4} x_k x_{m+k})^2$.

Then we have

COROLLARY 3.5. *For the homogeneous convex cone V defined by* (3.2) *or* (3.3), *the sectional curvature of V attains the values of both signs.*

PROOF. From the correspondence between homogeneous convex cones and T-algebras due to Vinberg [13] we can see that a simply transitive subalgebra $\mathfrak{t}$ of $\mathfrak{g}(V)$ (cf. (1.3)) for the cone V defined by (3.2) is bigraded with subspaces $\mathfrak{A}_{ij}$ $(1 \leqq i \leqq j \leqq 4)$ satisfying $n_{12} = n_{13} = n_{23} = 1$, $n_{14} = l$ and $n_{24} = n_{34} = 0$. Therefore $n_1 = 2 + l/2 > n_2 = 2$. For the cone V defined by (3.3), $\mathfrak{t}$ is bigraded with subspaces $\mathfrak{A}_{ij}$ $(1 \leqq i \leqq j \leqq 3)$ such that $n_{12} = 1$, $n_{13} = m$ and $n_{23} = n$. Thus, $n_3 = 1 + m/2 + n/2 > n_2 = 1 + 1/2 + n/2$. q.e.d.

§ 4. Homogeneous convex cones of non-positive sectional curvature.

In this section we will consider homogeneous convex cones of non-positive sectional curvature and determine such cones of rank 3 or of low dimensions by making use of the results obtained in the previous sections.

4.1. Let V be a homogeneous convex cone of rank r and $\mathfrak{t} = \sum_{1 \leqq i \leqq j \leqq r} \mathfrak{A}_{ij}$ a simply transitive $\boldsymbol{R}$-triangular solvable subalgebra of $\mathfrak{g}(V)$ given by (1.3). If the sectional curvature of the canonical metric on V is non-positive, then by Theorem 3.4 we have

$$(4.1) \qquad \max\{n_i, n_k\} \leqq n_j$$

for every triple (i, j, k) satisfying the conditions $n_{ij}n_{jk}\neq 0$ and $1\leqq i<j<k\leqq r$.

As an application of Theorem 3.4 we have the following

THEOREM 4.1. *If the sectional curvature of a homogeneous convex cone of rank r is non-positive and $n_{ij}\neq 0$ for every i, j with $1\leqq i<j\leqq r$, then n_i is constant for $1\leqq i\leqq r$.*

PROOF. Clearly the assertion holds for the case of $r\leqq 2$. So, we may assume that $r\geqq 3$. By (4.1) we have $\max\{n_i, n_{i+2}\}\leqq n_{i+1}$ for $1\leqq i\leqq r-2$. Therefore $n_1, n_r\leqq n_2=n_3=\cdots=n_{r-1}$ holds. On the other hand, by (1.4) and (1.5) we have $2(n_2-n_1)=\sum_{3\leqq i\leqq r}(n_{2i}-n_{1i})\leqq 0$ and $2(n_{r-1}-n_r)=\sum_{1\leqq i\leqq r-2}(n_{ir-1}-n_{ir})\leqq 0$. Hence, it follows that $n_1=n_2$ and $n_{r-1}=n_r$. q.e.d.

From the above theorem we have

COROLLARY 4.2. *Let V be a homogeneous convex cone of rank r with $r\geqq 3$ satisfying the following condition: For every j with $2\leqq j\leqq r$, the number n_{ij} is positive and independent of i $(1\leqq i\leqq j-1)$. Then the sectional curvature of V is non-positive if and only if V is self-dual.*

PROOF. In view of the well known result by Rothaus [8], we have only to prove the necessary condition. Let us put $m_j=n_{ij}$ $(1\leqq i\leqq j-1, 2\leqq j\leqq r)$. Then by Theorem 4.1 we have $n_i=n_{i+1}$ $(1\leqq i\leqq r-1)$, and by (1.5), $2(n_i-n_{i+1})=(i-1)(m_i-m_{i+1})=0$ for $2\leqq i\leqq r-1$. Therefore $m_2=m_3=\cdots=m_r$ and n_{ij} is constant independent of i, j with $1\leqq i<j\leqq r$. q.e.d.

Next we show the following

COROLLARY 4.3. *Let V be a homogeneous convex cone of rank r with $r\geqq 3$ satisfying the following two conditions:*

(1) n_{ij} is positive and constant for every i, j with $1\leqq i<j\leqq r-1$.

(2) The sectional curvature of V is non-positive.

Then V is self-dual or otherwise n_{ir} is positive and constant for $1\leqq i\leqq r-2$, $n_{r-1\,r}=0$ and $(r-3)n_{1r}\leqq(r-2)n_{12}$.

PROOF. If $n_{ir}=0$ for every i with $1\leqq i\leqq r-1$, then V is the direct product of a homogeneous self-dual cone of rank $r-1$ and the cone of positive real numbers. So, by the condition (1.5) we may assume that $n_{1r}=m>0$. Now we consider the case of $n_{r-1\,r}\neq 0$. Then from Theorem 4.1 it follows that the equalities $n_1=n_2=\cdots=n_r$ hold. By (1.4) we have $2n_i=2+(r-2)n_{12}+n_{ir}$ $(1\leqq i\leqq r-1)$ and $2n_r=2+n_{1r}+n_{2r}+\cdots+n_{r-1\,r}$. Therefore $n_{1r}=n_{2r}=\cdots=n_{r-1\,r}=m$, $n_{12}=m$ and V is self-dual. Next we consider the case of $n_{1r}=m>0$, $n_{r-1r}=0$ and $r\geqq 4$. Analogously as in the proof of Theorem 4.1 we can show that the equalities $n_1=n_2=\cdots=n_{r-2}$ hold. By these equalities and the condition (1.5) we have $n_{ir}=m$ for $1\leqq i\leqq r-2$. By the condition $n_r\leqq n_1$ we have $(r-3)n_{1r}\leqq(r-2)n_{12}$. We remark that in the case of $r=3$, $(r-3)n_{1r}<(r-2)n_{12}$ is trivially valid. q.e.d.

Furthermore we have

COROLLARY 4.4. *Let V be a homogeneous convex cone of rank 3 satisfying the condition $n_{12}n_{23}\neq 0$. Then the sectional curvature of V is non-positive if and only if V is self-dual.*

PROOF. By Theorem 4.1 we have $n_1=n_2=n_3$. From the conditions (1.4) and (1.5) it follows that $n_{12}=n_{23}=n_{13}$. Hence, V is self-dual. q.e.d.

4.2. Now we want to determine homogeneous convex cones of rank 3 with non-positive sectional curvature. Let us take positive integers m, n and define two irreducible homogeneous non-self-dual cones $V_{m,n}$ and $V'_{m,n}$ by the following:

$$V_{m,n}=\{x=(x_1,x_2,\cdots,x_{m+n+3})\in \boldsymbol{R}^{m+n+3};\ x_3>0,\ P_1(x)>0,\ P_2(x)>0\}\ , \tag{4.2}$$

where $P_1(x)=x_1x_3-\sum_{4\leqq k\leqq m+3}x_k^2$, $P_2(x)=x_2x_3-\sum_{m+4\leqq k\leqq m+n+3}x_k^2$, and

$$V'_{m,n}=\{x=(x_1,x_2,\cdots,x_{m+n+3})\in \boldsymbol{R}^{m+n+3};\ x_3>0,\ Q_1(x)>0,\ Q_2(x)>0\}\ , \tag{4.3}$$

where $Q_1(x)=P_1(x)$, $Q_2(x)=x_2Q_1(x)-x_3\sum_{m+4\leqq k\leqq m+n+3}x_k^2$. Then we have

PROPOSITION 4.5. *Let V be a homogeneous convex cone which is linearly isomorphic to $V_{m,n}$ or $V'_{m,n}$ defined above. Then the sectional curvature of V is non-positive.*

PROOF. The convex cone $V'_{m,n}$ defined by (4.3) is linearly isomorphic to the dual cone of $V_{m,n}$ defined by (4.2) (cf. [13], [4]). Therefore these cones are isometric (cf. [5], [9]). So, it is sufficient to prove the statement for the case of (4.2). The simply transitive subalgebra $\mathfrak{t}$ of $\mathfrak{g}(V)$ given by (1.3) is bigraded with subspaces $\mathfrak{A}_{ij}$ $(1\leqq i\leqq j\leqq 3)$ such that $n_{12}=0$, $n_{13}=m$ and $n_{23}=n$. Put $i=1$, $j=2$ and $k=r=3$ in Lemma 3.2. Then by using the Bianchi's identity and the formulas (1)-(6) in Lemma 3.2, for every

$$X=x_1e_1+x_2e_2+x_3e_3+x_{13}+x_{23} \quad\text{and}\quad Y=y_1e_1+y_2e_2+y_3e_3+y_{13}+y_{23}$$

in $\mathfrak{t}$ we have the following:

$$R(e_i,x_{i3},Y)=(y_i/4n_i)x_{i3}-(y_3/4\sqrt{n_in_3})x_{i3}+\langle x_{i3},y_{i3}\rangle((1/4\sqrt{n_in_3})e_3-(1/4n_i)e_i)\ ,$$

$$R(e_3, x_{i3}, Y)=(y_3/4n_3)x_{i3}-(y_i/4\sqrt{n_in_3})x_{i3}+\langle x_{i3}, y_{i3}\rangle((1/4\sqrt{n_in_3})e_i-(1/4n_3)e_3)\,,$$
$$R(x_{i3}, y_{i3}, Y)=(1/4n_i+1/4n_3)(\langle x_{i3}, y_{i3}\rangle y_{i3}-\langle y_{i3}, y_{i3}\rangle x_{i3}) \quad (i=1, 2)$$

and

$$R(x_{i3}, y_{j3}, Y)=(1/4n_3)(\langle x_{i3}, y_{i3}\rangle y_{j3}-\langle y_{j3}, y_{j3}\rangle x_{i3}) \quad ((i, j)=(1, 2), (2, 1))\,.$$

For every element a in t, we put $\|a\|=\langle a, a\rangle^{1/2}$. Then directly from the above formulas we have the following:

$$\begin{aligned}\langle R(X, Y)Y, X\rangle = &-\|(1/2\sqrt{n_1})(x_1y_{13}-y_1x_{13})+(1/2\sqrt{n_3})(y_3x_{13}-x_3y_{13})\|^2\\ &-\|(1/2\sqrt{n_2})(x_2y_{23}-y_2x_{23})+(1/2\sqrt{n_3})(y_3x_{23}-x_3y_{23})\|^2\\ &-(1/4n_1+1/4n_3)(\|x_{13}\|^2\|y_{13}\|^2-\langle x_{13}, y_{13}\rangle^2)\\ &-(1/4n_2+1/4n_3)(\|x_{23}\|^2\|y_{23}\|^2-\langle x_{23}, y_{23}\rangle^2)\\ &-(1/4n_3)(\|x_{13}\|^2\|y_{23}\|^2+\|x_{23}\|^2\|y_{13}\|^2-2\langle x_{13}, y_{13}\rangle\langle x_{23}, y_{23}\rangle)\,.\end{aligned}$$

Therefore, $\langle R(X, Y)Y, X\rangle\leqq 0$. q.e.d.

Combining the above proposition with Corollary 4.4, we have

THEOREM 4.6. *Let V be a homogeneous convex cone of rank 3. Then the sectional curvature of V is non-positive if and only if V is self-dual or linearly isomorphic to the one defined by* (4.2) *or* (4.3).

PROOF. A reducible homogeneous convex cone of rank 3 is self-dual. On the other hand, an irreducible homogeneous convex cone V of rank 3 satisfies the condition $n_{12}n_{23}\neq 0$ or V is linearly isomorphic to $V_{m,n}$ or $V'_{m,n}$ (cf. [4], [13], [14]). If $n_{12}n_{23}\neq 0$, then by Corollary 4.4, V is self-dual. The sufficient conditions in the above statement follow from Proposition 4.5 and a result by Rothaus [8]. q.e.d.

4.3. For homogeneous convex cones of low dimensions we have the following

THEOREM 4.7. *Let V be a homogeneous convex cone in $\boldsymbol{R}^n$ with $n\leqq 6$. Then the sectional curvature of V is non-positive.*

PROOF. If a homogeneous convex cone V is linearly isomorphic to the direct product of homogeneous convex cones V_1 and V_2, then by using (1.1) or the property (1.2) we can see that the Riemannian manifolds (V, g_V) and $(V_1, g_{V_1})\times(V_2, g_{V_2})$ are isometric. On the other hand, from the classification of homogeneous convex cones of low dimensions (cf. [4]) it can be seen that V is self-dual, or otherwise V is linearly isomorphic to one of the following: $V_{1,1}$, $V'_{1,1}$, $V_{2,1}$, $V'_{2,1}$, $V_{1,1}\times\boldsymbol{R}^+$, $V'_{1,1}\times\boldsymbol{R}^+$.

Thus, we obtain the above statement. q.e.d.

From the above theorem, Proposition 2.1 and the Sasaki's result stated in § 1, we have the following

COROLLARY 4.8. *Let S be a homogeneous hyperbolic affine hypersphere in $\boldsymbol{R}^n$ with $n \leqq 6$. Then the sectional curvature of S with respect to the affine metric is non-positive.*

Finally we would like to add corrections to the misstatements contained in the note [11]. The above Theorem 4.7 is a correct form of Theorem 2 in [11]. A homogeneous convex cone in Theorem 3 of [11] should be read as an *irreducible* homogeneous convex cone.

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