TOKYO J. MATH.
VOL. 5, NO. 2, 1982

Moduli Space of Polarized del Pezzo Surfaces and its Compactification

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(Communicated by K. Ogiue)

Introduction

A non-singular rational projective surface X over an algebraically closed field k is called a del Pezzo surface if the inverse of the canonical sheaf ω_X^{-1} is ample. It is called a del Pezzo surface of degree d if $\omega_X^{-1}\cdot\omega_X^{-1}=d$. It is known that the degree of a del Pezzo surface is at most 9 and the surface is isomorphic to $\boldsymbol{P}^2$ if $d=9$, $\boldsymbol{P}^1\times\boldsymbol{P}^1$ or $\boldsymbol{F}_1$ if $d=8$, the image of $\boldsymbol{P}^2$ under a monoidal transformation with center $(9-d)$-closed points in general position (cf. Definition 1) if $1\leqq d\leqq 7$ ([2]). Let X be a del Pezzo surface of degree $d\leqq 7$ and $f: X\to\boldsymbol{P}^2$ be a monoidal transformation of $\boldsymbol{P}^2$ with center $(9-d)$-points in general position. We call the sheaf $f^*\mathcal{O}_{P^2}(1)$ a contraction sheaf on X.

In this article we first construct the moduli space of del Pezzo surfaces of degree d $(1\leqq d\leqq 7)$ together with a contraction sheaf, and next construct its compactification in the sense of Definition 7.

In §1, we realize the moduli space of del Pezzo surfaces of degree $d(1\leqq d\leqq 7)$ as the geometric quotient by $PGL(2)$ of the open subspace U_d of $\mathrm{Sym}^{9-d}\boldsymbol{P}^2$, where U_d consists of the points which represent $(9-d)$-points in general position in $\boldsymbol{P}^2$.

In §2, to construct a "good" compactification of our moduli space, we take a blowing up of the subspace containing U_d with the center outside of U_d and next take its universal categorical quotient. Then we see that a point on the boundary corresponds to an irreducible reduced surface with possible A_1-singularities, which is isomorphic to the image of monoidal transformation of $\boldsymbol{P}^2$ with the center $(9-d)$-points allowing at most double points, where double point means a subscheme defined by an maximal primary ideal $\mathcal{I}$ in $\mathcal{O}_{P^2}$ such that $\dim_k\mathcal{O}_{P^2}/\mathcal{I}=2$ as a k-vector space.

Received February 14, 1981

Throughout this note we fix an algebraically closed field k of arbitrary characteristic. Terminologies "stable", "semistable" etc. will be used according to [4].

§1. Construction of the moduli space of polarized del Pezzo surfaces.

DEFINITION 1. A finite set of closed points in P^2 is said to be in general position if no three of these points lie on one line, no six of them lie on one conic and there are no cubics which pass through seven of them and have a double point at the eighth point. If X is a del Pezzo surface of degree d $(1 \leqq d \leqq 7)$ and $f: X \to P^2$ is the blowing up of P^2 with the center $(9-d)$-points in general position, we call this morphism a canonical contraction of X.

DEFINITION 2. Let X be a del Pezzo surface of degree $d \leqq 7$ and $f: X \to P^2$ be a canonical contraction. We call the sheaf $f^* \mathcal{O}_{P^2}(1)$ a contraction sheaf on X.

Henceforth, we fix $1 \leqq d \leqq 7$ and put $n = 9-d$.

DEFINITION 3. Let X be any scheme. A geometric point of X means a morphism $\operatorname{Spec} K \to X$ where K is an algebraically closed field. For a geometric point x of X and a morphism $f: Y \to X$, we denote the fiber product $\operatorname{Spec}(K) \times_X Y$ by Y_x.

DEFINITION 4. A smooth projective morphism $\mathscr{X} \to S$ with a line bundle $\mathscr{L}$ on $\mathscr{X}$ is called a contractably polarized family of del Pezzo surfaces of degree d if it satisfies the followings;

(i) for any geometric point s of S, $\mathscr{X}_s$ is a del Pezzo surface of degree d, and

(ii) $\mathscr{L}|_{\mathscr{X}_s}$ is a contraction sheaf on $\mathscr{X}_s$.

Let $(\mathscr{X} \xrightarrow{\pi} S, \mathscr{L})$, $(\mathscr{Y} \xrightarrow{\pi'} S, \mathscr{L}')$ be two contractably polarized families. We say these families are isomorphic if there are an isomorphism $f: \mathscr{X} \xrightarrow{\sim} \mathscr{Y}$ over S and a line bundle $\mathscr{L}_0$ on S such that $\mathscr{L} = \pi^* \mathscr{L}_0 \otimes f^* \mathscr{L}'$.

If S is the spectrum of a field R, a contractably polarized family of del Pezzo surfaces over S is called simply a del Pezzo surface over R.

DEFINITION 5. For any Noetherian k-scheme S, define $\mathscr{M}_d(S)$ to be the set of all isomorphism classes of contractably polarized families of del Pezzo surfaces of degree d on S.

Note that the collection of sets $\mathscr{M}_d(S)$ form a contravariant functor from the category of Noetherian k-schemes to the category of sets:

i.e., given any morphism $f\colon T\to S$, a map $\mathscr{M}_d(f)\colon \mathscr{M}_d(S)\to\mathscr{M}_d(T)$ is defined by associating to a family $\mathscr{X}\to S$, the "pull-back" family $\mathscr{X}\times_S T\to T$.

DEFINITION 6. For a Noetherian k-scheme S define $F_d(S)$ to be the set of all isomorphism classes of commutative diagrams

$$\begin{array}{ccc} \mathscr{X} & & \\ \downarrow & \searrow^{f} & \\ & & \boldsymbol{P}^2\times S\,, \\ & \swarrow_{p_2} & \cup \\ S & & Z \end{array}$$

where $p_2|_Z$ is étale and finite of degree n, such that for any geometric point s of S, Z_s is a set of points in general position in $\boldsymbol{P}^2_s$ and $f_s\colon \mathscr{X}_s\to\boldsymbol{P}^2_s$ is the blowing up with the center Z_s. Here we say two such diagrams are isomorphic if each object of one diagram is isomorphic to the corresponding object of the other and the morphisms are compatible.

PROPOSITION 1. *F_d is representable by an open subspace U_d of* $\mathrm{Hilb}^n\boldsymbol{P}^2$ *consisting of the points that represent n points in general position in $\boldsymbol{P}^2$.*

PROOF. For any element of $F_d(S)$, the subscheme $Z\subset\boldsymbol{P}^2\times S$ has Hilbert polynomial n so gives an element of $\mathrm{Hom}_k(S, U_d)$. Conversely, for any element of $\mathrm{Hom}_k(S, U_d)$, let $Z\to S$ be the pull-back family of the universal family on the Hilbert scheme and $f\colon\mathscr{X}\to\boldsymbol{P}^2\times S$ be the blowing up with center Z. Then the diagram

$$\begin{array}{ccc} \mathscr{X} & & \\ \downarrow & \searrow^{f} & \\ & & \boldsymbol{P}^2\times S \\ & \swarrow_{p_2} & \cup \\ S & & Z \end{array}$$

will give an element of $F_d(S)$ by the following lemma. It will be easily checked that the triple $\boldsymbol{P}^2\times S$, Z, S satisfy the condition of the lemma.

LEMMA 1. *Let $f\colon X\to Y$ be a flat morphism of k-schemes and Z be a closed subscheme of X. Denote the ideal of Z in $\mathscr{O}_X$ by $\mathscr{I}$. If $\mathscr{O}_X/\mathscr{I}^n$ is flat over $\mathscr{O}_Y$ for every $n>0$, then* $\mathrm{Proj}\,(\bigoplus\mathscr{I}^n)$ *is flat over Y and for any field K and K-valued point* $\mathrm{Spec}\,K\to Y$, *the fiber product*

$\operatorname{Proj}(\oplus \mathscr{I}^n) \times_Y \operatorname{Spec} K$ *is isomorphic to the blowing up of* $X \times_Y \operatorname{Spec} K$ *with center* $Z \times_Y \operatorname{Spec} K$.

THEOREM 1. *If there exists the geometric quotient of* U_d *by* $PGL(2)$, *it is the coarse moduli space of the functor* $\mathscr{M}_d$.

PROOF. There is a natural morphism of functors;

$$\eta: F_d \longrightarrow \mathscr{M}_d .$$

Let $\mathscr{P}\mathscr{G}\mathscr{L}(2)$ be the functor represented by $PGL(2)$. Then the action of the algebraic group $PGL(2)$ on U_d induces an action of the functor $\mathscr{P}\mathscr{G}\mathscr{L}(2)$ on the functor F_d which we write

$$\sigma: \mathscr{P}\mathscr{G}\mathscr{L}(2) \times F_d \rightarrow F_d .$$

Then it is obvious that

$$\eta \circ \sigma = \eta \circ p_2 . \tag{1}$$

For a Noetherian k-scheme S, let $\mathscr{M}_d'(S)$ be the quotient of the set $\operatorname{Hom}_k(S, U_d)$ by the action of the group $\operatorname{Hom}_k(S, PGL(2))$. Let $\mathscr{M}_d'$ be the functor defined by this collection of sets and by the obvious maps between them. According to (1), the morphism η factors:

$$F_d \xrightarrow{\eta'} \mathscr{M}_d' \xrightarrow{I} \mathscr{M}_d .$$

If we show the following lemma, the theorem follows in just the same way as Proposition 5.4 of [3].

LEMMA 2. *I is injective and for any* $\alpha \in \mathscr{M}_d(S)$, *there is an open covering* $\{U_i\}$ *of* S, *such that the restriction of* α *to* $\mathscr{M}_d(U_i)$, *for all* i, *is in the image of* I.

PROOF OF LEMMA. To prove the first statement, let ϕ_1 and ϕ_2 be morphisms of S to U_d and let

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{f} & P^2 \times S \\ & {\scriptstyle \pi}\searrow \; \swarrow & \\ & S & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathscr{X}' & \xrightarrow{f'} & P^2 \times S \\ & {\scriptstyle \pi'}\searrow \; \swarrow & \\ & S & \end{array}$$

be the corresponding diagrams. Suppose $\mathscr{X}$ and $\mathscr{X}'$ are isomorphic over S;

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{g} & \mathscr{X}' \\ & \searrow \; \swarrow & \\ & S & \end{array} .$$

Put $\mathscr{L}$, $\mathscr{L}'$ the line bundles on $\mathscr{X}$, $\mathscr{X}'$ respectively, which give polarizations. Then $g^*\mathscr{L}'=\mathscr{L}\otimes\pi^*\mathscr{L}_0$ for a suitable invertible sheaf $\mathscr{L}_0$ on S. Then we get a commutative diagram

$$\begin{array}{ccccc} P(\pi_*\mathscr{L}) & \xrightarrow{\sim} & P((\pi_*\mathscr{L})\otimes\mathscr{L}_0) & \xrightarrow{\sim} & P(\pi_*g^*\mathscr{L}') \, . \\ {\scriptstyle f}\uparrow & & & \nearrow {\scriptstyle f'\cdot g} & \\ \mathscr{X} & & & & \end{array}$$

This shows us that ϕ_1 and ϕ_2 give the same element in $\mathscr{M}_d'(S)$.

To prove the second statement, let α correspond to the contractably polarized family $(\mathscr{X}\to S, \mathscr{L})$. Since $h^1(\mathscr{X}_s, \mathscr{L}|_{\mathscr{X}_s})=0$ and $h^0(\mathscr{X}_s, \mathscr{L}|_{\mathscr{X}_s})=3$, for any geometric point s of S, $\pi_*\mathscr{L}$ is a locally free sheaf of rank 3 on S (see, for example, [1], Theorem 12.11. p. 290). Thus we get an open covering $\{U_i\}$ such that $\pi_*\mathscr{L}|_{U_i}$ is free for all i. There is a commutative diagram

$$\begin{array}{ccc} \mathscr{X}|_{U_i} & \xrightarrow{f_i} & P^2\times U_i \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

for every i. Here f_i is a contraction of a divisor to the subscheme Z of $P^2\times U_i$ which is finite over U_i. We can see this comes from $F_d(U_i)$.

THEOREM 2. *There exists the geometric quotient of U_d by the action of $PGL(2)$.*

PROOF. Since U_d consists of only one orbit for $d\geqq 5$, we have only to show the cases $1\leqq d\leqq 4$. There is the canonical birational $PGL(2)$-linear morphism Φ: $\mathrm{Hilb}^n P^2\to\mathrm{Sym}^n P^2$, which is an isomorphism on U_d. Let Ψ: $(P^2)^n\to\mathrm{Sym}^n P^2$ be the canonical finite morphism. We will show that any point of $\Phi(U_d)$ is $PGL(2)$-stable with respect to an ample $PGL(2)$-linearized invertible sheaf. To this end, we have only to show $\mu^{p_i^*(\mathscr{O}_{P^2}(1))}(x, \lambda)>0$ for any $x\in\Psi^{-1}\cdot\Phi(U_d)$ and any $1-PS$ λ of $SL(3)$, where p_i: $(P^2)^n\to P^2$ is the projection to the i-th factor. Here;

$$\mu^{p_i^*(\mathscr{O}_{P^2}(1))}(x, \lambda)=\sum_{i=1}^{n}\mu^{\mathscr{O}(1)}(x^{(i)}, \lambda) \, . \tag{2}$$

Write λ: $G_m\to SL(3)$ by $\lambda(t)=\begin{bmatrix} t^\alpha & 0 & 0 \\ 0 & t^\beta & 0 \\ 0 & 0 & t^{-(\alpha+\beta)} \end{bmatrix}$ where $\alpha\geqq\beta\geqq-(\alpha+\beta)$ by choosing a suitable coordinates $\{z_0, z_1, z_2\}$ of P^2. Let H be the line on P^2 defined by $z_2=0$. Then, for a point $z\in P^2$, $\mu^{\mathscr{O}(1)}(z, \lambda)=\alpha+\beta$ if $z\notin H$,

$\mu^{\mathcal{O}(1)}(z, \lambda)=-\beta$ if $z\in H$ and $z\neq(1, 0, 0)$, and $\mu(z, \lambda)=-\alpha$ if $z=(1, 0, 0)$ by the virtue of Proposition 2.3 of [3]. Now, if we put $m=$(number of i such that $x^{(i)}\in H$ and $x^{(i)}\neq(1, 0, 0)$) and $r=$(number of i such that $x^{(i)}=(1, 0, 0)$), then, by (2);

$$\mu^{p_i^*(\mathcal{O}(1))}(x, \lambda)=(\alpha+\beta)(n-(m+r))-\beta m-\alpha r . \tag{3}$$

Since $x\in\Psi^{-1}\Phi(U_d)$, we have $m+r\leqq 2$ and $r\leqq 1$. Therefore we get an inequality $\mu(x, \lambda)\geqq 3(\alpha+\beta)-\beta m-\alpha r$. We can easily check that the right hand side is positive for any possible values of m and r.

REMARK 1. By the equation (3) of the proof of Theorem 2, we summerize stable (or semi-stable) points in $\mathrm{Sym}^n \boldsymbol{P}^2$ in table (4) below, which will be used in next section. We denote "stable" (resp. "semi-stable", "unstable") by "s" (resp. "ss", "u").

TABLE (4). Stability of the point $x=(x^{(1)}, \cdots, x^{(9-d)})$

	d	1	2	3	4
the cases any points of P^2 appear in $\{x^{(i)}\}$ at most once	no three of $\{x^{(i)}\}$ lie on one line	s	s	s	s
	three of them may lie on one line	s	s	s	s
	four of them may lie on one line	s	s	ss	u
	five of them may lie on one line	s	u	u	u
	six or more of them may lie on one line	u	u	u	—
the cases some points of P^2 may appear in $\{x^{(i)}\}$ at most twice	three of them may lie on one line	s	s	ss	u
	four of them may lie on one line	s	s	ss	u
	five of them may lie on one line	s	u	u	u
	six or more of them may lie on one line	u	u	u	—
the cases three times or more		u	u	u	u

§ 2. A compactification of the moduli space of del Pezzo surfaces.

We denote by $\mathscr{DP}_d$ the moduli space of contractably polarized del Pezzo surfaces of degree d and by D_x the del Pezzo surface corresponding to a k-valued point x. Since $\mathscr{DP}_d$ is one point for $d=5, 6, 7$ we may restrict our discussion to the cases $1\leqq d\leqq 4$ to construct a compactification of $\mathscr{DP}_d$.

DEFINITION 7. A proper scheme $\mathscr{D}_d$ of finite type over k is a good compactification of the moduli space $\mathscr{DP}_d$, if $\mathscr{D}_d$ contains $\mathscr{DP}_d$ as an open dense subspace and $\mathscr{D}_d$ satisfies the following conditions;

(1) any k-valued point x of $\mathscr{D}_d-\mathscr{DP}_d$ corresponds to a surface D_x

(2) let X be any del Pezzo surface of degree d over $K=k((t))$ and f: Spec $K\to\mathscr{DP}_d$ be the morphism corresponding to X. For any extension $\bar{f}$: Spec $k[[t]]\to\mathscr{D}_d$ of f, there exists a flat projective morphism π: $\mathscr{X}\to$ Spec $k[[t]]$ and a finite morphism ρ: Spec $k[[t]]\to$ Spec $[[t]]$ such that (i) the generic fiber of π is isomorphic to the pull-back of X by ρ and (ii) the geometric fiber is D_x, where x is the image of the closed point by $\bar{f}$.

Before proving the main theorem of this section, we prepare a lemma.

LEMMA 3. *Let X be a non-singular projective surface over k and U an open subspace of* Hilb^n X *for $n>0$. Assume that any closed point of U represents a zero dimensional subscheme of X allowing at most double points, where double point means a subscheme defined by a maximal primary ideal whose codimension in $\mathscr{O}_X$ is 2 as a k-vector space.*

Then the restriction Z on U of the universal family on Hilb^n X *satisfies the condition of Lemma* 1.

PROOF OF LEMMA. Denote the ideal sheaf of Z in $X\times U$ by $\mathscr{I}$. Since Hilb^n X is non-singular, U is reduced a fortiori. We have only to show $\dim(\mathscr{O}_{X\times U}/\mathscr{I}^m)\otimes k(s)$ is constant for all closed points s of U. If Z_s consists of r double points and $(n-2r)$ non-singular points, then we have

$$\begin{aligned}\dim(\mathscr{O}_{X\times U}/\mathscr{I}^m)\otimes k(s) &= (n-2r)\cdot\dim(k[x, y]/(x, y)^m)\\ &\quad +r\cdot\dim(k[x, y]/(x, y^2)^m)\\ &=\frac{m(m+1)}{2}(n-2r)+m(m+1)r\\ &=\frac{m(m+1)}{2}n\,.\end{aligned}$$

Hence the dimension is independent of r.

REMARK 2. If Z_s has triple or higher multiple points, the assertion of Lemma 3 is not true.

Henceforth we fix d $(1\leqq d\leqq 4)$ and put $n=9-d$. To simplify notations we will denote $\mathrm{Sym}^n \boldsymbol{P}^2$ by S. Let $\mathscr{L}$ be an ample $SL(3)$-linearized invertible sheaf on S which we took in the proof of Theorem 2. Let $\mathscr{M}$ be an ample $SL(3)$-linearized invertible sheaf on $\Phi^{-1}(S^{ss}(\mathscr{L}))$ (it exists, since $\mathrm{Hilb}^n \boldsymbol{P}^2$ has an ample $SL(3)$-linearized invertible sheaf).

Theorem 3. *There exists a number* $N>0$ *such that*

$$\Phi^{-1}(S^s(\mathscr{L}))\subset(\Phi^{-1}(S^{ss}(\mathscr{L})))^s(\mathscr{L}^N\otimes\mathscr{M}) .$$

Moreover, for this N, *the quotient* $\mathscr{D}=(\Phi^{-1}(S^{ss}(\mathscr{L})))^{ss}(\mathscr{L}^N\otimes\mathscr{M})/PGL(2)$ *is a good compactification of* $\mathscr{DP}_d$, *and the morphism from* $\mathscr{D}$ *to* $S^{ss}(\mathscr{L})/PGL(2)$ *induced by* Φ *is surjective.*

Proof of Theorem 3. We may assume $\mathscr{M}$ is relatively very ample with respect to Φ. The first statement is shown in Proposition 2.18 of [3]. Let $\Phi^{-1}(S^{ss}(\mathscr{L}))\hookrightarrow P=\boldsymbol{P}(\Phi_*(\mathscr{L}^N\otimes\mathscr{M})|_{S^{ss}(\mathscr{L})})$ be the $PGL(2)$-linear closed immersion defined by the relatively very ample $PGL(2)$-linearized line bundle $\mathscr{L}^N\otimes\mathscr{M}$. There exists a rational map;

$$p: P\longrightarrow\bar{P}=\mathrm{Proj}\,(\mathscr{S}(\Phi_*(\mathscr{L}^N\otimes\mathscr{M}))^{PGL(2)}|_{S^{ss}(\mathscr{L})})$$

which is a morphism on the open subspace $P-F$, where F is the set of unstable points with respect to $\mathscr{O}_P(1)$ and $\mathscr{S}(*)$ means the symmetric algebra of $*$. On the other hand there exists a morphism $\Phi'\colon\bar{P}\rightarrow S^{ss}(\mathscr{L})/PGL(2)$. By Nagata's theorem (see, for example, [4]), $(\mathscr{S}(\Phi_*(\mathscr{L}^N\otimes\mathscr{M})))^{PGL(2)}|_{S^{ss}(\mathscr{L})}$ is an $(\mathscr{O}_{S^{ss}(\mathscr{L})})^{PGL(2)}$-algebra of finite type, so Φ' is projective. Since $\mathscr{D}$ is closed in $\bar{P}$, the restriction $\Phi|_{\mathscr{D}}\colon\mathscr{D}\rightarrow S^{ss}(\mathscr{L})/PGL(2)$ is a projective morphism. By the first statement, $\Phi'|_{\mathscr{D}}$ is dominating, so $\Phi'|_{\mathscr{D}}$ is surjective to $S^{ss}(\mathscr{L})/PGL(2)$. Now we will show that $\mathscr{D}$ is a good compactification of $\mathscr{DP}_d$. Since $\mathscr{D}$ is projective over $S^{ss}(\mathscr{L})/PGL(2)$ and the latter is projective over k, $\mathscr{D}$ is a projective k-scheme. It is clear that $\mathscr{D}$ contains $\mathscr{DP}_d$ as an open dense subset. Now we have to determine a suitable surface D_x for a closed point $x\in\mathscr{D}-\mathscr{DP}_d$. Let

$$\begin{array}{ccc} Z & \hookrightarrow & \boldsymbol{P}^2\times\Phi^{-1}(S^{ss}(\mathscr{L})) \\ & \searrow & \downarrow \\ & & \Phi^{-1}(S^{ss}(\mathscr{L})) \end{array}$$

be the restriction on $\Phi^{-1}(S^{ss}(\mathscr{L}))$ of the universal family on $\mathrm{Hilb}^n \boldsymbol{P}^2$, and $\tilde{P}$ be the blowing up of $\boldsymbol{P}^2\times\Phi^{-1}(S^{ss}(\mathscr{L}))$ with center Z. Then by Lemma 1, Lemma 3 and table (4), $\tilde{P}$ is flat over $\Phi^{-1}(S^{ss}(\mathscr{L}))$. For any closed point x of $\mathscr{D}-\mathscr{DP}_d$, take a closed point $\bar{x}$ of $\Phi^{-1}(S^{ss}(\mathscr{L}))-F$ which corresponds to x and belongs to the minimal orbit. Denote the fiber $\tilde{P}_{\bar{x}}$ by D_x. This is our desired surface corresponding to x. In fact, given a del Pezzo surface π_0: $X \to \operatorname{Spec} K$ over $K = k((t))$, we get the corresponding morphism f: $\operatorname{Spec} K \to \mathscr{D}\mathscr{P}_d$. Let $\bar{f}$: $\operatorname{Spec} k[[t]] \to \mathscr{D}$ be an extension of f. Then, by Shah (Proposition 2.1 of [5]), there is a finite morphism ρ: $\operatorname{Spec} k[[t]] \to \operatorname{Spec} k[[t]]$ and a section g: $\operatorname{Spec} k[[t]] \to \Phi^{-1}(S^{ss}(\mathscr{L})) - F$ such that $\phi \circ g = \bar{f} \circ \rho$ where ϕ: $\Phi^{-1}(S^{ss}(\mathscr{L})) - F \to \mathscr{D}$ is the canonical projection. Moreover his proposition gives us the image of the geometric point by g is contained in the minimal orbit. The base change $\tilde{P}_{\operatorname{Spec} k[[t]]}$ by g: $\operatorname{Spec} k[[t]] \to \Phi^{-1}(S^{ss}(\mathscr{L})) - F$ is a flat family satisfying the condition of Definition 7.

REMARK 3. D_x is an irreducible reduced surface with at worst singular points A_1-type. The number of singular points is at most four if $d=1$, three if $d=2, 3$ and zero if $d=4$. In fact, by Lemma 1 and Lemma 3, D_x is the blowing up of $\boldsymbol{P}_k^2$ with center $Z_{\bar{x}}$, where $\bar{x}$ is lying over x and belongs to the minimal orbit in $\Phi^{-1}(S^{ss}(\mathscr{L})) - F$. Notice that the blowing up of non-singular rational surface with center a double point has one singular point of A_1-type. Hence we get the above assertion by table (4).

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