Some Properties of Vector Bundles on the Flag Variety $Fl(r, s; n)$

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Introduction

In this article we would like to begin to study vector bundles over flag varieties. It is well-known that every vector bundle over a projective line P_1 is a direct sum of line bundles. Thus for a given vector bundle over a variety F, after restricting it to each line contained in F, by examining which line bundles appear as direct summands, we can analyze its properties. This method was developed in Van de Ven [12], Barth [1], and Hartshorne [6], when F is a projective space.

In § 1 in this article, we construct explicitly a component $\hat{X}$ of the Hilbert scheme parametrizing "straight lines" in the flag variety $F = Fl(r, s; n) = \{(P_r, P_s) | P_r \subset P_s \subset P_n\}$ $(0 \leq r \leq s < n)$. (In our case the union Q of two copies of P_1 intersecting transversally at one point appears over a divisor $\hat{\Delta}$ in $\hat{X}$ as a deformation of the usual straight line. Thus it is necessary that we call Q "a line" as well. However, vector bundles over Q have simple properties and it is not an obstacle for our purposes.)

In § 2 we give a theorem concerned with Schubert varieties in F and $\hat{X}$. Now according to § 1, we can define tautological homogeneous vector bundles not only on F but also on $\hat{X}$. We will show in Theorem 2.1 that there is a systematic correspondence between Chern classes of tautological vector bundles on F and those on $\hat{X}$.

Let $\hat{\Gamma}$ be the universal family of subvarieties of F over $\hat{X}$. We have a diagram

$$
\begin{array}{ccc}
 & \hat{\Gamma} & \\
{\scriptstyle\hat{\alpha}}\swarrow & & \searrow{\scriptstyle\hat{\beta}} \\
\hat{X} & & F
\end{array}
$$

such that for every point $x \in \hat{X}$, $\hat{\beta}\hat{\alpha}^{-1}(x)$ is the subvariety corresponding to $x \in \hat{X}$. Let $\mathscr{M}$ be a vector bundle on F with $c_1(\mathscr{M}) = 0$ such that

for some point $x_0 \in \hat{X}$, the restriction $\hat{\beta}^* \mathcal{M}|_{\hat{\alpha}^{-1}(x_0)}$ is trivial. Let J be the set of jumping lines, i.e., $J = \{x \in \hat{X} \mid \hat{\beta}^* \mathcal{M}|_{\alpha^{-1}(x)}$ is *not* trivial$\}$. In § 3, we show the next equality in the first Chow group $A^1 \hat{X}$

$$[J] = \hat{\alpha}_* \hat{\beta}^* c_2(\mathcal{M}) \ .$$

In the case where F is a projective space, this equality has been given in Barth [1], which plays an important role in the theory of vector bundles on complex projective spaces. (Cf. Okonek [8].) Our theorem is a generalization of Theorem 2 in Barth [1]. However our proof is entirely different from his one. The relative Riemann-Roch theorem plays an important role in our proof.

Every variety is assumed to be defined over an algebraically closed ground field k of arbitrary characteristic. A locally free coherent sheaf is called a vector bundle. For linear subspaces G, H of an n-dimensional projective space P_n, the minimum linear subspace of P_n containing both G and H is called the join of G and H, and it is denoted by $G \vee H$. For a variety Z with the structure sheaf $\mathcal{O}_Z$ and for a vector space W, we denote $W_Z = W \otimes_k \mathcal{O}_Z$ for simplicity.

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§ 1. The space parametrizing straight lines in F.

Let

$$F = Fl(r, s; n) = \{(P_r, P_s) \mid P_r \subset P_s \subset P_n\} \quad 0 \leq r \leq s < n$$

be the flag variety parametrizing flags in P_n made up of two linear subspaces. (If $r = s$, F is the Grassmann variety $Gr(r; n)$. In particular if $r = s = 0$, F coincides with the projective space P_n.) In this article we study on F. We use the following convenient abbreviation:

$$F = \left\{{}_{G^r}\diagup{}^{H^s}\right\} \ .$$

Where G and H denote linear subspaces in P_n, indices r and s denote the dimension respectively, the bar — denotes the inclusion relation and the brackets $\{\ \}$ means that the set of all linear subspaces of P_n satisfying the condition indicated in them are under consideration.

By this abbreviation we define varieties Γ and X.

$$\Gamma = \left\{ \begin{matrix} C^{r+1} & \!\!\!\!\!\!\!\! \underline{\quad\quad} \!\!\!\!\!\!\!\! & D^{s+1} \\ | & & | \\ G^r & \!\!\!\!\!\!\!\! \underline{\quad\quad} \!\!\!\!\!\!\!\! & H^s \\ | & & | \\ A^{r-1} & \!\!\!\!\!\!\!\! \underline{\quad\quad} \!\!\!\!\!\!\!\! & B^{s-1} \end{matrix} \right\} \qquad X = \left\{ \begin{matrix} C^{r+1} & \!\!\!\!\!\!\!\! \underline{\quad\quad} \!\!\!\!\!\!\!\! & D^{s+1} \\ | & & | \\ | & & | \\ A^{r-1} & \!\!\!\!\!\!\!\! \underline{\quad\quad} \!\!\!\!\!\!\!\! & B^{s-1} \end{matrix} \right\} .$$

A, B, C, D, G and H are linear subspaces of P_n. If $r=0$, A denotes the empty set. In case $r=s$, the above definition is equal to the following.

$$\Gamma = \left\{ \begin{matrix} C^{r+1} \\ | \\ G^r \\ | \\ A^{r-1} \end{matrix} \right\} \qquad X = \left\{ \begin{matrix} C^{r+1} \\ | \\ A^{r-1} \end{matrix} \right\} .$$

Canonical morphisms

$$\Gamma \xrightarrow{\ \ \alpha\ \ } X \qquad\qquad \Gamma \xrightarrow{\ \ \beta\ \ } F$$
$$\omega \qquad\qquad\qquad \omega \qquad\qquad\quad \omega \qquad\qquad\qquad \omega$$
$$\begin{pmatrix} C & D \\ G & H \\ A & B \end{pmatrix} \longmapsto \begin{pmatrix} C & D \\ A & B \end{pmatrix} \qquad \begin{pmatrix} C & D \\ G & H \\ A & B \end{pmatrix} \longmapsto (G \quad H)$$

are defined.

Subvarieties Δ, Λ ($X \supset \Delta \supset \Lambda$) are defined as

$$\Delta = \left\{ \left. \begin{matrix} C & \!\!\!\!\!\! \underline{\quad} \!\!\!\!\!\! & D \\ | & & | \\ A & \!\!\!\!\!\! \underline{\quad} \!\!\!\!\!\! & B \end{matrix} \right| B \cap C \ne A \right\}$$

$$\Lambda = \left\{ \begin{matrix} & & & & D \\ & & & B \!\!\nearrow & \\ & & C \!\!\nearrow & & \\ A \!\!\nearrow & & & & \end{matrix} \right\} .$$

By an easy calculation we have $\dim \Delta = \dim X - 1$ and $\dim \Lambda = \dim X - 4$. By the definition of Γ we have for a closed point $x \in X$

if $x \in X - \Delta$, $\alpha^{-1}(x) \cong P_1$

if $x \in \Delta - \Lambda$, $\alpha^{-1}(x) \cong 2$ copies of P_1 intersecting
transversally at one point

and

if $x \in \Lambda$, $\alpha^{-1}(x) \cong P_1 \times P_1$.

Let

$\pi: \hat{X} \longrightarrow X$: the blowing-up with center Λ

and

$\hat{\pi} \colon \hat{\Gamma} \longrightarrow \Gamma$: the blowing-up with center $\alpha^{-1}(\Lambda)$

$\hat{\alpha} \colon \hat{\Gamma} \longrightarrow \hat{X}$: the induced map

$\hat{\beta} \colon \hat{\Gamma} \longrightarrow F$ be the composition $\hat{\beta} = \beta \circ \hat{\pi}$.

We have the next diagram.

(1.1)

Now let V be the dual vector space of $H^0(\boldsymbol{P}_n, \mathscr{O}_{\boldsymbol{P}_n}(1))$. A vector subbundle $\mathscr{G}$ of $V_F = V \otimes \mathscr{O}_F$ is defined by putting for a closed point $x = (G, H) \in F$,

$$\mathscr{G}_x \otimes k(x) = \tilde{G} \subset V = V_F \otimes k(x) ,$$

where the inclusion $\tilde{G} \subset V$ is the dual morphism of the surjection $H^0(\boldsymbol{P}_n, \mathscr{O}_{\boldsymbol{P}_n}(1)) \to H^0(G, \mathscr{O}_G(1))$ associated with the inclusion $G \subset \boldsymbol{P}_n$.

Analogously associating with H, we can define a vector bundle $\mathscr{H}$ and the inclusion relation $G \subset H \subset \boldsymbol{P}_n$ induces canonical morphisms

$$\mathscr{G} \longrightarrow \mathscr{H} \longrightarrow V_F .$$

Here both arrows are injective homomorphisms between vector bundles, i.e., the cokernels are again locally free. Thus we have an exact commutative diagram of vector bundles over F.

(1.2)

Analogously we have a commutative (not exact) diagram of vector bundles over X

(1.3)
$$
\begin{array}{ccc}
\mathscr{A} & \longrightarrow & \mathscr{C} \\
\downarrow & & \downarrow \\
\mathscr{B} & \longrightarrow & \mathscr{D} \longrightarrow V_X .
\end{array}
$$

THEOREM 1.1. (A) *Let* $\hat{\Delta}$ *be the strict inverse image of* Δ *by* π $(=\overline{\pi^{-1}(\Delta-\Lambda)})$. *For every closed point* $\hat{x} \in \hat{X}$,

$$if \quad \hat{x} \in \hat{X} - \hat{\Delta}, \quad \hat{\alpha}^{-1}(x) \cong P_1 \quad and$$

$$if \quad \hat{x} \in \hat{\Delta}, \quad \hat{\alpha}^{-1}(x) \cong two \ copies \ of \ P_1 \ intersecting$$

$$transversally \ at \ one \ point .$$

(B) $\hat{X}$ *is one connected component of the Hilbert scheme of* F. (Cf. Grothendieck [3].)

(C) $\hat{\Gamma}$ *is the graph of the correspondence associating every point of* $\hat{X}$ *with the subvariety of* F.

Whole this section is devoted to verify Theorem 1.1.

(A) We put $\hat{\Lambda} = \pi^{-1}(\Lambda)$. This $\hat{\Lambda}$ is the exceptional divisor of π. For a closed point $\hat{x} \in \hat{X} - \hat{\Lambda}$, (A) is obvious. Thus we consider a point $\hat{x} \in \hat{\Lambda}$.

On X the composition $\mathscr{C} \to \mathscr{D} \to \mathscr{D}/\mathscr{B}$ defines an element

$$\omega \in H^0(X, \mathscr{Hom}(\mathscr{C}/\mathscr{A}, \mathscr{D}/\mathscr{B})) .$$

LEMMA 1.2. *The zero locus of* $\omega = \Lambda$.

PROOF. By definition, it is easy to see that they are equal as sets. Thus we have only to show that the zero locus of ω has the reduced scheme structure.

We fix linear subspaces $A_0^r \subset B_0^s \subset D_0^{s+2} \subset P_n$ and set

$$
\Sigma = \left\{
\begin{array}{c}
\left(
\begin{array}{ccc}
C^{r+1} & \!\!\!\!\!\text{---}\!\!\!\!\! & D_0 \\
| & & | \\
A_0 & \!\!\!\!\!\text{---}\!\!\!\!\! & B_0
\end{array}
\right) \ C
\end{array}
\right\} \subset X .
$$

Obviously Σ is isomorphic to the Grassmann variety $\mathrm{Gr}(1; s-r+1)$. Let $\bar{\omega}$ be the restriction of ω to Σ. We have {the zero locus of ω}$\cap \Sigma =$ {the zero locus of $\bar{\omega}$}$= Z$. By choosing local coordinates on Σ and by an easy calculation we know that Z coincides with

$$\left\{ \begin{matrix} & & & D_0 \\ & & B_0 \nearrow & \\ & C \nearrow & & \\ A_0 \nearrow & & & \end{matrix} \right\} \subset \Sigma \ .$$

Thus Z and the zero-locus of ω itself are reduced.   Q.E.D.

COROLLARY 1.3. $N_{A/X} \cong \mathcal{H}om\,(\mathcal{C}/\mathcal{A},\ \mathcal{D}/\mathcal{B})\,|_A.$

PROOF.   By Lemma 1.2 and by the following Lemma 1.4, it is obvious.   Q.E.D.

The next lemma is well-known and we omit the proof.

LEMMA 1.4.   Let $\mathcal{F}$ be a vector bundle of rank r over a variety Z. Let $f \in H^0(Z,\ \mathcal{F})$ be a section and Π be the zero locus of f. If $\mathrm{codim}_Z \Pi = r$, then we have $N_{\Pi/Z} \cong \mathcal{F}\,|_{\Pi}$.

On Γ, we have a canonical surjective homomorphism $\alpha^*(\mathcal{D}/\mathcal{B}) \to \alpha^*\mathcal{D}/\beta^*\mathcal{H}$ and a canonical injective homomorphism $\beta^*\mathcal{C}/\alpha^*\mathcal{A} \to \alpha^*(\mathcal{C}/\mathcal{A})$. Let

$$\gamma\colon \alpha^*\,\mathcal{H}om\,(\mathcal{C}/\mathcal{A},\ \mathcal{D}/\mathcal{B}) \longrightarrow \mathcal{H}om\,(\beta^*\mathcal{C}/\alpha^*\mathcal{A},\ \alpha^*\mathcal{D}/\beta^*\mathcal{H})$$

be the homomorphism induced by them.

LEMMA 1.5.   For $\alpha^*\omega \in H^0(\Gamma,\ \alpha^*\mathcal{H}om(\mathcal{C}/\mathcal{A},\ \mathcal{D}/\mathcal{B}))$, we have $\gamma(\alpha^*\omega) = 0$.

PROOF.   It is obvious by definition.   Q.E.D.

COROLLARY 1.6. $N_{\alpha^{-1}(A)/\Gamma} \cong \mathrm{Ker}\ \gamma\,|_{\alpha^{-1}(A)}.$

LEMMA 1.7. $\alpha^{-1}(A)$ is a smooth variety.

PROOF.   Set-theoretically we have

$$\alpha^{-1}(A) = \left\{ \begin{matrix} & & & & & D \\ & & & & H \nearrow & \\ & & & B \nearrow & & \\ & & C \nearrow & & & \\ & G \nearrow & & & & \\ A \nearrow & & & & & \end{matrix} \right\} \ .$$

Thus the underlying reduced variety $\alpha^{-1}(A)_{\mathrm{red}}$ is smooth. On the other hand, by Lemma 1.2, $\alpha^{-1}(A)$ is the zero-locus of $\alpha^*\omega$. Thus it is enough to show that $\hat{\omega} = \alpha^*\omega$ has a reduced zero-locus when we regard it as an element of $H^0(\Gamma,\ \mathrm{Ker}\ \gamma)$.

We fix A_0, B_0, D_0 and H_0. Let

$$\Theta = \left\{ \left. \begin{matrix} C^{r+1} \!\!\!\!\!\!-\!\!\!\!\!\!- D_0^{s+1} \\ | \qquad\qquad | \\ G^r \!\!\!\!\!\!-\!\!\!\!\!\!-\!\!\!\!\!\!- H_0^s \\ | \qquad\qquad | \\ A_0^{r-1} \!\!\!\!\!\!-\!\!\!\!\!\!- B_0^{s-1} \end{matrix} \right| \ C, G \right\} \subset \Gamma \ .$$

By choosing an appropriate local system of coordinates on Θ and by representing $\hat{\omega}$ by it we know that $\Theta \cap \{$the zero-locus of $\hat{\omega}\}$ and thus the zero-locus of $\hat{\omega}$ itself are reduced. (See the following figure.)   Q.E.D.
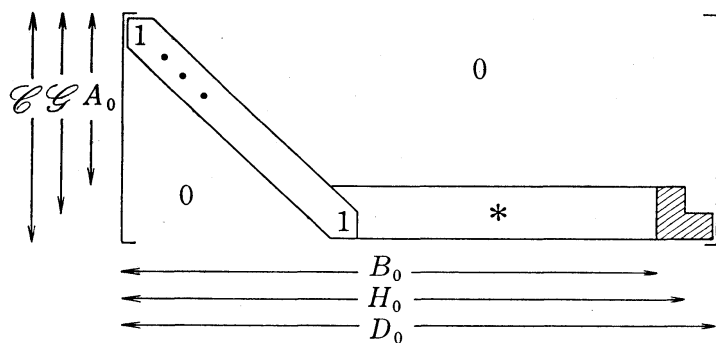
COROLLARY 1.8. $\hat{X}$ and $\hat{\Gamma}$ are smooth varieties.

COROLLARY 1.9.   The map $\hat{\alpha} \colon \hat{\alpha}^{-1}(\hat{\Lambda}) \to \hat{\Lambda}$ can be identified with the map $P(\mathrm{Ker}\ \gamma|_{\alpha^{-1}(\Lambda)}) \to P(\mathscr{H}\!om\,(\mathscr{C}/\mathscr{A},\ \mathscr{D}/\mathscr{B})|_\Lambda)$ induced by an injective homomorphism of vector bundles

$$\mathrm{Ker}\ \gamma|_{\alpha^{-1}(\Lambda)} \to \alpha^*(\mathscr{H}\!om\,(\mathscr{C}/\mathscr{A},\ \mathscr{D}/\mathscr{B})|_\Lambda) \ .$$

COROLLARY 1.10.   Let $x = \begin{pmatrix} C & D \\ A & B \end{pmatrix} \in \Lambda$ be a closed point. Then,

(1) $\pi^{-1}(x)$ can be identified with

$$P = P((C/A)^\vee \otimes (D/B)) \ . \quad (^\vee \text{ denotes the dual linear space.})$$

(2) Moreover if $\hat{x} \in \pi^{-1}(x) \subset \hat{\Lambda}$ corresponds to the point $u \in P$,

$\hat{\alpha}^{-1}(\hat{x}) \cong P_1$ if $u \notin P((C/A)^\vee) \times P(D/B)$

$\hat{\alpha}^{-1}(\hat{x}) \cong 2$ copies of P_1 intersecting at one point transversally
$\qquad\qquad$ if $u \in P((C/A)^\vee) \times P(D/B)$.

Here we regard

$$P((C/A)^\vee) \times P(D/B) \subset P((C/A)^\vee \otimes (D/B))$$

by the Segre embedding.

Now by the above discussion, if we set

$$X_1 = \{x \in \hat{X} \mid \hat{\alpha}^{-1}(x) \cong P_1\}$$

$X_2 = \{x \in \hat{X} \mid \hat{\alpha}^{-1}(x) \cong 2 \text{ copies of } P_1 \text{ intersecting transversally at a point}\}$,

then we have

$$X_1 \cup X_2 = \hat{X}$$

$$\hat{\Delta} \cup \hat{\Lambda} \supset X_2 \supset \hat{\Delta} - \hat{\Lambda}$$

and $X_2 \not\supset \hat{\Lambda}$. Since X_2 is a closed set and since $\hat{\Delta}$ and $\hat{\Lambda}$ are irreducible, we know $X_2 = \hat{\Delta}$. Thus we get (A).

(B), (C). First of all, $\hat{\alpha}$ is flat since $\hat{\Gamma}$ and $\hat{X}$ are smooth and since every fibre $\hat{\alpha}^{-1}(x)$ of $\hat{\alpha}$ has the same dimension.

By the construction of $\hat{X}$ and by Corollary 1.9, we know that for $x, x' \in \hat{X}$

$$x = x' \iff \beta \alpha^{-1}(x) = \beta \alpha^{-1}(x') .$$

Thus by the universality of the Hilbert scheme, we have an injective morphism (Grothendieck [5])

$$\hat{X} \longrightarrow \text{Hilb } (F) .$$

We would like to show that this is an isomorphism onto an component. The next lemma treats a general situation.

LEMMA 1.11 (Kodaira [7]). *Let L be a locally complete intersection subvariety of a smooth projective variety F and let $[L] \in \text{Hilb } (F) = H$ be the corresponding point. If for the normal bundle $N_{L/F}$, $H^1(L, N_{L/F}) = 0$, then H is smooth at $[L]$ and the dimension of H at $[L]$ is equal to $\dim H^0(L, N_{L/F})$.*

PROOF. For a complete Noetherian local ring A with the residue field k, we set

$S(A) = \{G \subset \text{Spec } (A) \times F$ such that

(1) G is a subvariety,

(2) the restriction of the projection to the first factor

$pr_1 : G \to \text{Spec } (A)$ is flat and proper,

(3) the fibre of pr_1 over the closed point $o \in \text{Spec } (A)$ is $L\}$.

Then, the correspondence $A \to S(A)$ defines a covariant functor.

According to Kodaira [7], if $H^1(L, N_{L/F}) = 0$, then functor $S(\ )$ is

represented by a certain pair $(k[[T^\vee]], \zeta)$ with $\zeta \in S(k[[T^\vee]])$. Here $k[[T^\vee]]$ is the completion of the local ring at the origin of the affine space $T = H^0(L, N_{L/F})$. (Cf. Schlessinger [10].) (Though Kodaira [7] treats a more restricted case, it is easy to generalize it to the above-mentioned statement.)

On the other hand, by the universality of the Hilbert schemes, the functor $S(\ )$ is also represented by $\widehat{\mathscr{O}_{[L],H}}$. (The roof $\frown$ denotes the completion.)

By the uniqueness of the representing object, we have $\widehat{\mathscr{O}_{[L],H}} \cong k[[T^\vee]]$, which implies our lemma. Q.E.D.

By Lemma 1.11, if we can show the following (1.4), it follows that the morphism $\hat{X} \to \mathrm{Hilb}\,(F)$ is injective and surjective onto an smooth component and thus it is an isomorphism.

(1.4)   "For every point $x \in \hat{X}$, setting $L = \beta \alpha^{-1}(x)$, we have
dim $H^0(L, N_{L/F}) = \dim \hat{X}$ and $H^1(L, N_{L/F}) = 0$".

We will show (1.4).

First of all, we set $\pi(x) = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$ and

$$L = \beta \alpha^{-1}(x) = \left\{ \begin{array}{c} C\text{---}D \\ | \quad | \\ G\text{---}H \\ | \quad | \\ A\text{---}B \end{array} \middle| G,\ H \right\}.$$

Let F_1 be the zero-locus of the composite morphism $\mathscr{H} \to V_F \to (V/D)_F$. Then, we can write that

$$F_1 = \left\{ \begin{array}{c} D^{s+1} \\ | \\ G^r\text{---}H^s \end{array} \middle| G,\ H \right\}$$

and $\dim F_1 = \dim \mathrm{Gr}\,(s;\,s+1) + \dim \mathrm{Gr}\,(r;\,s)$. Thus since $\dim F - \dim F_1 =$ rank $\mathscr{Hom}\,(\mathscr{H},\,(V/D)_F)$, we know

(1.5) $N_{F_1/F} \cong \mathscr{Hom}\,(\mathscr{H},\,(V/D)_F)|_{F_1}$.

Let F_2 be the zero-locus of $A_{F_1} \to D_{F_1} \to D_{F_1}/\mathscr{G}|_{F_1}$. We have

$$F_2 = \left\{ \begin{array}{c} D^{s+1} \\ | \\ G^r\text{---}H^s \\ | \\ A^{r-1} \end{array} \middle| G,\ H \right\}$$

and $\dim F_1 - \dim F_2 = r(s-r+1) = \operatorname{rank} \mathscr{H}\!\mathit{om}\,(A_{F_1}, D_{F_1}/\mathscr{G}\,|_{F_1})$. Therefore, we have

(1.6) $$N_{F_2/F_1} \cong \mathscr{H}\!\mathit{om}\,(A_{F_1}, (D_{F_1}/\mathscr{G}\,|_{F_1}))|_{F_2}\,.$$

Setting $F_3 =$ the zero-locus of $\mathscr{G}\,|_{F_2}/A_{F_2} \to (D/A)_{F_2} \to (D/C)_{F_2}$, we have

$$F_3 = \left\{ \begin{array}{c} \begin{array}{ccc} C^{r+1} & \longrightarrow & D^{s+1} \\ | & & | \\ G^r & \longrightarrow & H^s \\ | & & \\ A^{r-1} & & \end{array} \end{array} \;\middle|\; G,\, H \right\}$$

and $\dim F_2 - \dim F_3 = s - r = \operatorname{rank} \mathscr{H}\!\mathit{om}\,(\mathscr{G}\,|_{F_2}/A_{F_2}, (D/C)_{F_2})$. Thus we get

(1.7) $$N_{F_3/F_2} \cong \mathscr{H}\!\mathit{om}\,(\mathscr{G}\,|_{F_2}/A_{F_2}, (D/C)_{F_2})|_{F_3}\,.$$

Case 1. Assume $x \in \hat{X} - (\hat{\varDelta} \cup \hat{A})$. Then, $L \cong P_1$, L coinsides with the zero-locus of $(B/A)_{F_3} \to (D/A)_{F_3} \to D_{F_3}/\mathscr{H}\,|_{F_3}$, and $\dim F_3 - \dim L = s - r$. Therefore

(1.8) $$N_{L/F_3} \cong \mathscr{H}\!\mathit{om}\,((B/A)_{F_3}, D_{F_3}/\mathscr{H}\,|_{F_3})|_L\,.$$

Now we have an exact sequence

$$0 \longrightarrow B_L \longrightarrow \mathscr{H}\,|_L \longrightarrow \mathscr{O}_{P_1}(-1) \longrightarrow 0\,.$$

Since $\operatorname{Ext}^1(\mathscr{O}_{P_1}(-1), B_L) \cong H^1(\mathscr{O}_{P_1}(1)^s) \cong 0$, this sequence splits and we have $\mathscr{H}\,|_L \cong \mathscr{O}_{P_1}^s \oplus \mathscr{O}_{P_1}(-1)$. Thus by (1.5), we get

(1.9) $$N_{F_1/F}|_L \cong \mathscr{O}_{P_1}^{s(n-s-1)} \oplus \mathscr{O}_{P_1}(1)^{n-s-1}\,.$$

The exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & C_L/\mathscr{G}\,|_L & \longrightarrow & D_L/\mathscr{G}\,|_L & \longrightarrow & (D/C)_L & \longrightarrow 0 \\ & \| \mathrel{\mkern-5mu} \wr & & & & \| \mathrel{\mkern-5mu} \wr & \\ & \mathscr{O}_{P_1}(1) & & & & \mathscr{O}_{P_1}^{s-r} & \end{array}$$

also splits. Thus we have $D_L/\mathscr{G}\,|_L \cong \mathscr{O}_{P_1}^{s-r} \oplus \mathscr{O}_{P_1}(1)$. By (1.6),

(1.10) $$N_{F_2/F_1}|_L \cong \mathscr{O}_{P_1}^{r(s-r)} \oplus \mathscr{O}_{P_1}(1)^r\,.$$

By (1.7) and (1.8) it is easy to see that

(1.11) $$N_{F_3/F_2}|_L \cong \mathscr{O}_{P_1}(1)^{s-r}$$

(1.12) $$N_{L/F_3} \cong \mathscr{O}_{P_1}(1)^{s-r}\,.$$

By (1.9)–(1.12) and exact sequences

$$0 \longrightarrow N_{F_2/F_1}|_L \longrightarrow N_{F_2/F}|_L \longrightarrow N_{F_1/F}|_L \longrightarrow 0$$

$$0 \longrightarrow N_{F_3/F_2}|_L \longrightarrow N_{F_3/F}|_L \longrightarrow N_{F_2/F}|_L \longrightarrow 0$$

$$0 \longrightarrow N_{L/F_3} \longrightarrow N_{L/F} \longrightarrow N_{F_3/F}|_L \longrightarrow 0 ,$$

we get

$$\dim H^0(L, N_{L/F}) = 2(n+s-r-1) + s(n-s-1) + r(s-r) = \dim \hat{X} \quad \text{and}$$
$$H^1(L, N_{L/F}) = 0 .$$

Case 2. Assume that $x \in \hat{\Lambda} - \hat{\Lambda}$. In this case $\pi(x) = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$ satisfies $A \subset C \subset B \subset D$. Equalities (1.5), (1.6), (1.7), (1.9), (1.10) and (1.11) hold as well. But (1.8) and (1.12) do not hold.

The composed morphism $B_{F_3} \longrightarrow D_{F_3} \longrightarrow D_{F_2}/\mathscr{H}|_{F_3}$ defines a global section ω of $\mathscr{H}om\,(B_{F_3}/\mathscr{G}|_{F_3}, D_{F_2}/\mathscr{H}|_{F_3}) \equiv \Omega$. Let F_4 be the zero-locus of ω. Since

$$F_4 = \left\{ \begin{array}{c} \\ \\ \end{array} \middle| G, H \right\} \cong P_1 \times P_1$$

it is easy to see the equality $\dim F_3 - \dim F_4 = \mathrm{rank}\,\Omega$ and thus

$$N_{F_4/F_3} \cong \mathscr{H}om\,(B_{F_3}/\mathscr{G}|_{F_3}, D_{F_3}/\mathscr{H}|_{F_3})|_{F_4} .$$

We get

(1.12)* $N_{F_4/F_3}|_L \cong \mathscr{O}_{P_1} \oplus \mathscr{O}_{P_1}(1)^{s-r-2} .$

Obviously

(1.13) $N_{L/F_4} \cong \mathscr{O}_{P_1}(2) .$

By (1.9), (1.10), (1.11), (1.12)* and (1.13), we easily obtain (1.4) in our case.

Case 3. Assume $x \in \hat{\Lambda} - \hat{\Lambda}$. We set $\pi(x) = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$. We can write $L = \beta\alpha^{-1}(x) = L_1 \cup L_2$, where

$$L_1 = \left\{ \begin{array}{c} \\ \\ \end{array} \middle| G \right\} \cong P_1$$

$$L_2 = \left\{ \begin{array}{c} C\text{------}D \\ C\cap B\text{------}H \\ A\text{------}B \end{array} \middle| H \right\} \cong P_1 .$$

Equalities (1.5)–(1.8) hold as well in this case. However since $\mathscr{H}|_{L_1} \cong \mathscr{O}_{P_1}^{s+1}$ and $D_{L_1}/\mathscr{G}|_{L_1} \cong \mathscr{O}_{P_1}^{s-r} \oplus \mathscr{O}_{P_1}(1)$, we have

$$N_{F_1/F}|_{L_1} \cong \mathscr{O}_{P_1}^{(s+1)(n-s-1)}$$

$$N_{F_2/F_1}|_{L_1} \cong \mathscr{O}_{P_1}^{r(s-r)} \oplus \mathscr{O}_{P_1}(1)^r$$

$$N_{F_3/F_2}|_{L_1} \cong \mathscr{O}_{P_1}(1)^{s-r}$$

$$N_{L/F_3}|_{L_1} \cong \mathscr{O}_{P_1}^{s-r}$$

and consequently

$$N_{L/F}|_{L_1} \cong \mathscr{O}_{P_1}^{(s+1)(n-s-1)+(r+1)(s-r)} \oplus \mathscr{O}_{P_1}(1)^s .$$

Analogously we have

$$N_{F_1/F}|_{L_2} \cong \mathscr{O}_{P_1}^{s(n-s-1)} \oplus \mathscr{O}_{P_1}(1)^{n-s-1}$$

$$N_{F_2/F_1}|_{L_2} \cong \mathscr{O}_{P_1}^{r(s-r+1)}$$

$$N_{F_3/F_2}|_{L_2} \cong \mathscr{O}_{P_1}^{s-r}$$

$$N_{L/F_3}|_{L_2} \cong \mathscr{O}_{P_1}(1)^{s-r}$$

and thus

$$N_{L/F}|_{L_2} \cong \mathscr{O}_{P_1}^{r(s-r)+s(n-s)} \oplus \mathscr{O}_{P_1}(1)^{n-r-1} .$$

Now by using the exact sequence

$$0 \longrightarrow N_{L/F} \longrightarrow N_{L/F}|_{L_1} \oplus N_{L/F}|_{L_2} \longrightarrow \mathscr{O}_{P_1}/m_p^t \longrightarrow 0$$

where $t = \operatorname{rank} N_{L/F}$ and m_p is the ideal sheaf of the point $p = L_1 \cap L_2$, we obtain (1.4) in this case as well.

Case 4. Assume $x \in \hat{\mathit{\Delta}} \cap \hat{\mathit{\Lambda}}$. This is the last case. We set $\pi(x) = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$ with $A \subset C \subset B \subset D$. In this case we have G_0 and H_0 corresponding with x, and we can write $L = \beta\alpha^{-1}(x) = L_1 \cup L_2$ where

$$L_1 = \left\{ \begin{array}{c} D \\ H_0 \\ B \\ C \\ G \\ A \end{array} \middle| G \right\} \cong P_1$$

$$L_2 = \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \quad \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \quad H \cong P_1 .$$

As in Case 1 and Case 2, we can define a sequence of subvarieties $F \supset F_1 \supset F_2 \supset F_3 \supset F_4 \supset L$ such that

$$N_{F_1/F}|_{L_1} \cong \mathcal{O}_{P_1}^{(s+1)(n-s-1)}$$

$$N_{F_2/F_1}|_{L_1} \cong \mathcal{O}_{P_1}^{r(s-r)} \oplus \mathcal{O}_{P_1}(1)^r$$

$$N_{F_3/F_2}|_{L_1} \cong \mathcal{O}_{P_1}(1)^{s-r}$$

$$N_{F_4/F_3}|_{L_1} \cong \mathcal{O}_{P_1}(-1) \oplus \mathcal{O}_{P_1}^{s-r-2}$$

$$N_{L/F_4}|_{L_1} \cong \mathcal{O}_{P_1}(1)$$

$$N_{F_1/F}|_{L_2} \cong \mathcal{O}^{s(n-s-1)} \oplus \mathcal{O}(1)^{n-s-1}$$

$$N_{F_2/F_1}|_{L_2} \cong \mathcal{O}^{r(s-r+1)}$$

$$N_{F_3/F_2}|_{L_2} \cong \mathcal{O}_{P_1}^{s-r}$$

$$N_{F_4/F_3}|_{L_2} \cong \mathcal{O}_{P_1}(1)^{s-r-1}$$

$$N_{L/F_4}|_{L_2} \cong \mathcal{O}_{P_1}(1) .$$

By these equalities we have

$$\dim H^0(N_{L/F}|_{L_1}) = t+s$$

$$\dim H^0(N_{L/F}|_{L_2}) = t+n-r-1$$

$$H^1(N_{L/F}|_{L_1}) = 0 \quad \text{and}$$

$$H^1(N_{L/F}|_{L_2}) = 0$$

with $t = \text{rank } N_{L/F}$. It follows that

$$\dim H^0(N_{L/F}) = (t+s) + (t+n-r-1) - t = \dim \hat{X} \quad \text{and}$$

$$H^1(N_{L/F}) = 0 .$$

They conclude the proof of Theorem 1.1. Q.E.D.

§ 2. Correspondence of Chern classes.

For a smooth variety Z, we denote by $A^k Z$ the k-th Chow group, i.e., the group of cycles on Z of codimension k modulo rational equivalence. For a cycle M on Z, $[M] \in A^{\cdot}Z$ denotes its equivalence class. We put $\hat{\mathscr{A}} = \pi^* \mathscr{A}$, $\hat{\mathscr{B}} = \pi^* \mathscr{B}$, $\hat{\mathscr{C}} = \pi^* \mathscr{C}$, $\hat{\mathscr{D}} = \pi^* \mathscr{D}$ and $\hat{\mathscr{E}} = \mathcal{O}_{\hat{X}}(-\hat{A})$. They are vector bundles over $\hat{X}$. Chern classes of them are defined. (Chevalley,

Grothendieck, Serre [3])

In this section we will show the following.

THEOREM 2.1. *Consider the map*

$$\phi = \hat{\alpha}_* \hat{\beta}^* \colon A^k F \longrightarrow A^{k-1} \hat{X} .$$

We have

$$\phi(c_k(\mathscr{G})) = -c_{k-1}(\hat{\mathscr{A}}) \quad 1 \leq k \leq r+1$$

$$\phi(c_k(\mathscr{H})) = -c_{k-1}(\hat{\mathscr{B}}) \quad 1 \leq k \leq s+1$$

$$\phi(c_k(\mathscr{I})) = c_{k-1}(V_{\hat{x}}/\hat{\mathscr{C}}) \quad 1 \leq k \leq n-r$$

$$\phi(c_k(\mathscr{J})) = c_{k-1}(V_{\hat{x}}/\hat{\mathscr{D}}) \quad 1 \leq k \leq n-s$$

and

$$\phi(c_2(\mathscr{X})) = -c_1(\hat{\mathscr{E}}) .$$

COROLLARY 2.2. *The map*

$$\phi = \hat{\alpha}_* \hat{\beta}^* \colon A^2 F \longrightarrow A^1 \hat{X}$$

is an isomorphism and we have

$$\phi(c_2(\mathscr{G})) = -c_1(\hat{\mathscr{A}})$$

$$\phi(c_2(\mathscr{H})) = -c_1(\hat{\mathscr{B}})$$

$$\phi(c_2(\mathscr{I})) = -c_1(\hat{\mathscr{C}})$$

$$\phi(c_2(\mathscr{J})) = -c_1(\hat{\mathscr{D}})$$

$$\phi(c_2(\mathscr{X})) = -c_1(\hat{\mathscr{E}}) .$$

That is, ϕ transfers vertices of the square

$$
\begin{array}{ccc}
\mathscr{G} & \rightarrow\rightarrow & \mathscr{I} \\
\downarrow & & \downarrow \\
\mathscr{H} & \rightarrow\rightarrow & \mathscr{J}
\end{array}
$$

to ones of the square

$$
\begin{array}{ccc}
\hat{\mathscr{A}} & \longrightarrow & \hat{\mathscr{C}} \\
\downarrow & & \downarrow \\
\hat{\mathscr{B}} & \longrightarrow & \hat{\mathscr{D}}
\end{array}
$$

and the exceptional one $\mathscr{X}$ to the exceptional one $\hat{\mathscr{E}}$.

PROOF. It is easy to show that non-zero elements in $c_2(\mathscr{G}) - c_2(\mathscr{X})$ are basis of $A^2 F$ and that non-zero elements in $c_1(\hat{\mathscr{A}}) - c_1(\hat{\mathscr{E}})$ are basis

of $A^1\hat{X}$. Thus our corollary follows from Theorem 2.1.   Q.E.D.

We begin with a lemma in Piene [9].

TRANSVERSALITY LEMMA 2.3.   *Let Z be a reduced equidimensional variety, $g: Z \to G = \mathrm{Gr}\,(m;\, n)$ a morphism. Fix a Schubert condition $(a_1, a_2, \cdots, a_{n-m})$. Then for a general flag $Fl = \{T_1 \subset \cdots \subset T_{n-m} \subset P_n\}$ with $\dim T_i = a_i - 1$, the corresponding Schubert variety $\Sigma = \Sigma(a;\, Fl)$ satisfies the following conditions:*

(i) *$g^{-1}\Sigma$ is either empty or equidimensional with $\mathrm{codim}\,(g^{-1}\Sigma,\, Z) = \mathrm{codim}\,(\Sigma,\, G)$;*

(ii) *$g^{-1}\Sigma$ has no embedded components. If $\mathrm{char}\,k = 0$, $g^{-1}\Sigma$ is reduced;*

(iii) *given a Zariski open dense set $U \subset Z$, $g^{-1}\Sigma |_U$ is dense in $g^{-1}\Sigma$;*

(iv) *the cycle $g^*[\Sigma]$ is defined and is equal to $[g^{-1}\Sigma]$.*

Let T be an $(n-r+k-2)$-dimensional linear subspace of P_n with $1 \leq k \leq r+1$. We define a reduced subvariety $c(T)$ by

$$c(T) = \{(G,\, H) \in F \mid \dim\,(G \cap T) \geq k-1\}\,.$$

By definition of Chern classes we have in $A^k F$

$$(-1)^k c_k(\mathscr{G}) = [c(T)]\,.$$

Transversality lemma implies for any general T

$$(-1)^k \hat{\beta}^* c_k(\mathscr{G}) = [\hat{\beta}^{-1} c(T)]\,.$$

Now set

$$c'(T) = \left\{ \begin{pmatrix} C & D \\ A & B \end{pmatrix} \in X \;\middle|\; \dim\,(A \cap T) \geq k-2 \right\}\,.$$

We consider $c'(T)$ as a reduced subvariety of X. By definition we have in $A^{k-1} X$,

$$(-1)^{k-1} c_{k-1}(\mathscr{A}) = [c'(T)]\,.$$

By Transversality lemma we have in $A^{k-1}\hat{X}$

$$(-1)^{k-1} c_{k-1}(\hat{\mathscr{A}}) = [\hat{\pi^{-1} c'(T)}]$$

for any general T. Thus by following Lemma 2.4, we obtain the first equality in Theorem 2.1.

LEMMA 2.4. $\hat{\alpha}(\hat{\beta}^{-1}(c(T))) = \pi^{-1}(c'(T))$ *and $\hat{\alpha}$ is generically one-to-one on $\hat{\beta}^{-1}c(T)$.*

PROOF. It follows from the next Lemma 2.5 and formal calculations that

$$\pi^{-1}\pi\hat{\alpha}\hat{\beta}^{-1}(c(T))=\pi^{-1}c'(T) .$$

However, by Transversality lemma, we may assume that

$$\text{codim}\,(\pi^{-1}c'(T),\ \hat{X})=\text{codim}\,(\pi^{-1}c'(T)\cap\hat{A},\ \hat{A}) .$$

It implies that

$$\pi^{-1}(c(T))=\overline{\pi^{-1}(c(T))-\hat{A}} .$$

(The bar $^{-}$ denotes the closure.) Aanlogously

$$\hat{\beta}^{-1}(c(T))=\overline{\hat{\beta}^{-1}(c(T))-\hat{\alpha}^{-1}(\hat{A})} .$$

Thus, noting that $\hat{\alpha}$ is proper, we obtain that

$$\begin{aligned}
\pi^{-1}c'(T)&=\overline{\pi^{-1}\pi\hat{\alpha}\hat{\beta}^{-1}(c(T))-\hat{A}}\\
&=\overline{\hat{\alpha}\hat{\beta}^{-1}(c(T))-\hat{A}}\\
&=\overline{\hat{\alpha}(\hat{\beta}^{-1}(c(T))-\hat{\alpha}^{-1}(\hat{A}))}\\
&=\hat{\alpha}\overline{(\hat{\beta}^{-1}(c(T))-\hat{\alpha}^{-1}(\hat{A}))}\\
&=\hat{\alpha}\hat{\beta}^{-1}(c(T)) .
\end{aligned}$$

The latter half of the assertion follows from the next lemma.   Q.E.D.

LEMMA 2.5. $\alpha\beta^{-1}(c(T))=c'(T)$ and α is generically one-to-one on $\beta^{-1}(c(T))$.

PROOF. Recall that linear spaces we are considering satisfy the condition indicated by the next diagram.

(2.1)
$$\begin{array}{ccc}
C^{r+1} & \!\!\!\!-\!\!\!\! & D^{s+1}\\
| & & |\\
G^{r} & \!\!\!\!-\!\!\!\! & H^{s}\\
| & & |\\
A^{r-1} & \!\!\!\!-\!\!\!\! & B^{s-1}
\end{array}$$

Thus if $(G, H)\in c(T)$, i.e.,

(2.2) $\dim(G\cap T)\geqq k-1$,

$\begin{pmatrix} C & D\\ A & B \end{pmatrix}\in\alpha\beta^{-1}(c(T))$ satisfies

(2.3) $\dim(A\cap T)\geqq k-2$.

Conversely is $\begin{pmatrix} C & D \\ A & B \end{pmatrix} \in c'(T)$, i.e., the equality (2.3) is satisfied, then there exists $(G, H) \in F$ satisfying the conditions (2.1) and (2.2). Moreover if $\dim (A \cap T) = k-2$, C and T intersect transversally, i.e., $\dim (C \cap T) = k-1$ and $C \cap B = A$, then the element $(G, H) \in F$ satisfying (2.1) and (2.2) is unique. It implies our lemma. Q.E.D.

Analogously we can show the second equality in Theorem 2.1.

We proceed to the third equality. Let U be an $(n-r-k)$-dimensional linear subspace of P_n with $1 \leq k \leq n-r$. We define

$$c''(U) = \{(G, H) \in F \mid \dim (G \cap U) \geq 0\}$$

$$c'''(U) = \left\{ \begin{pmatrix} C & D \\ A & B \end{pmatrix} \in X \mid \dim (C \cap U) \geq 0 \right\}.$$

They are considered as subvarieties of F and X respectively. By definition we have

$$c_k(\mathscr{F}) = [c''(U)]$$

$$c_{k-1}(V_X/\mathscr{C}) = [c'''(U)].$$

By similar arguments as in the above, it is enough to show the following lemma, whose proof is an easy exercise of linear algebra.

LEMMA 2.6. $\alpha(\beta^{-1}(c''(U))) = c'''(U)$ and α is generically one-to-one on $\beta^{-1}(c(U))$.

Analogously we can show the fourth one.

Now we consider the fifth equality.

Set $X' = X - \varDelta$, $\varGamma' = \varGamma - \alpha^{-1}(\varDelta)$. On $\varGamma'$, $\alpha^*(\mathscr{B}/\mathscr{A}) \cong \alpha^*(\mathscr{D}/\mathscr{C}) \cong \beta^* \mathscr{K}$. Thus we have

$$\hat{\alpha}_* \hat{\beta}^* c_2(\mathscr{K}) |_{\frac{k}{2}-\frac{1}{2}-\frac{1}{2}} = \alpha_* \beta^* c_2(\mathscr{K}) |_{x'} = \alpha_* \alpha^* c_2(\mathscr{B}/\mathscr{A}) |_{x'}.$$

If $c_2(\mathscr{B}/\mathscr{A}) |_{x'} = [M]$ for a cycle M on X', then $\alpha^* c_2(\mathscr{B}/\mathscr{A}) |_{x'} = [\alpha^{-1}M]$ since $\alpha |_{\varGamma'}$ is flat. Thus since $\dim \alpha^{-1}M > \dim \alpha(\alpha^{-1}M) = \dim M$,

$$\alpha_* \alpha^* c_2(\mathscr{B}/\mathscr{A}) |_{x'} = 0.$$

It follows that

$$\hat{\alpha}_* \hat{\beta}^* c_2(\mathscr{K}) = a\pi^* c_1(\mathscr{O}_X(\varDelta)) + b c_1(\mathscr{O}_X(\hat{\varLambda}))$$

for some integers a and b.

LEMMA 2.7. $c_1(\mathscr{O}_X(\varDelta)) = c_1(\mathscr{A}) - c_1(\mathscr{B}) - c_1(\mathscr{C}) + c_1(\mathscr{D})$.

PROOF. The composition $\mathscr{C} \to \mathscr{D} \to \mathscr{D}/\mathscr{B}$ defines a global section ω of the sheaf $\mathscr{H}om\,(\mathscr{C}/\mathscr{A},\,\mathscr{D}/\mathscr{B})$. We set

$$\delta = \Lambda^2\omega \in \mathscr{H}om\,(\Lambda^2(\mathscr{C}/\mathscr{A}),\,\Lambda^2(\mathscr{D}/\mathscr{B})) = \mathscr{L}\,.$$

As sets Δ coincides with the zero-locus of δ. Moreover it is easy to show that the zero-locus of δ is reduced. Thus we have that $c_1(\mathscr{O}_X(\Delta)) = c_1(\mathscr{L})$, which implies the desired equality. Q.E.D.

In order to calculate a and b, we use the following lemma. (Chevalley, Grothendieck, Serre [3], Fulton [4])

LEMMA 2.8. We consider the map $\phi = \hat{\alpha}_*\hat{\beta}^*\colon A^2F \to A^1\hat{X}$. Suppose that for a smooth subvariety $\hat{\Xi}$ of $\hat{X}$, subvarieties $\hat{\Theta} = \hat{\alpha}^{-1}(\hat{\Xi})$, $\Sigma = \hat{\beta}\hat{\alpha}^{-1}(\hat{\Xi})$ are smooth. Let $\bar{\phi} = \bar{\alpha}_*\bar{\beta}^*\colon A^2\Sigma \to A^1\hat{\Xi}$ be the induced map where $\bar{\alpha} = \hat{\alpha}\,|_{\hat{\Theta}}$ and $\bar{\beta} = \hat{\beta}\,|_{\hat{\Theta}}$. Then, we have for every element $c \in A^2F$

$$\bar{\phi}(c\,|_\Sigma) = \phi(c)\,|_{\hat{\Xi}}\,.$$

We define subvarieties

$$\hat{\Xi}_1,\ \hat{\Xi}_2 \subset \hat{X}\,, \qquad \Xi_2 \subset X$$
$$\hat{\Theta}_1,\ \hat{\Theta}_2 \subset \hat{\Gamma}\,, \qquad \Theta_2 \subset \Gamma \quad \text{and}$$
$$\Sigma_1,\ \Sigma_2 \subset F \quad \text{as follows.}$$

(1) We fix a point $x_0 = \begin{pmatrix} C_0 & D_0 \\ A_0 & B_0 \end{pmatrix} \in \Delta \subset X$. The condition implies $A_0 \subset C_0 \subset B_0 \subset D_0$. We set

$$\hat{X} \supset \hat{\Xi}_1 = \pi^{-1}(x_0) \cong P_3$$
$$\hat{\Gamma} \supset \hat{\Theta}_1 = \hat{\alpha}^{-1}(\hat{\Xi}_1)$$
$$F \supset \Sigma_1 = \hat{\beta}(\hat{\Theta}_1) \cong P_1 \times P_1\,.$$

We obtain a diagram;

$$
\begin{array}{ccc}
 & \hat{\Theta}_1 & \\
\bar{\alpha}\swarrow & & \searrow\bar{\beta} \\
\hat{\Xi}_1 & & \Sigma_1
\end{array}
$$

Note that this diagram can be embedded in the next diagram.

$$
\begin{array}{ccccc}
 & & P_3 \times P_1 \times P_1 & & \\
\pi_1\swarrow & & \uparrow j & & \searrow\pi_2 \\
 & & A & & \\
 & \bar{\alpha}\swarrow & & \searrow\bar{\beta} & \\
P_3 & & & & P_1 \times P_1
\end{array}
$$

where π_1, π_2 are projections, A is a divisor of degree $(1, 1, 1)$ and j is the inclusion map.

LEMMA 2.9.

(i)
$$c_2(\mathscr{K})|_{\Sigma_1} = -c_1(\mathscr{G})c_1(\mathscr{H})|_{\Sigma_1}$$
$$= -c_1(\mathscr{O}_{P_1 \times P_1}(-1, 0))c_1(\mathscr{O}_{P_1 \times P_1}(0, -1))$$

(ii)
$$c_1(\widehat{\mathscr{E}})|_{\hat{\mathcal{S}}_1} \cong c_1(\mathscr{O}_{P_3}(1))$$

(iii)
$$\phi(c_2(\mathscr{K}))|_{\hat{\mathcal{S}}_1} = -c_1(\widehat{\mathscr{E}})|_{\hat{\mathcal{S}}_1}$$

(iv)
$$\pi^* c_1(\mathscr{O}_X(\varDelta))|_{\hat{\mathcal{S}}_1} = 0 .$$

PROOF. (i) First note

$$c_2(\mathscr{K}) = -c_2(\mathscr{G}) + c_2(\mathscr{H}) + c_1(\mathscr{G})^2 - c_1(\mathscr{G})c_1(\mathscr{H})$$

since the sequence $0 \to \mathscr{G} \to \mathscr{H} \to \mathscr{K} \to 0$ is exact. Since $\mathscr{G}|_{\Sigma_1}$ contains a trivial bundle $(A_0)_{\Sigma_1}$, $c_2(\mathscr{G})|_{\Sigma_1} = 0$. By the same reason $c_2(\mathscr{H})|_{\Sigma_1} = 0$. Since $\Sigma_1 \cong P_1 \times P_1$, we have $c_1(\mathscr{G})^2|_{\Sigma_1} = 0$. Thus we obtain (i).

(ii) It is obvious by definition of blowing-up.

(iii) By (i) and (ii) and by Lemma 2.16 it is enough to show that $\bar{\phi}(e) = c_1(\mathscr{O}_{P_3}(1))$ with $\bar{\phi} = \bar{\alpha}_* \bar{\beta}^*$ and $e = c_1(\mathscr{O}_{P_1 \times P_1}(-1, 0))c_1(\mathscr{O}_{P_1 \times P_1}(0, -1))$. Indeed,

$$(\bar{\alpha}_* \bar{\beta}^* e)c_1(\mathscr{O}_{P_3}(1))^2 \cap [P_3]$$
$$= \bar{\alpha}_* (\bar{\beta}^* e \cdot \bar{\alpha}^* c_1(\mathscr{O}_{P_3}(1))^2 \cap [A])$$
$$= \pi_{1*} j_* (j^*(\pi_2^* e \cdot \pi_1^* c_1(\mathscr{O}_{P_3}(1))^2) \cap [A])$$
$$= \pi_{1*} (\pi_{2*} e \cdot \pi_1^* c_1(\mathscr{O}_{P_3}(1))^2 \cap j_*[A])$$
$$= \pi_{1*}\{\{\pi_2^* c_1(\mathscr{O}_{P_1 \times P_1}(1, 0))\pi_2^* c_1(\mathscr{O}_{P_1 \times P_1}(0, 1)) \cdot \pi_1^* c_1(\mathscr{O}_{P_3}(1))^2\}$$
$$\times \{\pi_2^* c_1(\mathscr{O}_{P_1 \times P_1}(1, 0)) + \pi_2^* c_1(\mathscr{O}_{P_1 \times P_1}(0, 1)) + \pi_1^* c_1(\mathscr{O}_{P_3}(1))\}\} \cap [P_3 \times P_1 \times P_1]$$
$$= 1$$
$$= c_1(\widehat{\mathscr{E}}) \cdot c_1(\mathscr{O}_{P_3}(1))^2 \cap [P_3] ,$$

which implies the desired equality.   Q.E.D.

COROLLARY 2.10. $b = -1$.

(2) We fix the following linear subspaces

$$\begin{array}{ccc} C_1^{r+1} & \!\!\!\!\!\!\! \rule[0.5ex]{1.5cm}{0.4pt} \!\!\!\!\!\!\! & D_1^{s+1} \\ | & & \diagup \\ A_1^{r-1} & \!\!\!\!\!\!\! \rule[0.5ex]{1cm}{0.4pt} \!\!\!\!\!\!\! & S_1^{s-2} \end{array}$$

(Bars — denote inclusion relations.) with $C_1 \cap S_1 = A_1$. Subvarieties are defined as follows.

$$X \supset \Xi_2 = \left\{ \begin{array}{c} C_1 \text{—————} D_1 \\ | \quad\quad\quad | \\ | \quad\quad\quad | \\ A_1 \text{——} S_1 \text{——} B^{s-1} \end{array} \middle| B \right\} \cong P_2$$

$$\hat{X} \supset \hat{\Xi}_2 = \pi^{-1}(\Xi_2)$$

$$\Gamma \supset \Theta_2 = \alpha^{-1}(\Xi_2)$$

$$= \left\{ \begin{array}{c} C_1 \text{————————} D_1 \\ | \quad\quad\quad\quad | \\ | \quad\quad\quad\quad | \\ G^r \text{————————} H^s \\ | \quad\quad\quad\quad | \\ | \quad\quad\quad\quad | \\ A_1 \text{——} S_1^{s-2} \text{——} B^{s-1} \end{array} \middle| G, H, B \right\}$$

$$\hat{\Gamma} \supset \hat{\Theta}_2 = \hat{\pi}^{-1}(\Theta_2)$$

$$F \supset \Sigma_2 = \hat{\beta}(\hat{\Theta}_2)$$

$$= \left\{ \begin{array}{c} C_1^{r+1} \text{————————} D_1^{s+1} \\ | \quad\quad\quad\quad | \\ | \quad\quad\quad\quad | \\ G^r \text{————————} H^s \\ | \quad\quad\quad \diagup \\ | \quad\quad \diagup \\ A_1^{r-1} \text{——} S_1^{s-2} \end{array} \middle| G, H \right\}$$

LEMMA 2.11. $\Xi_2 \cap A = \phi$.

PROOF. If $\Xi_2 \cap A \neq \phi$, then we have a linear space B^{s-1} with $C_1 \subset B$ and $S_1 \subset B$. However,

$$s - 1 = \dim B \geqq \dim (C_1 \vee S_1) = \dim C_1 + \dim S_1 - \dim (C_1 \cap S_1)$$
$$= (r+1) + (s-2) - (r-1) = s ,$$

which is a contradiction.   Q.E.D.

COROLLARY 2.12. *Morphisms*

$$\pi : \hat{\Xi}_2 \longrightarrow \Xi_2$$
$$\hat{\pi} : \hat{\Theta}_2 \longrightarrow \Theta_2$$

are isomorphisms.

LEMMA 2.13. $\dim(G \vee S_1) = s - 1$.

PROOF. Since G, $S_1 \supset A^{r-1}$, either $G \cap S_1 = A$ or $G \cap S_1 = G$. If $G \cap S_1 = G$, then $G \subset S_1$ and $A_1 = C_1 \cap S_1 \supset G$, which implies $r - 1 = \dim A_1 \geqq \dim G = r$, a contradiction.   Q.E.D.
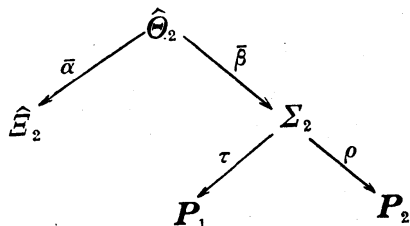
COROLLARY 2.14.   *The morphism* $\tau: \Sigma_2 \to P(C_1/A_1)$ *defined by* $\tau(G, H) = P(G/A_1) \in P(C_1/A_1)$ *gives to* Σ_2 *a structure of* P_1-*bundle over* P_1.

PROOF. The fibre of τ over the point $P(G/A_1)$ is $P(D_1/G \vee S_1)$, which is isomorphic to P_1 by Lemma 2.13.   Q.E.D.
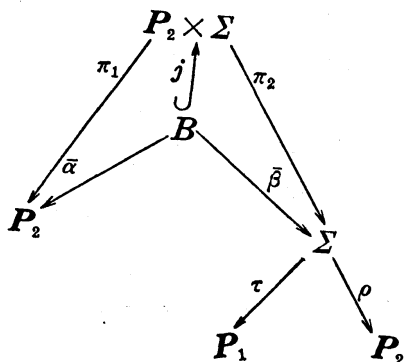
Analogously it is easy to show the next lemma.

LEMMA 2.15.   *The morphism* $\rho: \Sigma_2 \to P((D_1/S_1)^\vee)$ *defined by* $\rho(G, H) = P((H/S_1)^\vee)$ (*Here wedge* $^\vee$ *denotes the dual linear space.*) *is identified with the blowing-up of the point* $P((C_1 \vee S_1/S_1)^\vee)$.

Now we have a diagram;

$$
\begin{array}{ccc}
 & \widehat{\Theta}_2 & \\
{}^{\bar\alpha}\swarrow & & \searrow{}^{\bar\beta} \\
\widehat{\Xi}_2 & & \Sigma_2 \\
 & {}^{\tau}\swarrow \quad \searrow{}^{\rho} & \\
 & P_1 \qquad P_2 &
\end{array}
$$

The above discussion implies that this diagram can be embedded into the next commutative diagram.

$$
\begin{array}{ccc}
 & P_2 \times \Sigma & \\
{}^{\pi_1}\swarrow \; {}^{j}\uparrow \; \searrow{}^{\pi_2} & & \\
 & B & \\
{}^{\bar\alpha}\swarrow & & \searrow{}^{\bar\beta} \\
P_2 & & \Sigma \\
 & {}^{\tau}\swarrow \quad \searrow{}^{\rho} & \\
 & P_1 \qquad P_2 &
\end{array}
$$

where π_1 and π_2 are projections, B is a divisor, j is its inclusion, ρ is the blowing-up of a point of P_2, and τ is the canonical surjection.

LEMMA 2.16.

(i) $c_2(\mathscr{X})|_{\Sigma_2} = 0$

(ii)
$$c_1(\widehat{\mathscr{A}})|_{\hat{s}_2}=0$$

$$c_1(\widehat{\mathscr{B}})|_{\hat{s}_2}= -c_1(\mathscr{O}_{\hat{s}_2}(1))\cong -c_1(\mathscr{O}_{P_2}(1))$$

$$c_1(\widehat{\mathscr{C}})|_{\hat{s}_2}=0$$

$$c_1(\widehat{\mathscr{D}})|_{\hat{s}_2}=0$$

$$c_1(\widehat{\mathscr{E}})|_{\hat{s}_2}=0$$

(iii)
$$\pi^*c_1(\mathscr{O}_X(\varDelta))|_{\hat{s}_2}\neq 0 .$$

PROOF. (i) Recall

$$c_2(\mathscr{K})= -c_2(\mathscr{G})+c_2(\mathscr{H})+c_1(\mathscr{G})^2-c_1(\mathscr{G})c_1(\mathscr{H}) .$$

By definition of Σ_2, it is easy to show that $c_2(\mathscr{G})|_{\Sigma_2}=0$, $c_2(\mathscr{H})|_{\Sigma_2}\cong \tau^*c_1(\mathscr{O}_{P_2}(1))^2$, $c_1(\mathscr{G})^2|_{\Sigma_2}\cong \tau^*c_1(\mathscr{O}_{P_1}(1))^2=0$ and $c_1(\mathscr{G})c_1(\mathscr{H})|_{\Sigma_2}=\tau^*c_1(\mathscr{O}_{P_1}(1))\times\rho^* c_1(\mathscr{O}_{P_2}(1))$. They imply (i).

(ii) It is obvious by definition.

(iii) It follows from Lemma 2.7 and (ii).   Q.E.D.

COROLLARY 2.17. $a=0$.

Corollaries 2.10 and 2.17 imply the fifth equality in Theorem 2.1. Q.E.D.

§ 3. The set of jumping lines.

Main purpose of this section is to show the equality in (C) of Theorem 3.1, which is very similar to the equalities in Theorem 2.1. Theorem 3.1 is a generalization of a theorem for P_n in Barth [1]. The essential idea of the proof of (A) and (B) in Theorem 3.1 is due to him. However, our proof of (C) is entirely different from his one. The relative Riemann-Roch theorem due to Grothendieck (Borel, Serre [2]) will play a very important role in the proof.

Let $\mathscr{L}$ be a coherent sheaf over a variety Z. Let

$$\mathscr{O}_U^p \xrightarrow{\lambda} \mathscr{O}_U^q \longrightarrow \mathscr{L}|_U \longrightarrow 0 \qquad (*)$$

be a free resolution of $\mathscr{L}$ on a neighbourhood U of $z\in Z$. Consider the ideal Q_U in $\mathscr{O}_U$ generated by q-minors of the matrix representing λ. We can show that Q_U does not depend on the choice of the resolution $(*)$ and thus the ideal sheaf Q called *the 0-th Fitting ideal* of $\mathscr{L}$ is defined by setting $Q|_U=Q_U$ for $U\subset Z$ (cf. Teissier [11]).

For a vector bundle $\mathscr{M}$ over F we denote $\mathscr{M}(a, b)=\mathscr{M}\otimes(\varLambda^{n-r}\mathscr{F})^a\otimes(\varLambda^{n-s}\mathscr{F})^b$.

THEOREM 3.1. *Let $\mathscr{M}$ be a vector bundle over the flag variety F with*

$c_1(\mathscr{M}) = 0$. *Assume that* $\widehat{\beta}^* \mathscr{M}|_{\widehat{a}^{-1}(x_0)}$ *is trivial for some point* $x_0 \in \widehat{X}$. *Let*

$$J = \{x \in \widehat{X} \mid \widehat{\beta}^* \mathscr{M}|_{\widehat{a}^{-1}(x)} \text{ is not trivial}\} .$$

Then, we have

(A)
$$J = \text{Supp}\,(R^1 \widehat{a}_* \widehat{\beta}^* \mathscr{M}(0, -1))$$
$$= \text{Supp}\,(R^1 \widehat{a}_* \widehat{\beta}^* \mathscr{M}(-1, 0)) .$$

(B) *The 0-th Fitting ideal* Q *(resp.* Q'*) of* $R^1 \widehat{a}_* \widehat{\beta}^* \mathscr{M}(0, -1)$ *(resp.* $R^1 \widehat{a}_* \widehat{\beta}^* \mathscr{M}(-1, 0)$*) is locally principal and* $Q = Q'$.

By (A) *and* (B)*, we can give a structure of divisors with multiplicity to* J. *Then in* $A^1 \widehat{X}$.

(C)
$$[J] = \widehat{a}_* \widehat{\beta}^* c_2(\mathscr{M}) .$$

J *is called the set of jumping lines.*

PROOF. We denote $\widehat{\mathscr{M}} = \widehat{\beta}^* \mathscr{M}$ and $m = \text{rank}\,\mathscr{M}$.

(A) By symmetry, it is enough to show the first equality. The base-change theorem implies that

$$R^1 \widehat{a}_* \widehat{\mathscr{M}}(0, -1) \otimes k(x) \cong H^1(L, \widehat{\mathscr{M}}(0, -1)|_L)$$

with $L = \widehat{a}^{-1}(x)$ for any closed point $x \in \widehat{X}$, since the flat morphism $\widehat{a}$ has 1-dimensional fibre. Thus obviously the next proposition implies the desired equality.

PROPOSITION 3.2. *The following two conditions are equivalent for any closed point* $x \in \widehat{X}$. *Let* $L = \widehat{a}^{-1}(x)$.

(1) $\widehat{\mathscr{M}}|_L$ *is not trivial.*

(2) $H^1(L, \widehat{\mathscr{M}}(0, -1)|_L) \neq 0$.

PROOF. *Case 1.* Assume $x \in \widehat{X} - \widehat{\Delta}$. Then we have $L \cong P_1$. The vector bundle $\widehat{\mathscr{M}}|_L$ is a direct sum of line bundles.

$$\widehat{\mathscr{M}}|_L \cong \bigoplus_{i=1}^{m} \mathscr{O}_{P_1}(k_i) .$$

It follows that $\sum_{i=1}^{m} k_i = 0$ from assumption $c_1(\mathscr{M}) = 0$. Since $\widehat{\beta}^* \mathscr{O}_F(0, -1)|_L \cong \mathscr{O}_{P_1}(-1)$, it follows that

$$\widehat{\mathscr{M}}(0, -1)|_L \cong \bigoplus_{i=1}^{m} \mathscr{O}_{P_1}(k_i - 1)$$

and we have

$$H^1(\widehat{\mathscr{M}}(0, -1)|_L) \cong \bigoplus_{i=1}^{m} H^1(\mathscr{O}_{P_1}(k_i-1)) \cong \bigoplus_{i=1}^{m} H^0(\mathscr{O}_{P_1}(-k_i-1)) .$$

Assume that $\widehat{\mathscr{M}}|_L$ is trivial. Then, since $k_1 = \cdots = k_m = 0$, $H^1(\widehat{\mathscr{M}}(0, -1)|_L) = 0$ by the above equality.

Assume that $\widehat{\mathscr{M}}|_L$ is not trivial. Then, for some i, $k_i < 0$ and $H^1(\widehat{\mathscr{M}}(0, -1)|_L) \neq 0$.

Case 2. Assume $x \in \widehat{\varDelta}$. We denote $L = \widehat{a}^{-1}(x) = L_1 \cup L_2$ $(L_1 \cong L_2 \cong P_1)$.

LEMMA 3.3. *The following two conditions are equivalent.*
(1) $\widehat{\mathscr{M}}|_L$ *is trivial.*
(2) $\widehat{\mathscr{M}}|_{L_1}$ *and* $\widehat{\mathscr{M}}|_{L_2}$ *are trivial.*

PROOF. The implication (1)$\Rightarrow$(2) is obvious.
Assume (2) and consider the exact sequence

$$0 \longrightarrow \widehat{\mathscr{M}}|_L \longrightarrow \widehat{\mathscr{M}}|_{L_1} \oplus \widehat{\mathscr{M}}|_{L_2} \longrightarrow (\mathscr{O}/m_p)^m \longrightarrow 0$$

where m_p is the ideal sheaf of the point $p = L_1 \cap L_2$. It follows that $H^0(\widehat{\mathscr{M}}|_L) \to H^0(\widehat{\mathscr{M}}|_{L_i})$ $(i=1, 2)$ is an isomorphism. Thus we can choose m sections of $\widehat{\mathscr{M}}|_L$ which are independent at each point of L. Thus we obtain (1).   Q.E.D.

Now we set

$$\mathscr{M}|_{L_1} \cong \bigoplus \mathscr{O}_{P_1}(k_i') , \qquad \mathscr{M}|_{L_2} \cong \bigoplus \mathscr{O}_{P_1}(k_i'') .$$

We note that the sequence

$$0 \longrightarrow H^0(\widehat{\mathscr{M}}(0, -1)|_L) \longrightarrow \bigoplus_i H^0(\mathscr{O}_{P_1}(k_i')) \oplus H^0(\mathscr{O}_{P_1}(k_i''-1)) \overset{\gamma}{\longrightarrow} k^m$$

$$\longrightarrow H^1(\widehat{\mathscr{M}}(0, -1)|_L) \longrightarrow \bigoplus_i H^0(\mathscr{O}_{P_1}(-k_i'-2)) \oplus H^0(\mathscr{O}_{P_1}(-k_i''-1)) \longrightarrow 0$$

is exact.

Assume that $H^1(\widehat{\mathscr{M}}(0, -1)|_L) = 0$. Then we have that $H^0(\mathscr{O}_{P_1}(-k_i''-1)) = 0$ for every i by the above sequence. Thus $k_i'' \geq 0$. It implies that $k_i'' = 0$ for every i since $\sum k_i'' = 0$ by assumption $c_1(\mathscr{M}) = 0$. Note that now $\bigoplus_i H^0(\mathscr{O}_{P_1}(k_i''-1)) = 0$ and that γ is surjective. It implies that $k_i' \geq 0$. Thus we know that $k_i' = 0$ for every i since $\sum k_i' = 0$ by assumption $c_1(\mathscr{M}) = 0$. Therefore $\mathscr{M}|_{L_1}$ and $\mathscr{M}|_{L_2}$ are trivial. By Lemma 3.3, $\mathscr{M}|_L$ is trivial.

Conversely assume that $\mathscr{M}|_L$ is trivial. Then, $k_i' = k_i'' = 0$ for every i. By the above sequence we obtain that $H^1(\widehat{\mathscr{M}}(0, -1)|_L) = 0$.   Q.E.D.

(B) By the Serre's theorem of ample line bundles, there exists an exact sequence

$$0 \longrightarrow \mathscr{W} \longrightarrow \bigoplus_{i=1}^{t} \mathscr{O}_F(-r_i, -r_i-1) \longrightarrow \mathscr{M}(0, -1) \longrightarrow 0$$

with $r_i \geqq 0$. By (A) the set J is Zariski-closed and $x_0 \notin J$. By the base change theorem, we have a non-empty Zariski open set $U_1 \subset \hat{X} - J$ such that for every point $x \in U_1$,

$$\hat{\alpha}_* \hat{\beta}^* \mathscr{M}(0, -1) \otimes k(x) \cong H^0(\hat{\beta}^* \mathscr{M}(0, -1)|_{\hat{\alpha}^{-1}(x)})$$
$$\cong H^0(\mathscr{O}_{P_1}(-1))^m = 0 .$$

It follows that $\hat{\alpha}_* \hat{\beta}^* \mathscr{M}(0, -1) = 0$ since it is torsion-free and zero on U_1. Thus we obtain the exact sequence

(3.1) $$0 \longrightarrow R^1\hat{\alpha}_* \hat{\beta}^* \mathscr{W} \overset{\lambda}{\longrightarrow} \bigoplus R^1\hat{\alpha}_* \hat{\beta}^* \mathscr{O}_F(-r_i, -r_i-1) \longrightarrow \mathscr{L} \longrightarrow 0$$

with $\mathscr{L} = R^1\hat{\alpha}_* \hat{\beta}^* \mathscr{M}(0, -1)$.

LEMMA 3.4. $R^1\hat{\alpha}_* \hat{\beta}^* \mathscr{O}_F(-r, -r-1)$ $(r \geqq 0)$ is a locally free sheaf on $\hat{X}$.

PROOF. Recall that $\hat{\alpha}$ is flat. Thus by the base change theorem, the function $x \rightarrow \chi(\hat{\beta}^* \mathscr{O}_F(-r, -r-1)|_{\hat{\alpha}^{-1}(x)})$ ($\chi(\ )$ denotes the Euler-Poincaré characteristic.) is constant.

On the other hand, we have $\hat{\beta}^* \mathscr{O}_F(-r, -r-1)|_{\hat{\alpha}^{-1}(x)} \cong \mathscr{O}_{\hat{\alpha}^{-1}(x)}(-D_1)$ for some non-zero effective divisor D_1 on $\hat{\alpha}^{-1}(x)$. It follows that $h^0(\hat{\beta}^* \mathscr{O}_F(-r, -r-1)|_{\hat{\alpha}^{-1}(x)}) = h^0(\mathscr{O}_{\hat{\alpha}^{-1}(x)}(-D_1)) = 0$.

Thus the function

$$x \longrightarrow h^1(\hat{\beta}^* \mathscr{O}_F(-r, -r-1)|_{\hat{\alpha}^{-1}(x)}) = -\chi(\hat{\beta}^* \mathscr{O}_F(-r, -r-1)|_{\hat{\alpha}^{-1}(x)})$$

is constant. Applying the base change theorem again, we obtain our lemma.   Q.E.D.

LEMMA 3.5. $R^1\hat{\alpha}_* \hat{\beta}^* \mathscr{W}$ is locally free.

PROOF. By the same argument as in Lemma 3.4, we have only to show that $h^0(\hat{\beta}^* \mathscr{W}|_{\hat{\alpha}^{-1}(x)}) = 0$ for every $x \in \hat{X}$. Now since $0 \rightarrow \beta^* \mathscr{W}|_{\hat{\alpha}^{-1}(x)} \rightarrow \bigoplus \hat{\beta}^* \mathscr{O}_F(-r_i, -r_i-1)|_{\hat{\alpha}^{-1}(x)}$ is exact, the sequence $0 \rightarrow H^0(\hat{\beta}^* \mathscr{W}|_{\hat{\alpha}^{-1}(x)}) \rightarrow \bigoplus H^0(\hat{\beta}^* \mathscr{O}_F(-r, -r-1)|_{\hat{\alpha}^{-1}(x)})$ is exact. We have shown that the right-hand term of this sequence is zero in the proof of Lemma 3.4. Thus we obtain the desired result.   Q.E.D.

By Lemmas 3.4 and 3.5, we know that λ in the sequence (3.1) is a

homomorphism between vector bundles. Since $\mathscr{L}$ is a torsion sheaf by (A), the source and the target of λ have the same rank. Therefore the 0-th Fitting ideal of $\mathscr{L}$ is generated by the determinant of the square matrix representing λ and thus it is locally principal. It implies the former half of (B). The latter half will be taken up after (C).

(C) The following theorem is well-known (Borel, Serre [2]).

THE GROTHENDIECK-RIEMANN-ROCH THEOREM 3.6. *Let* Z *and* W *be smooth projective varieties. Let* $f: Z \to W$ *be a proper morphism. Consider the following diagram:*

$$
\begin{array}{ccc}
K(Z) & \xrightarrow{\mathrm{ch}} & A^{\cdot}Z \\
\downarrow{f_*} & & \downarrow{f_*} \\
K(W) & \xrightarrow{\mathrm{ch}} & A^{\cdot}W .
\end{array}
$$

Then, we have that for any coherent sheaf $\mathscr{R}$ *over* Z

$$f_*(\mathrm{ch}\,(\mathscr{R})\,\mathrm{td}\,(Z)) = \sum (-1)^i\,\mathrm{ch}\,(R^i f_* \mathscr{R})\,\mathrm{td}\,(W) .$$

We would like to apply this theorem to our situation.

Note that we can write that for Todd classes $\mathrm{td}\,(\hat{\Gamma}) = \hat{\alpha}^* \,\mathrm{td}\,(\hat{X})\cdot t$ with $t = 1 + t_1 + t_2 + \cdots$, $t_i \in A^i\hat{\Gamma}$, since the part of degree 0 of the Todd class is 1.

PROPOSITION 3.7. *For any vector bundle* $\mathscr{R}$ *over* F *with* $r = \mathrm{rank}\,\mathscr{R}$, *we have that*

$$\hat{\alpha}_*\left\{ rt_2 + \hat{\beta}^* c_1(\mathscr{R})\cdot t_1 + \frac{1}{2}\hat{\beta}^*(c_1(\mathscr{R})^2 - 2c_2(\mathscr{R}))\right\}$$

$$= c_1(R^0\hat{\alpha}_*\hat{\beta}^*\mathscr{R}) - c_1(R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{R}) .$$

PROOF. By Theorem 3.6 and by the above remark we have

$$\hat{\alpha}_*(\hat{\beta}^*\,\mathrm{ch}\,(\mathscr{R})\cdot t)\cdot\mathrm{td}\,(\hat{X}) = \hat{\alpha}_*(\hat{\beta}^*\,\mathrm{ch}\,(\mathscr{R})\cdot\hat{\alpha}^*\,\mathrm{td}\,(\hat{X})\cdot t)$$

$$= \hat{\alpha}_*(\hat{\beta}^*\,\mathrm{ch}\,(\mathscr{R})\cdot\mathrm{td}\,(\hat{\Gamma}))$$

$$= (\mathrm{ch}\,(R^0\hat{\alpha}_*\hat{\beta}^*\mathscr{R}) - \mathrm{ch}\,(R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{R}))\,\mathrm{td}\,(\hat{X})$$

since $\hat{\alpha}$ has 1-dimensional fibre everywhere. It implies that

$$\hat{\alpha}_*(\hat{\beta}^*\,\mathrm{ch}\,(\mathscr{R})\cdot t) = \mathrm{ch}\,(R^0\hat{\alpha}_*\hat{\beta}^*\mathscr{R}) - \mathrm{ch}\,(R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{R})$$

since $\mathrm{td}\,(\hat{X})$ is invertible. It is easy to deduce the desired equality from the above one, since

$$\mathrm{ch}\,(\mathscr{R})=r+c_1(\mathscr{R})+\frac{1}{2}(c_1(\mathscr{R})^2-2c_2(\mathscr{R}))+\cdots\,. \qquad \mathrm{Q.E.D.}$$

Set $\mathscr{R}=\mathscr{M}(0,\,-1)$. By our assumption $c_1(\mathscr{M})=0$, we have

$$c(\mathscr{R})=1-m\gamma+\Big(c_2(\mathscr{M})+\frac{m(m-1)}{2}\gamma^2\Big)+\cdots$$

with $\gamma=c_1(\mathscr{O}_F(0,1))$. Thus it is easy to show that

$$mt_2+\hat{\beta}^*c_1(\mathscr{R})+\frac{1}{2}\hat{\beta}^*(c_1(\mathscr{R})^2-2c_2(\mathscr{R}))=m\Big(\frac{1}{2}\hat{\gamma}^2-t_1\cdot\hat{\gamma}+t_2\Big)-\hat{\beta}^*c_2(\mathscr{M})$$

with $\hat{\gamma}=\hat{\beta}^*(\gamma)$.

Since we have seen that $R^0\hat{\alpha}_*\hat{\beta}^*\mathscr{M}(0,\,-1)=0$ in the proof of (B), Proposition 3.7 implies that

$$(*)\qquad \hat{\alpha}_*\hat{\beta}^*c_2(\mathscr{M})-m\hat{\alpha}_*\Big(\frac{1}{2}\hat{\gamma}^2-t_1\cdot\hat{\gamma}+t_2\Big)=c_1(R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{M}(0,\,-1))\,.$$

Therefore our desired equality (C) follows from the next two lemmas.
$$\mathrm{Q.E.D.}$$

LEMMA 3.8. $\qquad \hat{\alpha}_*\Big(\frac{1}{2}\hat{\gamma}^2-t_1\cdot\hat{\gamma}+t_2\Big)=0\,.$

PROOF. Set $\mathscr{M}=\mathscr{O}_F$. Obviously $c_2(\mathscr{M})=0$ and $J=\phi$ in this case. Thus it follows from (A) that $R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{M}(0,\,-1)=0$. The above equality (*) implies our lemma. $\qquad \mathrm{Q.E.D.}$

Let J be the variety defined by the 0-th Fitting ideal of $R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{M}(0,\,-1)$.

LEMMA 3.9. $\qquad [J]=c_1(R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{M}(0,\,-1))\,.$

PROOF. By the proof of (B), we have an exact sequence

$$0 \longrightarrow \mathscr{W}_1 \overset{\lambda}{\longrightarrow} \mathscr{W}_2 \longrightarrow \mathscr{L} \longrightarrow 0$$

where $\mathscr{W}_1$ and $\mathscr{W}_2$ are vector bundles over $\hat{X}$ with the same rank w and $\mathscr{L}=R^1\hat{\alpha}_*\hat{\beta}^*\mathscr{M}(0,\,-1)$. J is the zero-locus of the induced element $\Lambda^w\lambda\in\mathrm{Hom}\,(\Lambda^w\mathscr{W}_1,\Lambda^w\mathscr{W}_2)$. By Lemma 1.4, we have that

$$[J]=c_1(\mathscr{H}om\,(\Lambda^w\mathscr{W}_1,\Lambda^w\mathscr{W}_2))=c_1(\mathscr{W}_2)-c_1(\mathscr{W}_1)=c_1(\mathscr{L})\,. \qquad \mathrm{Q.E.D.}$$

Finally we verify the latter half of (B). Set $X'=\hat{X}-\hat{\Delta}$. Note that every point $x\in X'$ has a neighbourhood $x\in U\subset X'$ such that

$$\hat{\beta}^* \mathcal{O}_F(0, \ -1)|_{\hat{a}^{-1}(U)} \cong \hat{\beta}^* \mathcal{O}_F(-1, \ 0)|_{\hat{a}^{-1}(U)} \ .$$

Then we have that

$$R^1 \hat{a}_* \hat{\beta}^* \mathcal{M}(0, \ -1)|_U \cong R^1 \hat{a}_* \hat{\beta}^* \mathcal{M}(-1, \ 0)|_U \ .$$

Thus Fitting ideals Q and Q' have to coincide over X'. Let J (resp. J') be the divisor defined by Q (resp. Q'). Only the multiplicity of the component Δ in J and J' may be different. However, by the proof of (B),

$$[J] = [J'] = \hat{a}_* \hat{\beta}^* c_2(\mathcal{M}) \ .$$

Thus they have to coincide.   Q.E.D.

References

[1] W. BARTH, Some properties of stable rank-2 vector bundles on P_n, Math. Ann., **226** (1977), 125–150.

[2] A. BOREL et J.-P. SERRE, Le théorème de Riemann-Roch, Bull. Soc. Math. France, **86** (1958), 97–136.

[3] C. CHEVALLEY, A. GROTHENDIECK et J.-P. SERRE, Anneaux de Chow et applications, Séminaire C. Chevalley, 2e année, Secr. Math. Paris, 1958.

[4] W. FULTON, Rational equivalence on singular varieties, Publ. Math. I.H.E.S., **45** (1976), 147–167.

[5] A. GROTHENDIECK, Techniques de Construction et Théorèmes d'Existence en Géométrie Algebrique, IV, Les Schémas de Hilbert, Séminaire Bourbaki, n° 221, mai 1961, Institute Henri Poincaré, Paris.

[6] R. HARTSHORNE, Stable vector bundles of rank 2 on P^3, Math. Ann., **238** (1978), 229–280.

[7] K. KODAIRA, A theorem of completeness of characteristic system for analytic families of compact submanifolds of complex manifolds, Ann. of Math., **75** (1962), 146–162.

[8] C. OKONEK et al., Vector Bundles on Complex Projective Spaces, Progress in mathematics, **3**, Boston, Birkhaüser, 1980.

[9] R. PIENE, Polar classes of singular varieties, Ann. Sci. École. Norm. Sup., 4e série, t. **11** (1978), 247–276.

[10] M. SCHLESSINGER, Functors of Artin rings, Trans. Amer. Math. Soc., **130** (1968), 208–222.

[11] B. TEISSIER, The hunting of invariants in the geometry of discriminants, Proc. Nordic Summer School "Real and complex singularities", Oslo 1976, Sijthoff and Noordhoff, 1977.

[12] A. VAN DE VEN, On uniform vector bundles, Math. Ann., **195** (1972), 245–248.

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