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Homeomorphisms with the Pseudo Orbit Tracing Property of the Cantor Set

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Let X be a compact metric space with metric d, and f be a homeomorphism from X onto itself. A sequence $\{x_i\}_{i=-\infty}^{\infty}$ is said to be a *δ-pseudo-orbit* of f if $d(fx_i, x_{i+1})<\delta$ holds for all $i\in \mathbf{Z}$. (X, f) is said to have the *pseudo orbit tracing property* (abbrev. P.O.T.P.) if for every $\varepsilon>0$ there is $\delta>0$ such that, for every δ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}\subset X$, there exists an $x\in X$ such that $d(f^ix, x_i)<\varepsilon$ for all $i\in \mathbf{Z}$. Let $C\subset[0, 1]$ be the Cantor set: i.e. C is the set of the numbers $x\in[0, 1]$ with $x=3^{-1}a_1+3^{-2}a_2+\cdots$ ($a_i=0$ or 2 for $i\geqq 1$). We denote by $\mathscr{H}(C)$ the set of all homeomorphisms on C, and by $\mathscr{P}(C)$ the set of all homeomorphisms with the P.O.T.P.. Define the metric $\bar{d}$ on $\mathscr{H}(C)$ by $\bar{d}(f, g)=\max_{x\in C} d(fx, gx)$, $f, g\in\mathscr{H}(C)$. Then $\mathscr{H}(C)$ is a Banach space.

In this paper we prove:

THEOREM. *$\mathscr{P}(C)$ is dense in $\mathscr{H}(C)$.*

For $r\geqq 1$, we call the set $C\cap[3^{-r}i, 3^{-r}(i+1)]$ $(0\leqq i\leqq 3^r-1)$ a *Cantor subinterval with rank* r if $C\cap(3^{-r}i, 3^{-r}(i+1))\neq\emptyset$. We denote by $I(i, r)$, the i-th Cantor subinterval with the rank r from the left. Clearly $C=\bigcup_{i=1}^{2^r} I(i, r)$ and $I(i, r)=I(2i-1, r+1)\cup I(2i, r+1)$. We call $g\in\mathscr{H}(C)$ a *generalized permutation* if there exists $r\geqq 1$ such that the following i) and ii) hold:

i) For every $1\leqq i\leqq 2^r$, there exist $s=s(i)\geqq 1$ and $1\leqq j=j(i)\leqq 2^s$ such that $g(I(i, r))=I(j, s)$, and

ii) For every $1\leqq i\leqq 2^r$, there exists $k=k(i)\in \mathbf{R}$ such that $g(x)=3^{r-s(i)}x+k$, $x\in I(i, r)$.

Denote by $\mathscr{G}$ the set of all generalized permutations. Then $\mathscr{G}$ is dense in $\mathscr{H}(C)$. In fact, take $f\in\mathscr{H}(C)$ and $r\geqq 1$. Choose $s\geqq 1$ such that $d(x, y)<3^{-s}$ implies $d(fx, fy)<3^{-r}$. Then for every $1\leqq i\leqq 2^s$ there exists

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$1 \leqq i_j \leqq 2^r$ such that $f(I(j, s)) \subset I(i_j, r)$. Since f is onto, for every $1 \leqq i \leqq 2^r$ there exist $n \geqq 1$ and $j_1, \cdots, j_n > 0$ such that $f(\bigcup_{k=1}^n I(j_k, s)) = I(i, r)$. Since $I(i, r)$ is a disjoint union of n Cantor subintervals (say $I(i, r) = \bigcup_{k=1}^n I(i_k, r_k)$), we can construct $g \in \mathscr{G}$ with $g(I(j_k, s)) = I(i_k, r_k)$, $1 \leqq k \leqq n$. It is easy to check that $\bar{d}(f, g) < 3^{-r}$. Since r is arbitrary, our requirement is obtained.

Let S be the finite set with the discrete topology, and $\Sigma = S^{\boldsymbol{Z}}$ be the bilateral infinite product space with metric d' defined by

$$d'(s, t) = \max_{i \in \boldsymbol{Z}} \delta(s_i, t_i)/2^{|i|} \qquad (s = (s_i),\ t = (t_i) \in \Sigma)$$

where $\delta(s_i, t_i) = 1$ if $s_i \neq t_i$, and $= 0$ if $s_i = t_i$. Define the shift homeomorphism σ of Σ by $(\sigma(s))_i = s_{i+1}$, $i \in \boldsymbol{Z}$. If X is a closed subset of Σ with $\sigma X = X$, then (X, σ) is called a *subshift*. A subshift (X, σ) is said to be of *finite type* if there exist $L > 0$ and $B \subset S^L$ such that $X = \{s = (s_i) \in \Sigma$: $(s_i, \cdots, s_{i+L-1}) \in B$ for all $i \in \boldsymbol{Z}\}$. L is called the *order* of (X, σ). It is proved in P. Walters [5] that a subshift (X, σ) has the P.O.T.P. iff (X, σ) is of finite type.

PROOF OF THEOREM. Since $\mathscr{G}$ is dense in $\mathscr{H}(C)$, it is enough to prove that every $g \in \mathscr{G}$ has the P.O.T.P.. Take $g \in \mathscr{G}$, then there exists $r_0 \geqq 1$ such that for every $r \geqq r_0$, g satisfies i) and ii) in the definition of generalized permutation. Let $\varepsilon > 0$, choose $r \geqq r_0$ such that $3^{-r} < \varepsilon$. For every $x \in C$ and every $i \in \boldsymbol{Z}$, define $x_i \in \{1, \cdots, 2^r\}$ by $g^i(x) \in I(x_i, r)$. By the definition of generalized permutation, it follows that $\bigcap_{j=1}^n g^j(I(x_{-j}, r))$ $(n \geqq 1)$ are Cantor subintervals for all $x \in C$. Put $\Sigma = F^{\boldsymbol{Z}}$ where $F = \{1, \cdots, 2^r\}$ and let σ be the shift homeomorphism of Σ. For $x \in X$, define $h(x) = (x_i)$. Then $h: C \to \Sigma$ is a continuous map and $h \circ g = \sigma \circ h$ holds. Let us put

$$A = \{x \in C: \bigcap_{j=1}^\infty g^j(x_{-j}, r) \subsetneqq I(x_0, r)\} .$$

For $x \in A$, denote by $n(x)$ the minimum number such that

$$\bigcap_{j=1}^{n(x)} g^j(I(x_{-j}, r)) \subsetneqq I(x_0, r) ,$$

and for $x \in C \backslash A$, denote by $n(x)$ the minimum number such that

$$\bigcap_{j=1}^{n(x)} g^j(I(x_{-j}, r)) = \bigcap_{j=1}^\infty g^j(I(x_{-j}, r)) \supset I(x_0, r) .$$

Then we have

$$\max_{x \in C} \operatorname{rank} \bigcap_{j=1}^{n(x)} g^j(I(x_{-j}, r)) = \max_{i \in F} \operatorname{rank} g(I(i, r)) < \infty ,$$

so that $\{\bigcap_{j=1}^{n(x)} g^j(I(x_{-j}, r)): x \in C\}$ is finite. If $\bigcap_{j=1}^{n(x)} g^j(I(x_{-j}, r)) = \bigcap_{j=1}^{n(x')} g^j(I(x'_{-j},$

$r))$ $(x, x' \in C)$, then we get $x_{-j}=x'_{-j}$ $(1 \leqq j \leqq \min(n(x), n(x')))$. Since $n(x)$ and $n(x')$ are minimal, $n(x)=n(x')$ holds. This implies that $\{n(x): x \in C\}$ is finite. Put $N=\max_{x \in C} n(x)$ and

$$B=\{(i_0, i_{-1}, \cdots, i_{-N}) \in F^{N+1}: \bigcap_{j=0}^{N} g^j(I(i_{-j}, r)) \neq \emptyset\} .$$

Then $(h(C), \sigma)$ is a subshift of finite type of order $N+1$. To see this, set

$$\Sigma_B=\{(i_j) \in \Sigma: (i_j, i_{j-1}, \cdots, i_{j-N}) \in B \quad \text{for all} \quad j \in \boldsymbol{Z}\} .$$

Clearly $\sigma\Sigma_B=\Sigma_B$, (Σ_B, σ) is of finite type, and $h(C) \subset \Sigma_B$. To prove that $h(C) \supset \Sigma_B$, it is enough to show that

$$(*) \qquad \bigcap_{j=0}^{m} g^j(I(i_{-j}, r)) \neq \emptyset \quad (m \geqq N) \quad \text{if}$$
$$(i_{-j}, i_{-(j+1)}, \cdots, i_{-(j+N)}) \in B \quad \text{for all} \quad 0 \leqq j \leqq m-N .$$

We use induction with respect to m. When $m=N$, (*) is true. Assume that (*) holds for m. Take $(i_{-j})_{j=0}^{m+1} \in F^{m+1}$ with $(i_{-j}, i_{-(j+1)}, \cdots, i_{-(j+N)}) \in B$ for all $0 \leqq j \leqq (m+1)-N$. By assumption we have $\bigcap_{j=1}^{m+1} g^j(I(i_{-j}, r)) \neq \emptyset$, and so $\bigcap_{j=0}^{N} g^j(I(i_{-j}, r)) \neq \emptyset$. Take $x \in \bigcap_{j=0}^{N} g^j(I(i_{-j}, r))$. If $x \in A$, then $\bigcap_{j=1}^{N} g^j(I(i_{-j}, r)) \subsetneqq I(i_0, r)$. Therefore $\bigcap_{j=0}^{m+1} g^j(I(i_{-j}, r))=\bigcap_{j=1}^{m+1} g^j(I(i_{-j}, r)) \neq \emptyset$. When $x \notin A$, $\bigcap_{j=1}^{m+1} g^j(I(i_{-j}, r))=\bigcap_{j=1}^{N} g^j(I(i_{-j}, r)) \supset I(i_0, r)$, and so $\bigcap_{j=0}^{m+1} g^j(I(i_{-j}, r)) \supset I(i_0, r) \neq \emptyset$. Thus $(h(C), \sigma)$ is of finite type of order $N+1$. As before let d' be the metric of Σ and ε' be a number such that $d'(s, t)<\varepsilon'$ $(s=(s_i), t=(t_i) \in \Sigma)$ implies $s_0=t_0$. Since $(h(C), \sigma)$ is of finite type, there exists $\delta>0$ such that for every δ-pseudo-orbit $\{s^n\} \subset h(C)$, there is $s \in h(C)$ such that $d'(\sigma^n s, s^n)<\varepsilon'$. Choose $\eta>0$ such that $d(x, y)<\eta$ $(x, y \in C)$ implies $d'(h(x), h(y))<\delta$ and take an η-pseudo-orbit $\{x^n\} \subset C$ of g. Then $\{h(x^n)\}$ is a δ-pseudo-orbit of σ. Hence there exists $x \in C$ such that $d'(\sigma^n h(x), h(x^n))<\varepsilon'$. This shows that $h(g^n x)_0=h(x^n)_0$ $(n \in \boldsymbol{Z})$, and so $d(g^n x, x^n)<3^{-r}<\varepsilon$ $(n \in \boldsymbol{Z})$. The proof of the theorem is completed.

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