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On Certain Infinite Dimensional Contragredient Modules*

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To the memory of Professor Mikao MORIYA

Introduction

Let G be an arbitrary group and FG be the group algebra of G over a field F. Let M be an FG-module and $M^*=\mathrm{Hom}_F(M, F)$ be the contragredient module of M, that is, G operates on M^* by the following operation

$$(g\psi)(m)=(\psi g^{-1})(m)=\psi(g^{-1}m) \quad \text{for} \quad g\in G\,, \quad \psi\in M^* \quad \text{and} \quad m\in M\,.$$

We write FG^* for the contragredient module of the left regular module ${}_{FG}FG$. Then it is well known that in case G is finite $FG\cong FG^*$ as FG-module, but FG is not isomorphic to FG^* in general when G is infinite. Hence we expect that in order to generalize the representation theory of finite groups to the representation theory of infinite groups it is more natural to study the structure of the contragredient module FG^*.

In this note we will show the structure of $\mathrm{Hom}_{FG}(FG\otimes_{FH_2}L_2, (FG\otimes_{FH_1}L_1)^*)$ by embedding $FG\otimes_{FH_2}L_2$ and $(FG\otimes_{FH_1}L_1)^*$ into FG^*, where H_i is a subgroup of G and L_i is a one-dimensional FH_i-module ($i=1, 2$) (see Theorem 2.2). Though the proof is fairly elementary, as far as the author knows, Theorem 2.2 is a most generalized intertwining number theorem for abstract groups, which includes [2, Theorem (1.3)].

Now let G be a group with a BN-pair and W be the Weyl group of G, i.e., $W=N/B\cap N$. Let L be the trivial one-dimensional module of FB. Then as it was shown in [2] in most cases

$$\dim_F \mathrm{Hom}_{FG}(FG\otimes_{FB}L, FG\otimes_{FB}L)=1$$

when G is infinite. However in case of a infinite BN-pair G, the dimension of the Hecke algebra

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$$\mathrm{Hom}_{FG}(FG\otimes_{FB}L,\ FG\otimes_{FB}L) \quad \text{is equal to} \quad |W|\,,$$

the cardinal number of W, and Hecke algebras play a very important role for the study of both ordinary and modular representation theories of G (see, e.g., [C. W. Curtis, Bull. Amer. Math. Soc. N.S., **1**, (1979), 721–757] and [N. B. Tinberg, J. Algebra **61**, (1979), 508-526, and Canad. J. Math., 32, (1980), 714–733]). In Example 2.3 we show that when $|W|$ is finite we have

$$\dim_F \mathrm{Hom}_{FG}(FG\otimes_{FB}L,\ (FG\otimes_{FB}L)^*)=|W|\,,$$

which suggests us that the space $\mathrm{Hom}_{FG}(FG\otimes_{FB}L,\ (FG\otimes_{FB}L)^*)$ seems to be a reasonable generalization of the Hecke algebra of a finite BN-pair.

In Section 1 we show how to embed $FG\otimes_{FH_2}L_2$ and $(FG\otimes_{FH_1}L_1)^*$ into FG^* (see Propositions 1.2 and 1.3). In Section 2 we prove the theorem, which is followed by the example.

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§ 1. Preliminaries.

Let G be an arbitrary group and F be a field. We write FG for the group algebra of G over F. Let M be an FG-module and $M^*=\mathrm{Hom}_F(M, F)$. Then M^* becomes a left FG-module, which is called the contragredient module of M, under the following operation given by

$$(g\psi)(m)=(\psi g^{-1})(m)=\psi(g^{-1}m) \quad \text{for} \quad g\in G\,, \quad \psi\in M^* \quad \text{and} \quad m\in M\,.$$

Assume H be a subgroup of G and FH be the group algebra of H over F. Let L be a left FH-module and $G=\cup_m x_mH$ (disjoint union). Then we have $FG\otimes_{FH}L=\sum_m \oplus x_m\otimes L$ (direct sum), and every element of $FG\otimes_{FH}L$ can be expressed uniquely in the form $\sum_m x_m\otimes l_m$, where $l_m\in L$. Hence if we take a pair $(\sum_m x_mb_m, \psi)$ from $FG\times L^*$, where $\sum_m x_mb_m\in FG=\sum_m \oplus x_mFH$ (direct sum), we can assign the element $\varphi\in(FG\otimes_{FH}L)^*$ which takes an element $x=\sum_m x_m\otimes l_m\in FG\otimes_{FH}L$ to $\varphi(x)=\sum_m (b_m\psi)(l_m)$. It is easily checked that there exists an F-linear map η of $FG\otimes_{FH}L^*$ into $(FG\otimes_{FH}L)^*$ which makes the following diagram commutative.

$$\begin{array}{ccc} & FG\otimes_{FH}L^* & \\ \nearrow & \circlearrowright & \searrow{\scriptstyle \eta} \\ FG\times L^* & \longrightarrow & (FG\otimes_{FH}L)^* \\ \cup\!\!\!\!\!\!\;\;\text{ω} & & \text{ω} \\ (\sum_m x_mb_m,\ \psi) & \longmapsto & \varphi \end{array}$$

Thus we have the following proposition.

PROPOSITION 1.1 (See [1, Theorem (43.9)]). *Let FG, L and* η *etc. be as before, then*

$$\eta\colon FG\otimes_{FH}L^{*} \longrightarrow (FG\otimes_{FH}L)^{*}$$

is an injective FG-homomorphism.

Now let FG^* be the contragredient module of ${}_{FG}FG$. We write $g^*\in FG^*$ for the F-linear map of FG into F, where $g\in G$, which takes g to 1 and g' to 0 for all $g'\in G-\{g\}$. Then the map

$$\begin{array}{ccc} \iota\colon FG & \hookrightarrow & FG^{*} \\ \cup\!\!\!| & & \cup\!\!\!| \\ g & \longmapsto & g^{*} \end{array}$$

is an injective FG-homomorphism and essentially same as the above embedding η. By this embedding ι we consider FG as an FG-submodule of FG^*. In case G is a finite group, we have $\iota\colon FG\cong FG^*$. However when G is infinite, ι is not surjective.

Let H be a subgroup of G and $\psi\in FH^*$, then we define an extension $\bar{\psi}\in FG^*$ of ψ to be the F-linear map which takes $h\in H$ to $\psi(h)$ and $g\in G-H$ to 0. Then we can embed FH^* into FG^* by the following FH-homomorphism

$$\begin{array}{ccc} \rho\colon FH^{*} & \hookrightarrow & FG^{*} \\ \cup\!\!\!| & & \cup\!\!\!| \\ \psi & \longmapsto & \bar{\psi} \end{array} .$$

Thus we always assume FH^* be an FH-module of FG^* and use the same notation ψ for ψ and $\rho(\psi)$. Suppose $\lambda\colon H\to F^{\times}$ is a linear character of H into $F^{\times}=F-\{0\}$, then

$$\begin{array}{ccc} \lambda^{-1}\colon H & \longrightarrow & F^{\times} \\ \cup\!\!\!| & & \cup\!\!\!| \\ h & \longmapsto & \lambda(h)^{-1} \end{array}$$

is also a linear character of H into $F^{\times}$. We write $\hat{\lambda}$ for λ^{-1}. Since $\hat{\lambda}\in FH^*$ and $h\cdot\hat{\lambda}=\lambda(h)\hat{\lambda}$, $FH\hat{\lambda}$ is a one dimensional FH-module with F-basis $\{\hat{\lambda}\}$ which affords the linear character λ.

PROPOSITION 1.2. *Let G be a group and H be a subgroup of G. Let X be the set of all linear characters of H into* $F^{\times}$, *where F is a field and* $F^{\times}=F-\{0\}$.

(i) *Let* $\lambda\in X$, *then there exists a natural FG-isomorphism of*

$FG\otimes_{FH}FH\hat{\lambda}$ onto $FG\hat{\lambda}$ which takes $(\sum_{g\in G} t_g g)\otimes\hat{\lambda}$ to $(\sum_{g\in G} t_g g)\hat{\lambda}$, where $(\sum_{g\in G} t_g g)\in FG$.

(ii) *Let M be the FG-submodule of FG^* generated by $\{\hat{\lambda}\mid\lambda\in X\}$, then we have $M=\sum_{\lambda\in X}\oplus FG\hat{\lambda}$ (direct sum).*

PROOF. (i) Let $G=\cup_m x_m H$ (disjoint union). Since $FG\otimes_{FH}FH\hat{\lambda}$ has an F-basis $\{x_m\otimes\hat{\lambda}\}$ and $FG\hat{\lambda}$ has an F-basis $\{x_m\hat{\lambda}\}$, one can easily verify that the map is an FG-isomorphism.

(ii) Since $FG\hat{\lambda}$ has an F-basis $\{x_m\hat{\lambda}\}$ where $G=\cup_m x_m H$ (disjoint union), it is enough to prove that

$$(\sum_m t_m^{(1)}x_m\hat{\lambda}_1)+\cdots+(\sum_m t_m^{(n)}x_m\hat{\lambda}_n)=0$$

implies $\sum_m t_m^{(1)}x_m\hat{\lambda}_1=\cdots=\sum_m t_m^{(n)}x_m\hat{\lambda}_n=0$ for any elements $\sum_m t_m^{(1)}x_m\hat{\lambda}_1, \cdots,$ $\sum_m t_m^{(n)}x_m\hat{\lambda}_n$ from $FG\hat{\lambda}_1, \cdots$ and $FG\hat{\lambda}_n$ respectively, where $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_n$ are arbitrary finite number of elements from $\{\hat{\lambda}\mid\lambda\in X\}$.

Since

$$(\sum_m t_m^{(1)}x_m\hat{\lambda}_1+\cdots+\sum_m t_m^{(n)}x_m\hat{\lambda}_n)(x_{m_0}h)=t_{m_0}^{(1)}\hat{\lambda}_1(h)+\cdots+t_{m_0}^{(n)}\hat{\lambda}_n(h)=0$$

for any $x_{m_0}\in\{x_m\}$ and $h\in H$ and $\{\hat{\lambda}\mid\lambda\in X\}$ is linearly independent over F in FG^* (see, for example, S. Lang's Algebra, p. 209, Addison-Wesley), $t_m^{(1)}=t_m^{(2)}=\cdots=t_m^{(n)}=0$ for any m. Thus we have proved the proposition. Q.E.D.

PROPOSITION 1.3. *Let G, H, λ and $\hat{\lambda}$ be as in Proposition 1.2. Let $G=\cup_m x_m H$ (disjoint union). Since $x_m\hat{\lambda}$ takes the value 0 outside of the coset $x_m H$, for any scalar $c_m\in F$ we can also define an element $\sum_m c_m x_m\hat{\lambda}$ of FG^* to be*

$$(\sum_m c_m x_m\hat{\lambda})(x_{m'}h)=c_{m'}\hat{\lambda}(h)\quad\textit{where}\quad h\in H\quad\textit{and}\quad x_{m'}\in\{x_m\}\,.$$

We write $\mathscr{Y}$ for the set of all such elements $\sum_m c_m x_m\hat{\lambda}$. Then

(i) *$\mathscr{Y}$ is an FG-submodule of FG^*, and*

(ii) *$(FG\lambda)^*$ is isomorphic to $\mathscr{Y}$ by the following FG-isomorphism $f\colon (FG\lambda)^*\to\mathscr{Y}$ which takes $\varphi\in(FG\lambda)^*$ to $f(\varphi)=\sum_m\varphi(x_m\lambda)x_m\hat{\lambda}$.*

PROOF. (i) Let $h\in H$, then $(x_m\hat{\lambda})(x_m h)=\hat{\lambda}(h)$ for all x_m. Assume $x\notin x_m H$, then $x_m^{-1}x\notin H$ and we have $(x_m\hat{\lambda})(x)=0$. Thus it is reasonable to define $\sum_m c_m x_m\hat{\lambda}\in FG^*$ to be

$$(\sum_m c_m x_m\hat{\lambda})(x_{m'}h)=c_{m'}\hat{\lambda}(h)\quad\text{where}\quad h\in H\quad\text{and}\quad x_{m'}\in\{x_m\}\,.$$

Let $g \in G$, then the map

$$\begin{array}{ccc} g_L^{-1}\colon \{x_m H\} & \longrightarrow & \{x_m H\} \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ x_m H & \longmapsto & g^{-1}x_m H \end{array}$$

is a bijection of the set of all left cosets of H in G into itself. Since there exists a unique $x_{m_0}H \in \{x_m H\}$ such that $g^{-1}x_{m'}H = x_{m_0}H$ for any $x_{m'}H$, we have

$$\begin{aligned} g(\sum_m c_m x_m \hat{\lambda})(x_{m'}h) &= (\sum_m c_m x_m \hat{\lambda})(g^{-1}x_{m'}h) \\ &= c_{m_0}\hat{\lambda}(h_0) \quad \text{for some} \quad h_0 \in H \end{aligned}$$

such that $g^{-1}x_{m'}h = x_{m_0}h_0$. Assume $gx_{m_0} = x_{m'}h'$ for some $h' \in H$, then we have

$$(\sum_{m'} c_{m_0}\lambda(h')x_{m'}\hat{\lambda})(x_{m'}h) = c_{m_0}\lambda(h')\hat{\lambda}(h)\ .$$

Since $gx_{m_0} = x_{m'}hh_0^{-1}$, we have $c_{m_0}\lambda(h')\hat{\lambda}(h) = c_{m_0}\lambda(hh_0^{-1})\hat{\lambda}(h) = c_{m_0}\lambda(h_0^{-1}) = c_{m_0}\hat{\lambda}(h_0)$. Hence

$$g(\sum_m c_m x_m \hat{\lambda}) = \sum_{m'} c_{m_0}\lambda(h')x_{m'}\hat{\lambda} \in \mathscr{Y}\ ,$$

and $\mathscr{Y}$ is an FG-submodule of FG^*.

(ii) It is clear that f is a well-defined F-linear map, which is bijective. From (i) we have

$$gf(\varphi) = g(\sum_m \varphi(x_m\lambda)x_m\hat{\lambda}) = \sum_{m'} \varphi(x_{m_0}\lambda)\lambda(h')x_{m'}\hat{\lambda}\ ,$$

where $g \in G$, $g^{-1}x_{m'}H = x_{m_0}H$ and $gx_{m_0} = x_{m'}h'$ ($h' \in H$). On the other hand since

$$f(g\varphi) = \sum_{m'} \varphi(g^{-1}x_{m'}\lambda)x_{m'}\hat{\lambda} = \sum_{m'} \varphi(x_{m_0}h'^{-1}\lambda)x_{m'}\hat{\lambda} = \sum_{m'} \varphi(x_{m_0}\lambda)\lambda(h')x_{m'}\hat{\lambda}\ ,$$

we have shown that $f(g\varphi) = gf(\varphi)$ for any $\varphi \in (FG)^*$ and $g \in G$. Hence f is an FG-isomorphism. Q.E.D.

§ 2. The structure of $\mathrm{Hom}_{FG}(FG\otimes_{FH}L^*, (FG\otimes_{FH}L)^*)$.

Let G be a group, H a subgroup of G and F a field. In Section 1 we have shown that there exists an embedding of $FG\otimes_{FH}L^*$ into $(FG\otimes_{FH}L)^*$ (see Proposition 1.1). In general if we take G to be an infinite group, $FG\otimes_{FH}L^*$ and $(FG\otimes_{FH}L)^*$ are not isomorphic to each other. In this section we will show the structure of $\mathrm{Hom}_{FG}(FG\otimes_{FH}L^*, (FG\otimes_{FH}L)^*)$ when L is of one dimension.

Now let H_i be a subgroup of G and FH_i be the group algebra of H_i over F ($i=1, 2$). Let $\lambda_i\colon H_i \to F^\times$ be a linear character of H_i into $F^\times = F - \{0\}$, where $i=1, 2$. Thus we have $\hat{\lambda}_i \in FH_i^*$ and $h\hat{\lambda}_i = \lambda_i(h)\hat{\lambda}_i$ for all $h\in H_i$ $(i=1, 2)$. Let $Y_i=FG\hat{\lambda}_i$ $(i=1, 2)$, $G=\cup_m x_m H_2$ (disjoint) and $G=\cup_n y_n H_1$ (disjoint), then Y_2 has an F-basis $\{x_m\hat{\lambda}_2\}$ and Y_1 has an F-basis $\{y_n\hat{\lambda}_1\}$ (see Proposition 1.2). As in Proposition 1.3 we write $\mathscr{Y}_1$ for the set $\{\sum_n c_n y_n\hat{\lambda}_1\}$ where c_n's are arbitrary elements of F.

We shall use following notation, for $x, y\in G$:

$$y^x=x^{-1}yx\ , \qquad H_1^x=x^{-1}H_1x\ , \qquad H_1^{(x)}=H_1^x\cap H_2 \quad \text{and}$$
$$\begin{array}{cccc} \lambda_1^x\colon & H_1^{(x)} & \longrightarrow & F^\times \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & h & \longmapsto & \lambda_1(xhx^{-1})\ . \end{array}$$

We also write $H_2^{(x)}$ for $H_1\cap H_2^x$, where $H_2^x=x^{-1}H_2x$. Let $G=\cup_{i\in I}D_i$, where the D_i's are the distinct (H_1, H_2)-double cosets H_1xH_2 in G, and let $J\subset I$ be the set of indices j such that for some $x\in D_j$, $\lambda_1^x=\lambda_2$ on $H_1^{(x)}$. It can be easily checked that if $\lambda_1^x=\lambda_2$ on $H_1^{(x)}$ for some $x\in D_j$, then $\lambda_1^x=\lambda_2$ on $H_1^{(x)}$ for all $x\in D_j$.

Suppose $x\in G$ and $H_2=\cup_s H_1^{(x)}h_s$ (disjoint), then we have $H_1xH_2=\cup_s H_1xh_s$ (disjoint) and also $H_2x^{-1}H_1=\cup_s h_s^{-1}x^{-1}H_1$ (disjoint) (see [2, Lemma (1.2)]).

PROPOSITION 2.1 (Cf. [2, Theorem (1.3)]). *Let G, H_i, λ_i, Y_i and J etc. be as before. Let $g_j\in D_j^{-1}$ be a fixed representative of each double coset D_j^{-1} $(j\in J)$. Let*

$$H_2=\bigcup_s H_1^{(g_j^{-1})}h_s \quad (disjoint)\ .$$

We always assume that one of $\{h_s\}$, that is, h_{s_*} is 1 and also one of $\{x_m\}$ is 1. Then

(i) *Since $H_2g_jH_1=\cup_s h_s^{-1}g_jH_1$ (disjoint) and $G\supset\cup_j H_2g_jH_1$ (disjoint), we can assume $\{y_n\}\supset\{h_s^{-1}g_j\}$ and define an element $\sum_s\lambda_2(h_s)h_s^{-1}g_j\hat{\lambda}_1$ of $\mathscr{Y}_1$, as in Proposition 1.3, to be*

$$(\sum_s\lambda_2(h_s)h_s^{-1}g_j\hat{\lambda}_1)(h_{s'}^{-1}g_jh)=\lambda_2(h_{s'})\hat{\lambda}_1(h) \quad \textit{where} \quad h\in H_1 \quad \textit{and}$$
$$(\sum_s\lambda_2(h_s)h_s^{-1}g_j\hat{\lambda}_1)(x)=0 \quad \textit{if} \quad x\notin D_j^{-1}\ .$$

(ii) *Let $A_j(\hat{\lambda}_2)=\sum_s\lambda_2(h_s)h_s^{-1}g_j\hat{\lambda}_1$, then*

$$\begin{array}{cccc} A_j\colon & Y_2 & \longrightarrow & \mathscr{Y}_1 \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & x_m\hat{\lambda}_2 & \longmapsto & x_mA_j(\hat{\lambda}_2) \end{array}$$

is a well defined FG-homomorphism for each $j\in J$.

(iii) *$\{A_j\}_{j\in J}$ are linearly independent in* $\mathrm{Hom}_{FG}(Y_2, \mathscr{Y}_1)$.

PROOF. (i) From the definition of $\mathscr{Y}_1=\{\sum_n c_n y_n \hat{\lambda}_1\}$ where $c_n \in F$, it is straightforwards.

(ii) It is clear that A_j is a well-defined F-linear map. Let $g \in G$ and assume $gx_m\hat{\lambda}_2 = x_{m_0}h\hat{\lambda}_2 = \lambda_2(h)x_{m_0}\hat{\lambda}_2$ where $gx_m = x_{m_0}h \in x_{m_0}H_2$ for some $h \in H_2$. Then we have

$$A_j(gx_m\hat{\lambda}_2) = \lambda_2(h)x_{m_0}A_j(\hat{\lambda}_2) .$$

Now we shall show $gA_j(x_m\hat{\lambda}_2) = A_j(gx_m\hat{\lambda}_2)$. Since $h^{-1}H_2 = H_2 = \cup_s h^{-1}h_s^{-1}H_1^{(g_j^{-1})}$ (disjoint), there exists $r_s \in H_1^{(g_j^{-1})}$ such that $h^{-1}h_s^{-1} = h_{s'}^{-1}r_s$ for each s. Hence $gA_j(x_m\hat{\lambda}_2) = gx_mA_j(\hat{\lambda}_2) = x_{m_0}hA_j(\hat{\lambda}_2)$. Let $x \notin D_j^{-1}$, then $h^{-1}x \notin D_j^{-1}$ and we have $\lambda_2(h)A_j(\hat{\lambda}_2)(x) = 0 = hA_j(\hat{\lambda}_2)(x)$. Assume $h_s^{-1}g_jh_0 \in H_2g_jH_1 = D_j^{-1} = \cup_s h_s^{-1}g_jH_1$ where $h_0 \in H_1$, then we have

$$\lambda_2(h)A_j(\hat{\lambda}_2)(h_s^{-1}g_jh_0) = \lambda_2(h)\lambda_2(h_s)\hat{\lambda}_1(h_0)$$

and

$$\begin{aligned} hA_j(\hat{\lambda}_2)(h_s^{-1}g_jh_0) &= A_j(\hat{\lambda}_2)(h^{-1}h_s^{-1}g_jh_0) = A_j(\hat{\lambda}_2)(h_{s'}^{-1}r_sg_jh_0) = A_j(\hat{\lambda}_2)(h_{s'}^{-1}g_jr_s^{g_j}h_0) \\ &= \lambda_2(h_{s'})\hat{\lambda}_1(r_s^{g_j}h_0) = \lambda_2(h_{s'})\hat{\lambda}_1(h_0)(\lambda_1(r_s^{g_j}))^{-1} = \lambda_2(h_{s'})\hat{\lambda}_1(h_0)(\lambda_1^{g_j^{-1}}(r_s))^{-1} \\ &= \lambda_2(h_{s'})\hat{\lambda}_1(h_0)\lambda_2(r_s^{-1}) = \lambda_2(h_sh)\hat{\lambda}_1(h_0) . \end{aligned}$$

Therefore we have $hA_j(\hat{\lambda}_2) = \lambda_2(h)A_j(\hat{\lambda}_2)$ and thus

$$gA_j(x_m\hat{\lambda}_2) = A_j(gx_m\hat{\lambda}_2) \quad \text{for all} \quad g \in G \quad \text{and} \quad x_m\hat{\lambda}_2 ,$$

that is, A_j is a well defined FG-homomorphism for each $j \in J$.

(iii) Suppose $\sum_{j\in J} t_jA_j = 0$, where $t_j \in F$ and almost all $t_{j'}$s are zero, then $\sum_{j\in J} t_jA_j(\hat{\lambda}_2)(h_s^{-1}g_{j_0}h) = t_{j_0}\lambda_2(h_s)\hat{\lambda}_1(h) = 0$ for any $j_0 \in J$ and $h_s^{-1}g_{j_0}h \in D_{j_0}^{-1}$. Hence $t_j = 0$ for all $j \in J$ and $\{A_j\}_{j\in J}$ are linearly independent. Q.E.D.

Since $A_j(\hat{\lambda}_2) = \sum_s \lambda_2(h_s)h_s^{-1}g_j\hat{\lambda}_1 \in \mathscr{Y}_1$ and $A_j(\hat{\lambda}_2)$ vanishes outside of the coset D_j^{-1}, for any scalar $c_j \in F$ we can define an element $(\sum_{j\in J} c_jA_j)(\hat{\lambda}_2)$ of $\mathscr{Y}_1$ to be

$$(\sum_{j\in J} c_jA_j)(\hat{\lambda}_2) = \sum_{j\in J} (\sum_s c_j\lambda_2(h_s)h_s^{-1}g_j\hat{\lambda}_1) .$$

Since $\mathscr{Y}_1$ is an FG-module we can define $(\sum_{j\in J} c_jA_j)(x_m\hat{\lambda}_2)$ to be $x_m(\sum_{j\in J} c_jA_j)(\hat{\lambda}_2) \in \mathscr{Y}_1$ for each x_m. From (ii) of Proposition 1.3 we have

$$\begin{aligned} (\sum_{j\in J} c_jA_j)(x_m\hat{\lambda}_2) &= x_m(\sum_{j\in J} c_jA_j)(\hat{\lambda}_2) = x_m \sum_{j\in J} (\sum_s c_j\lambda_2(h_s)h_s^{-1}g_j\hat{\lambda}_1) \\ &= \sum_{j\in J} (\sum_s c_j\lambda_2(h_s)x_mh_s^{-1}g_j\hat{\lambda}_1) . \end{aligned}$$

We also define $(\sum_{j\in J} c_jA_j)(\sum_m t_mx_m\hat{\lambda}_2)$ to be $\sum_m t_m(\sum_{j\in J} c_jA_j)(x_m\hat{\lambda}_2)$

where almost all $t_{m',s} \in F$ are zero.

THEOREM 2.2. *Let* G, H_t, λ_t, Y_t, A_j $(j \in J)$ *and* $\mathscr{Y}_1$ *etc. be as before.*

(i) *Let* $\mathscr{J} = \{j \in J \mid |D_j^{-1}/H_1| < \infty\}$ *and* $E = \mathrm{Hom}_{FG}(Y_2, Y_1)$, *then* E *is an* F*-subspace of* $\mathrm{Hom}_{FG}(Y_2, \mathscr{Y}_1)$ *and* $\{A_j \mid j \in \mathscr{J}\}$ *forms an* F*-basis for* E.

(ii) *For any scalars* $\{c_j \in F \mid j \in J\}$, $\sum_{j \in J} c_j A_j$ *(see above) is a well defined* FG*-homomorphism of* Y_2 *into* $\mathscr{Y}_1$.

(iii) *Let* f *be an arbitrary element of* $\mathrm{Hom}_{FG}(Y_2, \mathscr{Y}_1)$, *then there exists unique scalar* $c_j \in F$ *for each* $j \in J$ *such that*

$$f = \sum_{j \in J} c_j A_j \, .$$

PROOF. (i) Since $Y_1 \subset \mathscr{Y}_1$, E is an F-subspace of $\mathrm{Hom}_{FG}(Y_2, \mathscr{Y}_1)$. From [2, Theorem (1.3)] it is clear that $\{A_j \mid j \in \mathscr{J}\}$ forms an F-basis for E.

(ii) We have defined $(\sum_{j \in J} c_j A_j)(\hat{\lambda}_2)$ to be $\sum_{j \in J} (\sum_s c_j \lambda_2(h_s) h_s^{-1} g_j \hat{\lambda}_1)$ and $(\sum_{j \in J} c_j A_j)(x_m \hat{\lambda}_2)$ to be $x_m(\sum_{j \in J} c_j A_j)(\hat{\lambda}_2)$ (see above). Since $\{x_m \hat{\lambda}_2\}$ forms an F-basis for Y_2, $\sum_{j \in J} c_j A_j$ is a well defined F-linear map of Y_2 into $\mathscr{Y}_1$. Now let $g \in G$ and $x_m \in \{x_m\}$, and assume $g x_m = x_{m_0} h \in x_{m_0} H_2$ for some $x_{m_0} \in \{x_m\}$ and $h \in H_2$. Then

$$\Big(\sum_{j \in J} c_j A_j\Big)(g x_m \hat{\lambda}_2) = \Big(\sum_{j \in J} c_j A_j\Big)(x_{m_0} h \hat{\lambda}_2) = \lambda_2(h) x_{m_0} \Big(\sum_{j \in J} c_j A_j\Big)(\hat{\lambda}_2) \, .$$

Since $g(\sum_{j \in J} c_j A_j)(x_m \hat{\lambda}_2) = g x_m (\sum_{j \in J} c_j A_j)(\hat{\lambda}_2) = x_{m_0} h (\sum_{j \in J} c_j A_j)(\hat{\lambda}_2)$, we only have to show $h(\sum_{j \in J} c_j A_j)(\hat{\lambda}_2) = \lambda_2(h)(\sum_{j \in J} c_j A_j)(\hat{\lambda}_2)$. Let $x \notin D_j^{-1}$ for any $j \in J$, then since $h^{-1}x \notin D_j^{-1}$ for any $j \in J$, either, we have $h\{(\sum_{i \in J} c_j A_j)(\hat{\lambda}_2)\}(x) = (\sum_{j \in J} (\sum_s \lambda_2(h_s) h_s^{-1} g_j \hat{\lambda}_1))(h^{-1}x) = 0 = \lambda_2(h)\{(\sum_{j \in J} c_j A_j)(\hat{\lambda}_2)\}(x)$. In case $x \in D_j^{-1}$ for some $j \in J$, then since $h^{-1}x$ also belongs to $D_j^{-1} = H_2 g_j H_1$, we have

$$\begin{aligned} h\Big\{\Big(\sum_{j \in J} c_j A_j\Big)(\hat{\lambda}_2)\Big\}(x) &= c_j A_j(\hat{\lambda}_2)(h^{-1}x) = c_j\{h A_j(\hat{\lambda}_2)\}(x) \\ &= c_j A_j(h \hat{\lambda}_2)(x) = c_j \lambda_2(h) A_j(\hat{\lambda}_2)(x) = \lambda_2(h)\Big(\sum_{j \in J} c_j A_j\Big)(\hat{\lambda}_2)(x) \, . \end{aligned}$$

Hence $\sum_{j \in J} c_j A_j$ is a well defined FG-homomorphism.

(iii) Let take a fixed representative g_i from each (H_2, H_1)-double coset D_i^{-1} $(i \in I)$ such that $\{g_i\}_{i \in I} \supset \{g_j\}_{j \in J}$. Let $H_2 = \cup_q H_1^{(g_i^{-1})} r_q$ (disjoint) for each $i \in I$, then we have $H_2 g_i H_1 = \cup_q r_q^{-1} g_i H_1$ (disjoint) and we can take an F-basis of Y_1 to be $\cup_{i \in I} \{r_q^{-1} g_i \hat{\lambda}_1 \mid H_2 = \cup_q H_1^{(g_i^{-1})} r_q \text{ (disjoint)}\}$. We always assume that one of $\{r_q\}$, that is, r_{q_*} is 1.

Let f be an arbitrary element of $\mathrm{Hom}_{FG}(Y_2, \mathscr{Y}_1)$, then we have $f(\hat{\lambda}_2) = \sum_{q,i} c_{q,i} r_q^{-1} g_i \hat{\lambda}_1$, where $c_{q,i} \in F$ and almost all $\{c_{q,i}\}$ are not necessarily zero. Since $f(h\hat{\lambda}_2) = \lambda_2(h) f(\hat{\lambda}_2)$ for any $h \in H_2$, we have $h \sum_{q,i} c_{q,i} r_q^{-1} g_i \hat{\lambda}_1 =$

$\sum_{q,i} c_{q,i} h r_q^{-1} g_i \hat{\lambda}_1 = \lambda_2(h) \sum_{q,i} c_{q,i} r_q^{-1} g_i \hat{\lambda}_1 = \sum_{q,i} c_{q,i} \lambda_2(h) r_q^{-1} g_i \hat{\lambda}_1$. Since $hH_2 g_i H_1 = H_2 g_i H_1 = \cup_q h r_q^{-1} g_i H_1 = \cup_q r_q^{-1} g_i H_1$, we have $\sum_q c_{q,i} h r_q^{-1} g_i \hat{\lambda}_1 = \lambda_2(h) \sum_q c_{q,i} r_q^{-1} g_i \hat{\lambda}_1$ for any $h \in H_2$, where $i \in I$. Let $r_{q_0} \in \{r_q\}$, then we have

$$\sum_q c_{q,i} r_{q_0} r_q^{-1} g_i \hat{\lambda}_1 = \lambda_2(r_{q_0}) \sum_q c_{q,i} r_q^{-1} g_i \hat{\lambda}_1 .$$

Thus $c_{q_0,i} = \lambda_2(r_{q_0}) c_{q*,i}$ for any $r_{q_0} \in \{r_q\}$, and we have

$$f(\hat{\lambda}_2) = \sum_i \sum_q c_{q*,i} \lambda_2(r_q) r_q^{-1} g_i \hat{\lambda}_1 = \sum_i c_{q*,i} (\sum_q \lambda_2(r_q) r_q^{-1} g_i \hat{\lambda}_1) .$$

Let $h \in H_2$, then there exists $h_q \in H_1^{(g_i^{-1})}$ such that $hr_q^{-1} = r_{q'}^{-1} h_q$, because $hH_2 = H_2 = \cup_q r_q^{-1} H_1^{(g_i^{-1})}$ (disjoint). Thus we have

$$\begin{aligned} c_{q*,i} \sum_q \lambda_2(r_q) h r_q^{-1} g_i \hat{\lambda}_1 &= c_{q*,i} \lambda_2(h) \sum_q \lambda_2(r_q) r_q^{-1} g_i \hat{\lambda}_1 = c_{q*,i} \sum_q \lambda_2(r_q) r_{q'}^{-1} h_q g_i \hat{\lambda}_1 \\ &= c_{q*,i} \sum_q \lambda_2(r_q) r_{q'}^{-1 g_i} (h_q)^{g_i} \hat{\lambda}_1 = c_{q*,i} \sum_q \lambda_2(r_q) r_{q'}^{-1} g_i \lambda_1 (h_q)^{g_i} \hat{\lambda}_1 \\ &= c_{q*,i} \sum_q \lambda_2(r_q) \lambda_1^{g_i^{-1}}(h_q) r_{q'}^{-1} g_i \hat{\lambda}_1 , \quad \text{for any} \quad i \in I . \end{aligned}$$

Hence we have

$$c_{q*,i} \lambda_2(r_q) \lambda_1^{g_i^{-1}}(h_q) = c_{q*,i} \lambda_2(h) \lambda_2(r_{q'}) = c_{q*,i} \lambda_2(h) \lambda_2(h_q r_q h^{-1}) = c_{q*,i} \lambda_2(h_q) \lambda_2(r_q) .$$

Therefore for any $h_0 \in H_1^{(g_i^{-1})}$ if we take $h = r_{q'}^{-1} h_0 r_q$ ($\in H_2$) for some q and q', $hr_q^{-1} = r_{q'}^{-1} h_0$ and $c_{q*,i} \lambda_1^{g_i^{-1}}(h_0) = c_{q*,i} \lambda_2(h_0)$. Hence $c_{q*,i} \neq 0$ only when $\lambda_1^{g_i^{-1}} = \lambda_2$ on $H_1^{(g_i^{-1})}$. Thus we have

$$f(\hat{\lambda}_2) = \sum_{j \in J} c_{q*,j} (\sum_q \lambda_2(r_q) r_q^{-1} g_j \hat{\lambda}_1) = (\sum_{j \in J} c_{q*,j} A_j)(\hat{\lambda}_2) .$$

Since $\sum_{j \in J} c_{q*,j} A_j$ is an FG-homomorphism from (ii), we have $f = \sum_{j \in J} c_{q*,j} A_j$. It is clear that the scalars $\{c_{q*,j} \mid j \in J\}$ are uniquely determined by f.

Q.E.D.

EXAMPLE 2.3. Let G be a group with a BN-pair, that is, G has subgroups B and N such that

(i) G is generated by B and N, and $B \cap N$ is normal in N;

(ii) let $W = N/B \cap N$, then there exists a subset S of W which generates W and every element of S is of order 2;

(iii) $\sigma B w \subset BwB \cup B\sigma wB$ for any $\sigma \in S$ and $w \in W$;

(iv) $\sigma B \sigma \not\subset B$ for any $\sigma \in S$.

It is well known that G has a Bruhat decomposition

(a) $G = \cup_{w \in W} BwB$

(b) $BwB = Bw'B \Rightarrow w = w'$ for any $w, w' \in W$.

Let ι be a trivial linear character of B into $F^\times$, then in case $|W|$, the cardinal number of W, is finite, we have

$$\dim_F \mathrm{Hom}_{FG}(FG\iota, (FG\iota)^*) = |W| .$$

Hence the space $\mathrm{Hom}_{FG}(FG\iota, (FG\iota)^*)$ seems to be a reasonable generalization of the Hecke algebra $\mathrm{Hom}_{FG}(FG\iota, FG\iota)$ of finite BN-pair G, because in most cases $\dim_F \mathrm{Hom}_{FG}(FG\iota, FG\iota)=1$ when G is infinite (see [2, Theorem (2.1)]).

References

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