TOKYO J. MATH.
VOL. 7, NO. 1, 1984

A Remark on a Theorem of B. T. Batikyan and E. A. Gorin

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Introduction

Let X be a compact Hausdorff space and $\tilde{X}=\beta(N\times X)$ be the Stone-Čech compactification of $N\times X$, the direct product of the space of natural numbers N and X. We consider a Banach space E which satisfies that E is a Banach space lying in $C(X)$ (resp. $C_R(X)$) with the norm $\|\cdot\|_E$ such that $\|u\|_\infty\leqq\|u\|_E$ for each u in E where $\|\cdot\|_\infty$ denotes the supremum norm and we also suppose that E separates the points of X and contains constant functions with $\|1\|_E=1$. Let $\tilde{E}=1^\infty(N, E)$ be the Banach space of all bounded sequences in E with the norm $\|(f_n)\|_{\tilde{E}}=\sup_n\|f_n\|_E$. For every (f_n) in $\tilde{E}$ we can suppose that (f_n) is a bounded continuous function on $N\times X$ defined as $(f_n)(m, x)=f_m(x)$ for (m, x) in $N\times X$. So we may suppose that $\tilde{E}$ is lying in $C(\tilde{X})$ (resp. $C_R(\tilde{X})$). We say that E is *ultra-separating* on X if $\tilde{E}$ separates the points of $\tilde{X}$ (cf. [2], [3], [4]).

§1. A characterization for ultraseparability.

We say that A is a Banach function algebra on X if A is a Banach algebra lying in $C(X)$ which separates the points of X and contains constant functions. It is shown in B. T. Batikyan and E. A. Gorin [2] that ultraseparability for a Banach function algebra A can be characterized as follows:

There exist a natural number m and $\delta>0$ such that for every pair of disjoint compact subsets Y_1 and Y_2 of X there exist functions $f_1, f_2, \cdots, f_m$ and $g_1, g_2\cdots, g_m$ in the unit ball of A which satisfy

$$\sum_{i=1}^{m}(|f_i|-|g_i|)\geqq\delta \quad on \quad Y_1$$

$$\sum_{i=1}^{m}(|f_i|-|g_i)\leqq-\delta \quad on \quad Y_2 .$$

Let $\operatorname{Re}E=\{u\in C_R(X)\colon \exists f\in E, \operatorname{Re}f=u\}$. Then $\operatorname{Re}E$ is also an above

Received April 25, 1983

type Banach space in $C_R(X)$ with the quotient norm $N(u)=\inf\{\|f\|_E: f\in E, \operatorname{Re} f=u\}$. It is well-known that E is ultraseparating on X if and only if $\operatorname{Re} E$ is so. Thus we may assume that E is lying in $C_R(X)$ throughout this paper. We show that ultraseparability for such E is characterized in the same way as the case of Banach function algebras.

THEOREM. *Let E be an above type Banach space in $C_R(X)$, and let h be a real valued continuous function on $[-1, 1]$ which is not the restriction of a polynomial. Then E is ultraseparating on X if and only if the following condition is satisfied:*

(*) *There are a natural number m and $\delta>0$ such that if Y_1 and Y_2 are disjoint compact subsets of X then we can choose $f_1, f_2, \cdots, f_m$ in the unit ball of E and real numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$ with $|\alpha_i|\leqq 1$ satisfying*

$$\sum_{i=1}^{m}\alpha_i h\circ f_i(x)\geqq\delta \quad on \quad Y_1$$

$$\sum_{i=1}^{m}\alpha_i h\circ f_i(x)\leqq -\delta \quad on \quad Y_2 .$$

To prove Theorem, we need the following lemma which is easily proved by the same way as in [5].

LEMMA. *Let E and h be as above. If $[h\circ E]$ is the uniform closure of the space of all linear combinations of $h\circ u$ for u in the unit ball of E, then $[h\circ E]=C_R(X)$.*

PROOF OF THEOREM. We prove Theorem as same way as the proof of Theorem in [2]. Assume that E is ultraseparating on X and yet the requirements formulated in (*) are not satisfied, that is, for any positive integer k there exists a pair of disjoint compact subsets $Y_{1,k}$ and $Y_{2,k}$ of X such that for every $f_1, f_2, \cdots, f_k$ in the unit ball of E and for every real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$ with $|\alpha_i|\leqq 1$ for $i=1, 2, \cdots, k$ and one of the following is not satisfied.

$$\sum_{i=1}^{k}\alpha_i h\circ f_i\geqq 1/k \quad \text{on} \quad Y_{1,k}$$

$$\sum_{i=1}^{k}\alpha_i h\circ f_i\geqq -1/k \quad \text{on} \quad Y_{2,k} .$$

There exists F_k in $C_R(X)$ with $\|F_k\|_\infty\leqq 1$ such that $F_k=1$ on $Y_{1,k}$ and $F_k=-1$ on $Y_{2,k}$. Put $\tilde{F}=(F_k)\in C_R(\tilde{X})$. By Lemma we can choose $\tilde{u}_i=(u_{i,j})=(u_{i,1}, u_{i,2}, \cdots, u_{i,j}, \cdots)\in\tilde{E}$ with $\|u_{i,j}\|_E\leqq 1$ for $i=1, 2, \cdots, n$ and real numbers $\beta_1, \beta_2, \cdots, \beta_n$ such that

$$\left|\sum_{i=1}^{n} \beta_i h\circ\tilde{u}_i - F\right| < 1/2 .$$

So, for any k,

$$\left|\sum_{i=1}^{n} \beta_i h\circ u_{i,k} - 1\right| < 1/2 \quad \text{on} \quad Y_{1,k}$$

and

$$\left|\sum_{i=1}^{n} \beta_i h\circ u_{i,k} + 1\right| < 1/2 \quad \text{on} \quad Y_{2,k} .$$

Thus

$$\sum_{i=1}^{n} (\beta_i/M) h\circ u_{i,k} > 1/2M \quad \text{on} \quad Y_{1,k}$$

and

$$\sum_{i=1}^{n} (\beta_i/M) h\circ u_{i,k} < -1/2M \quad \text{on} \quad Y_{2,k}$$

where $M=\max\{|\beta_1|, |\beta_2|, \cdots, |\beta_n|\}$ which is a contradiction for large k.

Now assume that there exist a real valued continuous function h on $[-1, 1]$ and a positive number δ and a positive integer m which satisfy $(*)$. Thus $\sum_{i=1}^{m} \alpha_i h\circ u_i > \delta/2$ on Y_1 and $\sum_{i=1}^{m} \alpha_i h\circ u_i < -\delta/2$ on Y_2. Let $\tilde{x}_1$ and $\tilde{x}_2$ be different points in $\tilde{X}$ and U_1 and U_2 be open neighborhoods of $\tilde{x}_1$ and $\tilde{x}_2$ respectively which have disjoint closures. Put

$$Y_{1,k} = \{x \in X : (k, x) \in \overline{(\{k\}\times X)\cap U_1}\}$$
$$Y_{2,k} = \{x \in X : (k, x) \in \overline{(\{k\}\times X)\cap U_2}\} .$$

We may suppose that $Y_{1,k}$ and $Y_{2,k}$ are disjoint compact subsets of X for every k. Thus for every positive integer k there exist $f_{1,k}, f_{2,k}, \cdots, f_{m,k}$ in the unit ball of E and real numbers $\alpha_{1,k}, \alpha_{2,k}, \cdots, \alpha_{m,k}$ with $|\alpha_{i,k}| \leqq 1$ for $i=1, 2, \cdots, m$ such that

$$\sum_{i=1}^{m} \alpha_{i,k} h\circ f_{i,k} > \delta/2 \quad \text{on} \quad Y_{1,k}$$

and

$$\sum_{i=1}^{m} \alpha_{i,k} h\circ f_{i,k} < -\delta/2 \quad \text{on} \quad Y_{2,k} .$$

We put $\tilde{\alpha}_i = (\alpha_{i,k}) = (\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,k}, \cdots)$ and $\tilde{f}_i = (f_{i,k}) = (f_{i,1}, f_{i,2}, \cdots,$

$f_{i,k}, \cdots)$ for every positive i, then $\tilde{\alpha}_i$ and $\tilde{f}_i$ are functions in $\tilde{E}$ and, in addition,

$$\sum_{i=1}^{m} \tilde{\alpha}_i h \circ \tilde{f}_i \geqq \delta/2 \quad \text{on} \quad U_1$$

and

$$\sum_{i=1}^{m} \alpha_i h \circ \tilde{f}_i \leqq -\delta/2 \quad \text{on} \quad U_2$$

especially $\sum_{i=1}^{m} \tilde{\alpha}_i h \circ \tilde{f}_i(\tilde{x}_1) \neq \sum_{i=1}^{m} \tilde{\alpha}_i h \circ \tilde{f}_i(\tilde{x}_2)$. Thus there exist j such that $\tilde{\alpha}_j h \circ \tilde{f}_j(\tilde{x}_1) \neq \tilde{\alpha}_j h \circ \tilde{f}_j(\tilde{x}_2)$ so $\tilde{\alpha}_j$ or $\tilde{f}_j$ separate $\tilde{x}_1$ and $\tilde{x}_2$. That is, $\tilde{E}$ separates $\tilde{x}_1$ and $\tilde{x}_2$.

REMARK. One can use $h(t)=|t|$ in Theorem to characterize ultra-separability for E in the same way as a theorem of B. T. Batikyan and E. A. Gorin.

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