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Weighted Norm Inequalities for Certain Pseudo-Differential Operators

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Introduction

Several authors (R. R. Coifman, C. Fefferman, R. A. Hunt, B. Mukenhoupt and R. L. Wheeden [1], [4], [6]) have shown that if T is the Hardy-Littlewood maximal operator or a classical singular integral operator, the weighted norm inequality

$$\int_{\boldsymbol{R}^n} |Tf(x)|^p w(x)dx \leqq C\int_{\boldsymbol{R}^n} |f(x)|^p w(x)dx$$

is valid if and only if the weight function w satisfies the A_p-condition (see (3) below). Recently, N. Miller [5] showed that the same thing is also true when T is the pseudo-differential operator of order 0. In this note, we shall show that even when T is the pseudo-differential operator whose symbol, $\sigma(x, \xi)$, satisfies the regularity condition on x weaker than in the symbol of the pseudo-differential operator of order 0, the same thing is also true.

A pseudo-differential operator σ with symbol $\sigma(x, \xi)$, defined initially on the Schwartz class $\mathscr{S}(\boldsymbol{R}^n)$, is given by

$$f \longrightarrow \sigma f(x) = \int_{\boldsymbol{R}^n} \sigma(x, \xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi \ ,$$

where $\hat{f}$ denote the Fourier transform of f defined by

$$\hat{f}(\xi) = \int_{\boldsymbol{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx \ .$$

We shall say that the function $\sigma(x, \xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$ is a symbol of order m if it satisfies

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$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}\sigma(x,\xi)\right|\leqq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}$$

for all multi-indices α and β. A symbol of order $-\infty$ is the one for which the above estimate is satisfied for all real number m.

Before we state our result we give some necessary definitions for symbols.

We call the modulus of continuity every function $\omega\colon \boldsymbol{R}^{+}\to\boldsymbol{R}^{+}$ which is continuous, increasing, concave and such that $\omega(0)=0$.

DEFINITION 1. Letting ω be a modulus of continuity, we denote $\sigma(x,\xi)\in\Sigma_{\omega}$, if $\sigma(x,\xi)\colon \boldsymbol{R}^{n}\times\boldsymbol{R}^{n}\to\boldsymbol{C}$ is the continuous function and for all $\alpha\in\boldsymbol{N}^{n}$, there exists a constant C_{α} for which we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\sigma(x,\xi)\right|\leqq C_{\alpha}(1+|\xi|)^{-|\alpha|}\,, \tag{1}$$

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\sigma(x+y,\xi)-\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\sigma(x,\xi)\right|\leqq C_{\alpha}\omega(|y|)(1+|\xi|)^{-|\alpha|}\,. \tag{2}$$

DEFINITION 2. Let $1<p<\infty$. A measurable function f is said to belong to the weighted L^{p}, $L^{p}(\boldsymbol{R}^{n}, wdx)$, with weight function w, if

$$\int_{\boldsymbol{R}^{n}}|f(x)|^{p}w(x)dx<\infty\,.$$

We denote the weighted L^{p} norm by

$$\|f\|_{p,w}=\left(\int_{\boldsymbol{R}^{n}}|f(x)|^{p}w(x)dx\right)^{1/p}\,.$$

For $1<p<\infty$, a positive weight function w is said to be in the class A_{p}, $w\in A_{p}$, or w satisfies the A_{p}-condition, if w is locally integrable and satisfies the condition

$$\sup_{Q}\left(\frac{1}{|Q|}\int_{Q}w(x)dx\right)\left(\frac{1}{|Q|}\int_{Q}w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\,, \tag{3}$$

where the supremum is taken over all cubes Q in $\boldsymbol{R}^{n}$.

Our result is stated as follows:

THEOREM 1. *Suppose* $1<p<\infty$. *If the modulus of continuity* ω *satisfies the condition* $j^{2}\omega(2^{-j})<C$, *for all* $j\in\boldsymbol{N}$, *every pseudo-differential operator* σ *with symbol* $\sigma(x,\xi)\in\Sigma_{\omega}$ *has a bounded extention to* $L^{p}(\boldsymbol{R}^{n}, wdx)$ *if and only if* $w\in A_{p}$.

To prove this theorem, we first introduce some notations. Let Q denote any cube in $\boldsymbol{R}^n$ and write $|Q|$ for the Lebesgue measure of Q. For a locally integrable function f, let f_Q denote the mean value of f over Q, that is,

$$f_Q=\frac{1}{|Q|}\int_Q f(x)dx .$$

We list the several operators we use later:

(i) Maximal function of f:

$$Mf(x)=\sup_Q \frac{1}{|Q|}\int_Q |f(y)|dy ,$$

where the supremum ranges over all cubes Q containing x.

(ii) Modified maximal function of f:

$$M_r f(x)=\sup_Q \left(\frac{1}{|Q|}\int_Q |f(y)|^r dy\right)^{1/r} ,$$

where the supremum ranges over all cubes Q containing x.

(iii) Dyadic maximal function of f:

$$f^*(x)=\sup_Q \frac{1}{|Q|}\int_Q |f(y)|dy ,$$

where the supremum ranges over all dyadic cubes Q containing x, with sides parallel to the axes.

(iv) Sharp function of f:

$$f^\sharp(x)=\sup_Q \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy ,$$

where the supremum ranges over all cubes Q containing x.

Throughout this note C will denote a constant not necessarily the same at each occurrence.

§ 1. Preliminaries.

In this section we shall descrive the known facts as lemmas.

LEMMA 1. *Let* $w \in A_p$, *then* $\mathscr{S}(\boldsymbol{R}^n)$ *is dense in* $L^p(\boldsymbol{R}^n, wdx)$, $1<p<\infty$. (N. Miller [5], Lemma 2.1)

LEMMA 2. *Let* σ *be a pseudo-differential operator with symbol* $\sigma(x, \xi) \in \Sigma_\omega$. *Then the following two conditions are equivalent*:

(4) $$\sum_{0}^{\infty}[\omega(2^{-j})]^2<\infty\ ,$$

(5) *for all* p, $1<p<\infty$, σ *is bounded on* $L^p(\boldsymbol{R}^n)$.

(R. R. Coifman and Y. Meyer [2], Theorem 9)

DEFINITION 3. (R. R. Coifman and Y. Meyer [2]) We say that $\sigma(x, \xi)\in\Sigma_\omega$ is a reduced symbol if there exist a constant $C>0$, a function $\phi\in C_0^\infty(\boldsymbol{R}^n)$ and a sequence m_j, $j\geqq 0$, of continuous functions on $\boldsymbol{R}^n$ such that

(6) $$\sigma(x, \xi)=\sum_{0}^{\infty} m_j(x)\phi(2^{-j}\xi)\ ,$$

where

(7) $$\|m_j\|_\infty\leqq C\ ,$$

(8) $$\|m_j(x+y)-m_j(x)\|_\infty\leqq C\omega(|y|)\ ,$$

(9) $$\phi\quad\text{is supported in}\quad \frac{1}{3}\leqq|\xi|\leqq 1\ ,$$

and

(10) $$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\phi(\xi)\right|\leqq C\quad\text{for}\quad |\alpha|\leqq 2n\ .$$

LEMMA 3. *For every symbol* $\sigma(x, \xi)\in\Sigma_\omega$, *we can find a sequence of reduced symbols* $\sigma_k(x, \xi)$, $k\in\boldsymbol{Z}^n$, *such that*

$$\sigma(x, \xi)=\tau(x, \xi)+\sum_{k\in\boldsymbol{Z}^n}(1+|k|^2)^{-n}\sigma_k(x, \xi)$$

and

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\tau(x, \xi)\right|\leqq C_\alpha\ ,$$

$$\tau(x, \xi)=0\quad if\quad |\xi|\geqq 1\ .$$

(R. R. Coifman and Y. Meyer [2], Proposition 5, p. 46)

LEMMA 4. *Suppose* $1<p<\infty$. *Let* ϕ *be a radial, decreasing, positive and integrable function. Set* $\phi_t(x)=t^{-n}\phi(x/t)$. *Then*

$$\sup_{t>0}|f*\phi_t(x)|\leqq CMf(x)\quad for\quad f\in\mathcal{S}(\boldsymbol{R}^n)\ .$$

([7], p. 63)

LEMMA 5. *Let ϕ be a given function in Definition 3. Then for $t \geqq 0$, there is a constant $C_t > 0$ such that the inequality*

$$|y|^t \left| \int_{R^n} \phi(2^{-j}\xi) e^{2\pi i y \cdot \xi} d\xi \right| \leqq C_t 2^{j(n-t)}$$

holds for all $y \in \boldsymbol{R}^n$ and every integer $j \geqq 0$.

This lemma is easily derived by integration by parts. For the details, see Lemma 2.9 in [5], where its analogue is found.

LEMMA 6. *There is a constant $C > 0$ such that*

$$\|f^*\|_{p,w} \leqq C \|f^\#\|_{p,w} \quad \text{for all} \quad f \in L^p(\boldsymbol{R}^n, w dx) \cap L^1(\boldsymbol{R}^n) .$$

([5], Lemma 2. See also [3], Theorem 5.)

§2. The main part of the proof of Theorem 1.

In order to prove Theorem 1, we first show the following result which we state as Theorem 2 and constitutes the main part of the proof of Theorem 1.

THEOREM 2. *Suppose $1 < r < \infty$ and let σ be a pseudo-differential operator with symbol $\sigma(x, \xi) \in \Sigma_\omega$. If the modulus of continuity ω satisfies the condition*

$$j^2 \omega(2^{-j}) < C \quad \text{for all} \quad j \in \boldsymbol{N} , \tag{11}$$

then there is a constant $C > 0$ such that the pointwise estimate

$$(\sigma f)^\#(x^0) \leqq C M_r f(x^0) \quad \text{for all} \quad x^0 \in \boldsymbol{R}^n, f \in \mathscr{S}(\boldsymbol{R}^n)$$

holds.

PROOF. The proof is based on the idea of the proof of Theorem 2.8 in [5].

Given $x^0 \in \boldsymbol{R}^n$, we let Q be a cube containing x^0, with center x' and diameter d. Let $\theta \in C_0^\infty(R^n)$ satisfy $0 \leqq \theta(x) \leqq 1$, be 1 when $|x - x'| \leqq 2d$, and vanish when $|x - x'| \geqq 3d$. Then for $f \in \mathscr{S}(\boldsymbol{R}^n)$

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |\sigma f(x) - (\sigma f)_Q| dx \\ &\quad \leqq \frac{2}{|Q|} \int_Q |\sigma(\theta f)(x)| dx \\ &\qquad + \frac{1}{|Q|} \int_Q |\sigma((1-\theta) f)(x) - [\sigma((1-\theta) f)]_Q| dx . \end{aligned}$$

We note that the condition (11) implies the condition (4). Letting Q' be the cube centered at x', with sides of length $7d$ parallel to those of Q, we see that the first term is dominated by

$$2\left(\frac{1}{|Q|}\int_Q |\sigma(\theta f)(x)|^r dx\right)^{1/r} \leqq C\left(\frac{1}{|Q|}\int_{R^n} |(\theta f)(x)|^r dx\right)^{1/r}$$

(by Lemma 2)

$$\leqq C\left(\frac{1}{|Q'|}\int_{Q'} |f(x)|^r dx\right)^{1/r}$$
$$\leqq CM_r f(x^0) .$$

To deal with the second term, we, for simplicity, write f for $(1-\theta)f$, and we assume that f has the support in the set $\{x: |x-x'|\geqq 2d\}$.

We begin by decomposing the symbol $\sigma(x, \xi)$ into the sum of simpler symbols by making use of Lemma 3. Then we can write

$$\begin{aligned}(\sigma f)(x) &= \int_{R^n} \hat{f}(\xi)\sigma(x, \xi)e^{2\pi i x\cdot\xi}d\xi \\ &= \int_{R^n} \hat{f}(\xi)\tau(x, \xi)e^{2\pi i x\cdot\xi}d\xi \\ &\quad + \int_{R^n} f(y)\int_{R^n}\sum_{k\in Z^n}(1+|k|^2)^{-n}\sigma_k(x, \xi)e^{2\pi i(x-y)\cdot\xi}d\xi dy \\ &= Tf(x) + \sum_{k\in Z^n}(1+|k|^2)^{-n}S_k f(x) ,\end{aligned}$$

say. T is a pseudo-differential operator whose symbol is $\tau(x, \xi)$; the ξ-support of this symbol is contained in the set $\{\xi: |\xi|\leqq 1\}$, and $\tau(x, \xi)$ has the property that

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha \tau(x, \xi)\right| \leqq C_\alpha , \quad \alpha\in N^n .$$

Thus we can write

$$Tf(x) = \int_{R^n} f(y)K(x, x-y)dy ,$$

where

$$K(x, y) = \int_{R^n} \tau(x, \xi)e^{2\pi i y\cdot\xi}d\xi .$$

Note that $K(x, y)$ has the property that

$$|K(x, y)| \leqq \frac{C_m}{(1+|y|)^m} \quad \text{for all} \quad x \in \boldsymbol{R}^n$$

where m is any integer greater than n, and C_m is a constant independent of x. In fact more generally we have

$$\begin{aligned} \left| y^\alpha \left(\frac{\partial}{\partial y}\right)^\beta K(x, y)\right| &= C_{\alpha\beta} \left| \int_{\boldsymbol{R}^n} \tau(x, \xi)\xi^\beta \left(\left(\frac{\partial}{\partial \xi}\right)^\alpha e^{2\pi i y\cdot\xi}\right) d\xi \right| \\ &\leqq C_{\alpha\beta} \int_{\boldsymbol{R}^n} \left| \left(\frac{\partial}{\partial \xi}\right)^\alpha \tau(x, \xi)\xi^\beta \right| d\xi \\ &\leqq C_{\alpha\beta} , \end{aligned}$$

with $C_{\alpha\beta}$ independent of x and y. The last inequality is due to the property of $\tau(x, \xi)$ mentioned above. Then because of Lemma 4, we have

$$\begin{aligned} |Tf(x)| &\leqq \int_{\boldsymbol{R}^n} |f(y)|\, |K(x, x-y)| dy \\ &\leqq C_m \int_{\boldsymbol{R}^n} \frac{|f(y)|}{(1+|x-y|)^m} dy \leqq C_m Mf(x) . \end{aligned}$$

Thus we have

$$(Tf)^\#(x^0) \leqq CM_r f(x^0) ,$$

and hence

$$(\sigma f)^\#(x^0) \leqq CM_r f(x^0) + \sum_{k \in \boldsymbol{Z}^n} (1+|k|^2)^{-n} (S_k f)^\#(x^0) .$$

Therefore our next task is to examine the operator S_k. We note that $\sigma_k(x, \xi)$ satisfies the condition (6) to (10) in Definition 3 with m_{jk}, ϕ_k in place of m_j, ϕ respectively, where C's are independent of k and also of j. Then for every k

$$\begin{aligned} S_k f(x) &= \int_{\boldsymbol{R}^n} f(y) \int_{\boldsymbol{R}^n} \sum_{j=0}^{\infty} m_{jk}(x) \phi_k(2^{-j}\xi) e^{2\pi i (x-y)\cdot\xi} d\xi dy \\ &= \sum_{j=0}^{\infty} R_{jk} f(x) , \end{aligned}$$

say. We now estimate $(S_k f)^\#(x^0)$.

$$\frac{1}{|Q|} \int_Q |R_{jk} f(x) - (R_{jk} f)_Q| dx \tag{12}$$

$$=\frac{1}{|Q|}\int_Q\left|\frac{1}{|Q|}\int_Q\{R_{jk}f(x)-R_{jk}f(z)\}dz\right|dx$$

$$=\frac{1}{|Q|}\int_Q\left|\frac{1}{|Q|}\int_Q\int_{R^n}f(y)\int_{R^n}\phi_k(2^{-j}\xi)\cdot\right.$$

$$\left.[m_{jk}(x)e^{2\pi i(x-y)\cdot\xi}-m_{jk}(z)e^{2\pi i(z-y)\cdot\xi}]d\xi dydz\right|dx\,.$$

To estimate this quantity, we distinguish two cases:
Case 1. $2^jd\geqq 1$. The last quantity is dominated by

$$2\sum_{p=1}^{\infty}\frac{1}{|Q|}\int_Q\int_{2^pd\leq|y-x'|\leq 2^{p+1}d}|f(y)|\cdot$$

$$\left|\int_{R^n}m_{jk}(x)\phi_k(2^{-j}\xi)e^{2\pi i(x-y)\cdot\xi}d\xi\right|dydx$$

$$\leqq C\sum_{p=1}^{\infty}\int_Q\frac{2^{np}}{|Q_p|}\int_{2^pd\leq|y-x'|\leq 2^{p+1}d}\frac{|f(y)|}{|x-y|^{n+1}}|x-y|^{n+1}\cdot$$

$$\left|\int_{R^n}\phi_k(2^{-j}\xi)e^{2\pi i(x-y)\cdot\xi}d\xi\right|dy|m_j^k(x)|dx\,,$$

where Q_p is the cube with center x' and with sides parallel to those of Q and with diameter $2^{p+2}d$. The last term is bounded by

$$C\sum_{p=1}^{\infty}d^n2^{np}(2^pd)^{-n-1}2^{-j}\frac{1}{|Q_p|}\int_{Q_p}|f(y)|dy$$

(by Lemma 5 with $t=n+1$, and the condition (7) of m_{jk})

$$\leqq C(2^jd)^{-1}Mf(x^0).$$

Case 2. $2^jd<1$. In this case, (12) is dominated by

$$\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{\infty}\int_{2^pd\leq|y-x'|\leq 2^{p+1}d}|f(y)|\left|\int_{R^n}\phi_k(2^{-j}\xi)[m_{jk}(x)e^{2\pi i(x-y)\cdot\xi}\right.$$

$$\left.-m_{jk}(x)e^{2\pi i(z-y)\cdot\xi}+m_{jk}(x)e^{2\pi i(z-y)\cdot\xi}-m_{jk}(z)e^{2\pi i(z-y)\cdot\xi}]d\xi\right|dydzdx$$

$$\leqq\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{\infty}\int_{2^pd\leq|y-x'|\leq 2^{p+1}d}|f(y)|\left|\int_{R^n}\phi_k(2^{-j}\xi)\cdot\right.$$

$$\left.[e^{2\pi i(x-y)\cdot\xi}-e^{2\pi i(z-y)\cdot\xi}]m_{jk}(x)d\xi\right|dydzdx$$

$$+\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{\infty}\int_{2^pd\leq|y-x'|\leq 2^{p+1}d}|f(y)|\left|\int_{R^n}\phi_k(2^{-i}\xi)\cdot\right.$$

$$\left.[m_{jk}(x)-m_{jk}(z)]e^{2\pi i(z-y)\cdot\xi}d\xi\right|dydzdx$$

$$=A+B\,.$$

We first estimate A.

$$A\leqq\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{\infty}\int_{2^pd\leqq|y-x'|\leqq 2^{p+1}d}|f(y)|\left|\int_{R^n}\phi_k(2^{-j}\xi)\cdot\right.$$

$$\left.\left\{\sum_{q=1}^{n}(x_q-z_q)\int_0^1 2\pi i\xi_q e^{2\pi i(x(t)-y)\cdot\xi}dt\right\}d\xi\right|dydz|m_{jk}(x)|dx$$

(where $x(t)=z+t(x-z)$)

$$\leqq\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{\infty}\int_{2^pd\leqq|y-x'|\leqq 2^{p+1}d}\frac{|f(y)|}{|y-x'|^{n+1/2}}\sum_{q=1}^{n}|x_q-z_q|\cdot$$

$$\int_0^1|x(t)-y|^{n+1/2}\left|\int_{R^n}\phi_k(2^{-j}\xi)\,2\pi i\xi_q e^{2\pi i(x(t)-y)\cdot\xi}d\xi\right|dtdydz|m_{jk}(x)|dx\,.$$

The integral with respect to ξ is handled just as in the proof of Lemma 5 with $t=n+1/2$ and we see that the last member is not greater than

$$C\sum_{p=1}^{\infty}(2^pd)^n(2^pd)^{-n-1/2}d2^{j/2}\frac{1}{|Q_p|}\int_{Q_p}|f(y)|dy$$

$$\leqq C(2^jd)^{1/2}Mf(x^0)\,.$$

Next we estimate B.

$$B\leqq\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{\infty}\int_{2^pd\leqq|y-x'|\leqq 2^{p+1}d}|f(y)|\cdot$$

$$\left|\int_{R^n}\phi_k(2^{-j}\xi)e^{2\pi i(z-y)\cdot\xi}d\xi\right|dy|m_{jk}(x)-m_{jk}(z)|dzdx$$

$$\leqq\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{j_0}\int_{2^pd\leqq|y-x'|\leqq 2^{p+1}d}|f(y)|\cdot$$

$$\left|\int_{R^n}\phi_k(2^{-j}\xi)\,e^{2\pi i(z-y)\cdot\xi}d\xi\right|dy|m_{jk}(x)-m_{jk}(z)|dzdx$$

$$+\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_{Q}\sum_{p=j_0+1}^{\infty}\int_{2^pd\leqq|y-x'|\leqq 2^{p+1}d}|f(y)|\cdot$$

$$\left|\int_{R^n}\phi_k(2^{-j}\xi)\,e^{2\pi i(z-y)\cdot\xi}d\xi\right|dy|m_{jk}(x)-m_{jk}(z)|dzdx$$

$$=B_1+B_2,$$

say, where j_0 is the integer which satisfies $2^{j_0}d<1\leqq 2^{j_0+1}d$.

$$B_1=\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=1}^{j_0}\int_{2^pd\leqq|y-x'|\leqq 2^{p+1}d}\frac{|f(y)|}{|z-y|^n}\cdot|z-y|^n\cdot$$

$$\left|\int_{R^n}\phi_k(2^{-j}\xi)e^{2\pi i(z-y)\cdot\xi}d\xi\right|dy|m_{jk}(x)-m_{jk}(z)|dzdx$$

$$\leqq C_1\sum_{p=1}^{j_0}(2^pd)^n(2^pd)^{-n}\omega(d)\frac{1}{|Q_p|}\int_{Q_p}|f(y)|dy$$

(by Lemma 5 with $t=n$ and the condition (8) of m_{jk})

$$\leqq C_1 j_0\omega(2^{-j_0})Mf(x^0)\ .$$

$$B_2=\frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\sum_{p=j_0+1}^{\infty}\int_{2^pd\leqq|y-x'|\leqq 2^{p+1}d}\frac{|f(y)|}{|z-y|^{n+1}}\cdot|z-y|^{n+1}$$

$$\left|\int_{R^n}\phi_k(2^{-i}\xi)e^{2\pi i(z-y)\cdot\xi}d\xi\right|dy|m_{jk}(x)-m_{jk}(z)|dzdx$$

$$\leqq C_2\sum_{p=j_0+1}^{\infty}(2^pd)^n(2^pd)^{-n-1}2^{-j}\omega(d)\frac{1}{|Q_p|}\int_{Q_p}|f(y)|dy$$

(by Lemma 5 with $t=n+1$ and the condition (8) of m_{jk})

$$\leqq C_2\sum_{p=j_0+1}^{\infty}(2^pd)^{-1}2^{-j}\omega(1)Mf(x^0)$$

$$\leqq C_2 2^{-j}Mf(x^0)\ .$$

Thus we have

$$B\leqq B_1+B_2\leqq\{C_1 j_0\omega(2^{-j_0})+C_2 2^{-j}\}Mf(x^0)\ .$$

Putting two cases together, we have shown that if Q is any cube containing x^0, then

$$\begin{aligned}(S_kf)^\sharp(x^0)&\leqq\sum_{j=0}^{\infty}(R_{jk}f)^\sharp(x^0)\\ &\leqq\Big\{C\sum_{2^jd\geqq 1}(2^jd)^{-1}+C'\sum_{2^jd<1}(2^jd)^{1/2}\\ &\quad+\sum_{2^jd<1}(C_1j_0\omega(2^{-j_0})+C_2 2^{-j})\Big\}Mf(x^0)\\ &\leqq\{C+j_0^2\omega(2^{-j_0})\}Mf(x^0)\\ &\leqq CMf(x^0)\quad\text{(by (11))}\\ &\leqq CM_rf(x^0)\ .\end{aligned}$$

We thus find that

$$\begin{aligned}(\sigma f)^\sharp(x^0)&\leqq CM_rf(x^0)+\sum_{k\in Z^n}(1+|k|^2)^{-n}CM_rf(x^0)\\ &\leqq CM_rf(x^0)\ .\end{aligned}$$

Summarizing, we have shown that if Q is any cube containg x^0, then

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(\sigma f)(x)-(\sigma f)_Q|dx &\leqq [\sigma(\theta f)]^\sharp(x^0)+[T((1-\theta)f)]^\sharp(x^0)+\sum_{k\in Z^n}(1+|k|^2)^{-n}[S_k((1-\theta)f)]^\sharp(x^0)\\ &\leqq CM_rf(x^0)+CM_r((1-\theta)f)(x^0)\\ &\leqq CM_rf(x^0),\end{aligned}$$

where the constant C is independent of Q, f and x^0. When we take the supremum of the left side over all cubes Q containing x^0, we finally obtain the desired inequality:

$$(\sigma f)^\sharp(x^0)\leqq CM_rf(x^0) \quad \text{for all} \quad x^0\in \boldsymbol{R}^n,\ f\in\mathscr{S}(\boldsymbol{R}^n). \qquad \text{Q.E.D.}$$

We shall show the sufficiency of the A_p-condition.

THEOREM 3. *If $1<p<\infty$ and $w\in A_p$, then any pseudo-differential operator σ with the type in Theorem 2 has a bounded extention to all of $L^p(\boldsymbol{R}^n, wdx)$.*

PROOF. We prove this in the same way as was used by N. Miller. If $f\in\mathscr{S}(\boldsymbol{R}^n)$, then since $\sigma f\in L^p(\boldsymbol{R}^n, wdx)\cap L^1(\boldsymbol{R}^n)$

$$\begin{aligned}\|\sigma f\|_{p,w}&\leqq\|(\sigma f)^*\|_{p,w}\leqq C\|(\sigma f)^\sharp\|_{p,w}\\ &\leqq C\|M_rf\|_{p,w} \quad \text{if} \quad 1<r<\infty\\ &\leqq C\|f\|_{p,w} \quad \text{if} \quad 1<r<p<\infty\end{aligned}$$

(cf. Theorem 2.12 in [5]).

Because of Lemma 1, we can extend σ to a bounded operator on $L^p(\boldsymbol{R}^n, wdx)$. Q.E.D.

§ 3. The final part of the proof of Theorem 1.

The A_p-condition is necessary even for the continuity of the best-behaved pseudo-differential operators. In fact, the following theorem due to N. Miller (Theorem 2.2 in [5]) shows:

Suppose w is a nonnegative locally integrable function whose zero-set has Lebesgue measure 0. If every pseudo-differential operator of order $-\infty$ is continuous on $L^p(\boldsymbol{R}^n, wdx)$, then $w\in A_p$, $1<p<\infty$.

It is obvious that a pseudo-differential operator of order $-\infty$ is one of the pseudo-differential operators considered in Theorem 2 and we find, by Theorem 2 and the last result, that a weight function w satisfies the

A_p-condition if and only if every pseudo-differential operator of the type in Theorem 2 is bounded on $L^p(\boldsymbol{R}^n, wdx)$. This completes the proof of Theorem 1.

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